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On ω_1-Weakly p^α-Projective Abelian p-Groups

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Abstract. For any infinite ordinal α, we consider the class of ω_1-weakly p^α-projective p-primary abelian groups and give its systematic study when $\alpha \leq \omega_1$. In particular, some applications are obtained for the ordinal $\alpha = \omega \cdot 2 + n$ with $n \in \mathbb{N}$. One of the basic theorems is that a subgroup of an n-summable group of countable length α is also n-summable exactly when this subgroup is a C_α n-summable group. Our results substantially generalize those established by P. Keef in his recent publications.

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1. Introduction and preliminary terminology

Throughout the text, let all groups into consideration be p-torsion abelian groups where p is a fixed arbitrary prime integer. Our further notions and notations are mainly standard and follow these from the existing literature–see, for instance, [8] and [9]; those that are new will be stated explicitly in what follows. For example, the abbreviation *a* Σ*-cyclic group* is used for *direct sums of cyclic groups*, while we abbreviate *a dsc group* for *direct sums of countable groups*. Note that each Σ-cyclic group is a dsc group.

Attempting to enlarge the classical theory of $p^{\omega+n}$-projective groups ($n \in \mathbb{N} \cup \{0\}$) due to Nunke [19], that are groups G for which there exists a p^n-bounded (nice) subgroup P such that G/P is Σ-cyclic or, equivalently, that are groups K which are isomorphic to S/L for some Σ-cyclic group S and its p^n-bounded subgroup L, Keef recently defined in [16] the so-called ω_1-$p^{\omega+n}$-projective groups.

Definition 1.1. *Let $n \geq 0$. The group G is said to be ω_1-$p^{\omega+n}$-projective if there exists a countable subgroup C of G such that G/C is a $p^{\omega+n}$-projective group. When C is finite, G is called finitely ω_1-$p^{\omega+n}$-projective or more appropriately ω-$p^{\omega+n}$-projective, and when C is balanced in G, the group G is called balanceable ω_1-$p^{\omega+n}$-projective.*

We point out that in the case of a finite subgroup C, it need not be a direct summand of G. Even more, in [16, Example 2.3], was constructed an ω-$p^{\omega+n}$-projective group G

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of length $\omega+n$ which is not $p^{\omega+n}$-projective; in fact, such a group G has the property that $p^{\omega}G \cong \mathbb{Z}(p^n)$ and that $G/p^{\omega}G$ is $p^{\omega+1}$-projective, and hence this quotient is $p^{\omega+n}$-projective for any $n \geq 1$. Nevertheless, $G/p^{\omega+1}G$ need not be $p^{\omega+1}$-projective because otherwise, appealing to [7], G must be the direct sum of a countable group and a $p^{\omega+1}$-projective group, whence it is $p^{\omega+1}$-projective itself being with the property $p^{\omega+1}G=\{0\}$. This is a contradiction, however, with the aforementioned Example 2.3.

Moreover, we also conjecture that there exists a non $p^{\omega+n}$-projective group G with a countable balanced subgroup B such that G/B is $p^{\omega+n}$-projective; thus this group G is proper balanceable ω_1-$p^{\omega+n}$-projective. We also conjecture that B even need not be a direct summand of G.

Actually, the (finite or infinite) countable subgroup C can also be chosen to be nice in G satisfying the inclusions $p^{\omega+n}G \subseteq C \subseteq p^{\omega}G$ (see, for example, Proposition 1.4 and the proof of Theorem 3.9 from [16]). It follows as well that separable ω_1-$p^{\omega+n}$-projective groups are themselves $p^{\omega+n}$-projective (see, e.g., [4], Theorem 4.2). Besides, there is an ω_1-$p^{\omega+n}$-projective group of length $\omega+n$ which is not $p^{\omega+n}$-projective (see, for instance, Example 2.3 in [16]); in fact, such a group is of necessity $p^{\omega+2n}$-projective. Likewise, Keef showed there some principal characterizations of these groups in different forms.

However, we extended this concept in [3] as follows:

Definition 1.2. *Let $n \geq 0$. The group G is said to be weakly ω_1-$p^{\omega+n}$-projective if there exists a countable nice subgroup N of G such that $N \subseteq p^{\omega}G$ and $G/N/p^{\omega+n}(G/N)$ is $p^{\omega+n}$-projective. The last is equivalent to asking that $G/(p^{\omega+n}G+N)$ is $p^{\omega+n}$-projective. When N is balanced in G, the group G is called weakly balanced ω_1-$p^{\omega+n}$-projective.*

Apparently, ω_1-$p^{\omega+n}$-projective groups are weakly ω_1-$p^{\omega+n}$-projective as well as balanceable ω_1-$p^{\omega+n}$-projective groups are weakly balanced ω_1-$p^{\omega+n}$-projective. Moreover, any group G for which $G/p^{\omega+n}G$ is $p^{\omega+n}$-projective is necessarily weakly $p^{\omega+n}$-projective. In particular, so are the pillared groups G that are groups such that $G/p^{\omega}G$ is Σ-cyclic. Likewise, each weakly ω_1-$p^{\omega+n}$-projective group G has the property that $G/p^{\omega}G$ is $p^{\omega+n}$-projective.

Also, if $p^{\omega+n}(G/N)=0$, or even if $p^{\omega+n}G$ is countable, weakly ω_1-$p^{\omega+n}$-projective groups are of necessity ω_1-$p^{\omega+n}$-projective.

On the other hand, Keef considered in [17] the class of *weakly $p^{\omega\cdot 2}$-projectives*, that is, the class consisting of all groups G with Σ-cyclic subgroups H such that G/H are Σ-cyclic. It is self-evident that, for any $n<\omega$, $p^{\omega+n}$-projective groups are weakly $p^{\omega\cdot 2}$-projective. Moreover, we will show below (Proposition 2.2) that ω_1-$p^{\omega+n}$-projective groups of length at most $\omega\cdot 2$ are precisely weak $p^{\omega\cdot 2}$-projective. Keef also defined in [18] the more general class of weakly p^{α}-projectives where $\alpha \leq \omega_1$. We will likewise explore in the sequel weakly p^{α}-projective and ω_1-weakly p^{α}-projective groups when $\omega \leq \alpha \leq \omega_1$ (Theorems 2.6, 2.9, 2.10, 2.12, 2.13 and 2.14) by considerably improving some major results of Keef from [18]. Finally, we will finish off the paper with some left-open problems and a new strategy for a successful improvement of the considered group classes.

2. Main concepts and results

We start here with a recollection of the following notion from [17].

Definition 2.1. *The group G is weakly $p^{\omega\cdot 2}$-projective if it has a Σ-cyclic subgroup S such that G/S is a Σ-cyclic group.*

Clearly, $p^{\omega+n}$-projective groups are weakly $p^{\omega\cdot 2}$-projective. It is also apparently seen that subgroups of weakly $p^{\omega\cdot 2}$-projectives inherit the same property. Moreover, weakly $p^{\omega\cdot 2}$-projective groups are $p^{\omega\cdot 2}$-bounded. Besides, it is clear that there is a p^{ω}-bounded (= separable) weakly $p^{\omega\cdot 2}$-projective group which is not ω_1-$p^{\omega+n}$-projective (= $p^{\omega+n}$-projective).

On the other hand, we may attempt to consider the following natural generalization of weakly $p^{\omega\cdot 2}$-projectives, namely: G is a group such that there is a Σ-cyclic subgroup $S \leq G$ with G/S a $p^{\omega+n}$-projective group for some $n \geq 0$. However, surprisingly, we always can take $n = 0$, that is, G/S is also Σ-cyclic. In other words, this $n > 0$ is irrelevant; in fact, since G/S is $p^{\omega+n}$-projective, G has a subgroup L with $p^nL \subseteq S \subseteq L$ and $G/S/L/S \cong G/L$ being Σ-cyclic. It follows from [8, Proposition 18.3], that L is also Σ-cyclic because so is S, so that G is weakly $p^{\omega\cdot 2}$-projective, as claimed.

Via the same idea, if both S and G/S are $p^{\omega+n}$-projective, we again can choose G/S to be Σ-cyclic, so that S is $p^{\omega+n}$-projective whereas G/S is Σ-cyclic - such a group G is weakly $p^{\omega\cdot 2+n}$-projective (see Definition 2.3).

By analogy with [16] we have the following concept from [18].

Definition 2.2. *The group G is ω_1-weakly $p^{\omega\cdot 2}$-projective if it has a countable subgroup C such that G/C is a weakly $p^{\omega\cdot 2}$-projective group.*

Evidently, ω_1-$p^{\omega+n}$-projective groups are ω_1-weakly $p^{\omega\cdot 2}$-projective. It is also obviously seen that subgroups of ω_1-weakly $p^{\omega\cdot 2}$-projectives retain the same property. Moreover, it is clear that weakly $p^{\omega\cdot 2}$-projective groups are ω_1-weakly $p^{\omega\cdot 2}$-projective. Referring to [7], a group G is termed ω-*totally* Σ-*cyclic* if each its separable subgroup is Σ-cyclic. It is known from Theorem 2.6 of [7] that ω-totally Σ-cyclic groups are precisely the direct sums of a countable group and a Σ-cyclic group. That is why, in the sense of Definition 1.1, ω-totally Σ-cyclic groups are precisely the ω_1-p^{ω}-projective ones.

The following statement gives several different classifications of the class of ω_1-weakly $p^{\omega\cdot 2}$-projectives. It was originally established in ([18], Proposition 3.11) in a slightly different but an equivalent form for $n = 0$. The proof here is, however, more transparent.

Theorem 2.1. *Suppose G is a group and $n < \omega$. The following are equivalent:*

(a) *G is ω_1-weakly $p^{\omega\cdot 2}$-projective;*
(b) *G is an extension of a Σ-cyclic group by an ω_1-$p^{\omega+n}$-projective factor (or even by an ω-totally Σ-cyclic factor);*
(c) *G is an extension of an ω-totally Σ-cyclic group by a $p^{\omega+n}$-projective factor (and even by a Σ-cyclic factor);*
(d) *G is a subgroup of a group of the form $H \oplus C$ where H is a $p^{\omega\cdot 2}$-bounded dsc group and C is countable;*
(e) *G is an extension of a weakly $p^{\omega\cdot 2}$-projective group by a countable factor.*

Proof. (a) implies (b): Let $G' = G/C$ be weakly $p^{\omega\cdot 2}$-projective for some countable subgroup C of G. Suppose $L \subseteq G'$ is Σ-cyclic such that G'/L is also Σ-cyclic. Define $K \subseteq G$ by the equation $K/C = L$. It follows immediately from [4] or [7] that K is ω-totally Σ-cyclic and moreover $G/K \cong G/C/K/C = G'/L$ is Σ-cyclic. But by what we have observed above K has a Σ-cyclic subgroup J such that K/J is countable. Since $G/J/K/J \cong G/K$ is Σ-cyclic, it follows at once from [4] or [7] that G/J is ω-totally Σ-cyclic, i.e., ω_1-p^{ω}-projective, so that G/J is actually ω_1-$p^{\omega+n}$-projective for any n, as asserted.

(b) implies (c): Let $J \subseteq G$ be Σ-cyclic such that the quotient $G" = G/J$ is ω_1-$p^{\omega+n}$-projective. Now $G"$ has a countable subgroup M such that $G"/M$ is $p^{\omega+n}$-projective. If T

is defined by the equality $T/J = M$, then an appeal to [4] or [7] deduces that T is ω-totally Σ-cyclic and $G/T \cong G/J/T/J = G"/M$ is $p^{\omega+n}$-projective, as required.

(c) implies (d): Let T be an ω-totally Σ-cyclic subgroup of G such that G/T is $p^{\omega+n}$-projective. Let X be a dsc group such that $p^{\omega+n}X = T$. Since G/T is $p^{\omega+n}$-projective, it follows according to [8] that the embedding $T \subseteq X$ extends to a homomorphism $\phi : G \to X$. Let Y be a $p^{\omega+n}$-bounded dsc group such that $G/T \subseteq Y$. Consider the homomorphism $\varphi : G \to X \oplus Y$ given by $\varphi(a) = (\phi(a), a+T)$. Since the kernel of φ is the intersection of T and the kernel of ϕ, and moreover ϕ is injective on T, it easily follows that φ is an embedding. Notice that $p^{\omega\cdot 2}X = p^{\omega}T$ will be countable, thus $X = C \oplus X'$, where C is countable and $p^{\omega\cdot 2}X' = \{0\}$. Letting $H = X' \oplus Y$, (d) now follows, as claimed.

(d) implies (e): Clearly, if $X = G \cap H$, then $X \subseteq H$ is a weakly $p^{\omega\cdot 2}$-projective group. Since G/X embeds in $(H \oplus C)/H \cong C$, it plainly follows that G/X is countable, as wanted.

(e) implies (a): Suppose X is a weakly $p^{\omega\cdot 2}$-projective group and G/X is countable. Let X be a subgroup of the $p^{\omega\cdot 2}$-bounded dsc group H, and suppose $H = \oplus_{i\in I} Y_i$, where each Y_i is countable. Next, let Z be a countable subgroup of G such that $G = X + Z$, and let I_1 be a countable subset of I such that $X \cap Z \subseteq \oplus_{i\in I_1} Y_i$. Letting $C = Z + (X \cap \oplus_{i\in I_1} Y_i)$, we see that C is countable. Observe that the isomorphism

$$G/C = (X+Z)/C = (X+C)/C \cong X/(X\cap C) = X/(X \cap \oplus_{i\in I_1} Y_i).$$

holds. In addition, since $X \cap (\oplus_{i\in I_1} Y_i)$ is the kernel of the composite homomorphism $X \to \oplus_{i\in I} Y_i \to \oplus_{i\in I\setminus I_1} Y_i$, it follows immediately that G/C is a weakly $p^{\omega\cdot 2}$-projective group, as desired. ∎

Note that in several of these conditions, the value of n is irrelevant. The last theorem also suggests the following question:

Question 2.1. Does it follow that pure-complete weakly $p^{\omega\cdot 2}$-projective groups are Σ-cyclic?

Note that if the answer is ”yes” this will strengthen the well-known fact that, for any natural number n, pure-complete $p^{\omega+n}$-projectives are Σ-cyclic (see, e.g., [12]).

The following gives another characterization of the class of ω_1-weakly $p^{\omega\cdot 2}$-projectives (compare with [18] as well) with a more direct proof.

Corollary 2.1. *The ω_1-weakly $p^{\omega\cdot 2}$-projective groups form the smallest class containing the weakly $p^{\omega\cdot 2}$-projective groups that is closed under ω_1-bijective homomorphisms.*

Proof. Definition 2.2 clearly ensures that every ω_1-weakly $p^{\omega\cdot 2}$-projective group must be in this minimal class, so we need to verify that the class of ω_1-weakly $p^{\omega\cdot 2}$-projective groups is closed under the formation of ω_1-bijective homomorphisms. To that end, assume that A is a subgroup of G such that G/A is countable. Owing to Lemma 1.9 of [16], it suffices to show that G is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if A has this property. The necessity being elementary, we will take care on the sufficiency. So, supposing A is ω_1-weakly $p^{\omega\cdot 2}$-projective, we can conclude by Theorem 2.1(e) that it has a subgroup $K \leq A$ which is weakly $p^{\omega\cdot 2}$-projective such that A/K is countable. It follows now that G/K will also be countable because so is $G/A \cong G/K/A/K$. Therefore, G also satisfies Theorem 2.1(e), as expected. ∎

The next necessary and sufficient condition reduces the study of ω_1-weakly $p^{\omega\cdot 2}$-projective groups to these of length $\omega \cdot 2$ and thereby is rather useful for applications.

Corollary 2.2. *The group G is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if $p^{\omega\cdot 2}G$ is countable and $G/p^{\omega\cdot 2}G$ is ω_1-weakly $p^{\omega\cdot 2}$-projective.*

Proof. If G is ω_1-weakly $p^{\omega\cdot 2}$-projective, then by virtue of Theorem 2.1(d) one may deduce that $p^{\omega\cdot 2}G \subseteq p^{\omega\cdot 2}C$ is countable, so this condition is valid in either direction. However, the natural homomorphism $G \to G/p^{\omega\cdot 2}G$ is then ω_1-bijective, so the result follows from Corollary 2.1. ∎

The first part of the next consequence strengthens Corollary 3.7 from [17] when $n = 0$, while the second part strengthens Proposition 1 from [2]. It is worthwhile noticing that if a group is *far from thick* it is not thick; the converse is however untrue–if a group is not thick, it need not be far from thick. The complete definition of far from thick groups is detailed given in [12].

Corollary 2.3. *Every ω_1-weakly $p^{\omega\cdot 2}$-projective group is far from thick. In particular, thick ω_1-weakly $p^{\omega\cdot 2}$-projective groups are bounded.*

Proof. Since each weakly $p^{\omega\cdot 2}$-projective group is far from thick by Corollary 3.7 of [17], and the collection of groups that are far from thick is closed under ω_1-bijective homomorphisms, the first result follows. The second one is its immediate implication. ∎

The last theorem, namely Theorem 2.1, allows us to refine Definition 2.2 like this:

Corollary 2.4. *The group G is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if it has a countable subgroup $C \subseteq p^\omega G$ with the property that G/C is a weakly $p^{\omega\cdot 2}$-projective group. In particular, $p^\omega G$ is ω-totally Σ-cyclic, and separable ω_1-weakly $p^{\omega\cdot 2}$-projective groups are weakly $p^{\omega\cdot 2}$-projective.*

Proof. In view of point (c) in Theorem 2.1, G/K is Σ-cyclic where $K = L \oplus S$ with L being countable and S being Σ-cyclic. Thus $p^\omega K$ is countable and $G/p^\omega K/K/p^\omega K \cong G/K$ is Σ-cyclic. But $K/p^\omega K$ is also Σ-cyclic, so that $G/p^\omega K$ is weakly $p^{\omega\cdot 2}$-projective. Finally, take $C = p^\omega K$. Since $p^\omega G \subseteq K$, the second part is immediate by the utilization of [7] taking into account the fact from [7] that a subgroup of an ω-totally Σ-cyclic group is again ω-totally Σ-cyclic. The final one follows because we can choose $C = 0$. ∎

Even a slightly more information can be obtained:

Proposition 2.1. *The group G is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if there exists a subgroup M such that $p^\omega M$ is countable and $G/p^\omega M$ is ω_1-weakly $p^{\omega\cdot 2}$-projective.*

Proof. Follows immediately from the proof of Corollary 2.4. ∎

Corollary 2.5. *Suppose G is a group such that $p^\omega G$ is countable. Then G is an ω_1-weakly $p^{\omega\cdot 2}$-projective group if and only if $G/p^\omega G$ is a weakly $p^{\omega\cdot 2}$-projective group.*

Proof. The natural map $G \to G/p^\omega G$ is ω_1-bijective, hence in virtue of Corollary 2.1, G is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if so is $G/p^\omega G$. Finally, Corollary 2.4 is applicable to deduce the claim. ∎

It certainly is not the case that the weakly $p^{\omega\cdot 2}$-projective groups are closed under ω_1-bijections. A very simple example is the following:

Example 2.1. Let G be a dsc group such that $p^{\omega\cdot 2}G$ is countable and non-zero. Since $G/p^{\omega\cdot 2}G$ is again a dsc group (cf. [8]), and hence it is ω_1-weakly $p^{\omega\cdot 2}$-projective (and even weakly $p^{\omega\cdot 2}$-projective), it follows from Corollary 2.2 that G is also ω_1-weakly $p^{\omega\cdot 2}$-projective. However, G need not be weakly $p^{\omega\cdot 2}$-projective because it is not contained in any dsc group of length $\omega\cdot 2$; in fact $p^{\omega\cdot 2}G \neq 0$.

Instead of the above example, the following is true:

Corollary 2.6. *If G is weakly $p^{\omega\cdot 2}$-projective and $p^{\omega}G$ is countable, then $G/p^{\omega}G$ is weakly $p^{\omega\cdot 2}$-projective.*

Proof. Follows directly from Corollary 2.5. ∎

The following arises quite naturally from Corollary 2.4.

Problem 2.1. *Is it true that ω_1-weakly $p^{\omega\cdot 2}$-projective groups of length $\leq \omega\cdot 2$ are themselves weakly $p^{\omega\cdot 2}$-projective groups?*

If the answer is "yes", the converse of Corollary 2.6 will follow, provided $p^{\omega}G$ is countable separable. However, Keef showed in ([18], Proposition 4.6) that there exists a $p^{\omega+1}$-bounded ω_1-weakly $p^{\omega\cdot 2}$-projective group which is not weakly $p^{\omega\cdot 2}$-projective, so that for inseparable groups the problem has a negative resolution. Nevertheless, the exhibited by Keef ω_1-weakly $p^{\omega\cdot 2}$-projective group is not ω_1-$p^{\omega+n}$-projective.

So, we will resolve the above problem in a weaker version via the following statement.

Proposition 2.2. *If G is an ω_1-$p^{\omega+n}$-projective group of length not exceeding $\omega\cdot 2$, then G is weakly $p^{\omega\cdot 2}$-projective.*

Proof. Follows in the same manner as Proposition 2.5 stated and proved below. ∎

About the validity of the converse implication, note that there exists a separable weakly $p^{\omega\cdot 2}$-projective group which is not ω_1-$p^{\omega+n}$-projective (= $p^{\omega+n}$-projective).

The best possible that we can offer in connection with Problem 2.1 is still the following:

Proposition 2.3. *If G is ω_1-weakly $p^{\omega\cdot 2}$-projective of length at most $\omega\cdot 2$, there exists a Σ-cyclic subgroup S of G such that G/S is weakly $p^{\omega\cdot 2}$-projective.*

Proof. Applying Corollary 2.4, there is a countable subgroup $C \leq p^{\omega}G$ such that G/C is weakly $p^{\omega\cdot 2}$-projective. But C is clearly separable, and thus a Σ-cyclic group. ∎

The following extends Proposition 2.4 from [16].

Proposition 2.4. *The direct sum of ω_1-weakly $p^{\omega\cdot 2}$-projective groups is ω_1-weakly $p^{\omega\cdot 2}$-projective if and only if all but a countably many of them are weakly $p^{\omega\cdot 2}$-projective.*

Proof. "$\Rightarrow$". Suppose $G = \oplus_{i\in I}G_i$ is an ω_1-weakly $p^{\omega\cdot 2}$-projective group. Since as noted above subgroups of ω_1-weakly $p^{\omega\cdot 2}$-projectives retain the same property, all G_i are ω_1-weakly $p^{\omega\cdot 2}$-projective groups too. In view of Definition 2.2 there exists a countable subgroup C of G such that $G/C = (\oplus_{i\in I}G_i)/C$ is weakly $p^{\omega\cdot 2}$-projective. Without loss of generality we may assume that $|I| > \aleph_0$; otherwise we are done. Thus there is a countable subset $J \subseteq I$ such that $C \subseteq \oplus_{j\in J}G_j$. Furthermore, $G/C \cong [(\oplus_{j\in J}G_j)/C] \oplus (\oplus_{i\in I\setminus J}G_i)$. But as observed above, subgroups of weakly $p^{\omega\cdot 2}$-projectives are again weakly $p^{\omega\cdot 2}$-projectives, whence so are all G_i with $i \in I\setminus J$.

"$\Leftarrow$". Write $G = \oplus_{i\in I}G_i$ where there is a countable subset $J \subseteq I$ such that all G_j are ω_1-weakly $p^{\omega\cdot 2}$-projective groups, while G_i are weakly $p^{\omega\cdot 2}$-projective groups for each $i \in I\setminus J$. By Definition 2.2 there exist countable subgroups C_j of G_j such that the quotients G_j/C_j

are weakly $p^{\omega\cdot 2}$-projective groups for any $j \in J$. Denoting $C = \oplus_{j\in J} C_j$, we have that C is at most countable and $G/C \cong [\oplus_{j\in J}(G_j/C_j)] \oplus [\oplus_{i\in I\setminus J} G_i]$. Since the direct sum of an arbitrary number (finite or infinite) of weakly $p^{\omega\cdot 2}$-projectives is obviously weakly $p^{\omega\cdot 2}$-projective, and therefore so is G/C, we can again apply Definition 2.2 to conclude that G is ω_1-weakly $p^{\omega\cdot 2}$-projective, as promised. ∎

As a direct consequence, we yield:

Corollary 2.7. *The countable direct sum of ω_1-weakly $p^{\omega\cdot 2}$-projective groups is an ω_1-weakly $p^{\omega\cdot 2}$-projective group.*

Using the equivalencies from Theorem 2.1, one can state the following two extensions of Definitions 2.1 and 2.2, respectively; cf. [18] too.

Definition 2.3. *The group G is said to be weakly $p^{\omega\cdot 2+n}$-projective if there exists a $p^{\omega+n}$-projective subgroup H of G such that G/H is Σ-cyclic.*

It is obvious that a subgroup of a weakly $p^{\omega\cdot 2+n}$-projective group is again weakly $p^{\omega\cdot 2+n}$-projective. Moreover, weakly $p^{\omega\cdot 2+n}$-projective groups are $p^{\omega\cdot 2+n}$-bounded.

Definition 2.4. *The group G is said to be ω_1-weakly $p^{\omega\cdot 2+n}$-projective if there exists a countable subgroup C of G such that G/C is weakly $p^{\omega\cdot 2+n}$-projective, that is, G is an ω_1-weakly $p^{\omega\cdot 2}$-projective group if there exists a subgroup A of G which is ω_1-$p^{\omega+n}$-projective (i.e., a subgroup of a direct sum of a countable group and a $p^{\omega+n}$-projective group) such that G/A is Σ-cyclic.*

It is self-evident that a subgroup of an ω_1-weakly $p^{\omega\cdot 2+n}$-projective group is also ω_1-weakly $p^{\omega\cdot 2+n}$-projective.

Using the methods alluded to above, all of the stated and proved assertions for weakly $p^{\omega\cdot 2}$-projectives and ω_1-weakly $p^{\omega\cdot 2}$-projectives can be generalized without any difficulty to weakly $p^{\omega\cdot 2+n}$-projective and ω_1-weakly $p^{\omega\cdot 2+n}$-projective groups, respectively. We will formulate only the most important of them. The next equivalencies are also initially documented in ([18], Proposition 3.11) in a slightly different but an equivalent form.

Theorem 2.2. *Suppose G is a group and $n < \omega$. The following are equivalent:*

(a) *G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective;*
(b) *G is an extension of a $p^{\omega+n}$-projective group by an ω_1-$p^{\omega+n}$-projective factor (or even by an ω-totally Σ-cyclic factor);*
(c) *G is an extension of an ω_1-$p^{\omega+n}$-projective group (i.e., of a subgroup of a direct sum of a countable group with a $p^{\omega+n}$-projective group) by a $p^{\omega+n}$-projective factor (and even by a Σ-cyclic factor);*
(d) *G is a subgroup of a group of the form $H \oplus C$ where H is a $p^{\omega\cdot 2+n}$-bounded dsc group and C is countable;*
(e) *G is an extension of a weakly $p^{\omega\cdot 2+n}$-projective group by a countable factor.*

Proof. Follows by the same token as Theorem 2.1 using some parallels in the theories of Σ-cyclic groups and $p^{\omega+n}$-projective groups. ∎

As noted above, by Theorem 1.5 of [16], it follows that any subgroup of the direct sum of a countable group and a $p^{\omega+n}$-projective group is ω_1-$p^{\omega+n}$-projective and vice versa; thus in point (c) the phrase "a subgroup of" cannot be eliminated, i.e., point (c) is not valid for direct sums of countable groups with $p^{\omega+n}$-projective groups.

Proposition 2.5. *Let G be a group such that $p^{\omega\cdot 2+n}G = 0$. If G is ω_1-$p^{\omega+n}$-projective, then G is weakly $p^{\omega\cdot 2+n}$-projective.*

Proof. In accordance with Theorem 1.2 (a1) from [16], there exists a p^n-bounded subgroup T of G such that G/T is the direct sum of a countable group and a Σ-cyclic group. Hence $p^\omega(G/T)$ is countable and $G/T/p^\omega(G/T) \cong G/T/A/T \cong G/A$ is Σ-cyclic where $A = \cap_{i<\omega}(p^iG+T)$. But $p^\omega(G/T) = A/T$ is countable and A is $p^{\omega+n}$-bounded. In fact, it is easily seen that $p^nA \subseteq p^\omega G$ whence $p^{\omega+n}A \subseteq p^{\omega\cdot 2+n}G = 0$. Therefore [4] applies to deduce that A is the direct sum of a countable group and a Σ-cyclic group, and thus A is of necessity $p^{\omega+n}$-projective. Finally, by Definition 2.3, G is weakly $p^{\omega\cdot 2+n}$-projective, as stated. ∎

We will say that the group G is *an ω-elongation* of the separable group A if $A \cong G/p^\omega G$.

The positive resolution of the next question will generalize ([16], Theorem 3.10) as well as will also give another useful characterization of weak $p^{\omega\cdot 2}$-projectivity in terms of ω-elongations.

Question 2.2. Suppose A is a separable group whose final rank satisfies the countability condition (in particular, the GCH holds). Then whether or not the following are equivalent:

(a) A is weakly $p^{\omega\cdot 2}$-projective;
(b) Every $p^{\omega+n}$-bounded ω-elongation of A is weakly $p^{\omega\cdot 2+n}$-projective;
(c) Every $p^{\omega+n}$-bounded ω-elongation of A is ω_1-weakly $p^{\omega\cdot 2+n}$-projective.

We will now study a little different group class.

Definition 2.5. *A group is called ω-totally weak $p^{\omega\cdot 2+n}$-projective if each its separable subgroup is weakly $p^{\omega\cdot 2+n}$-projective.*

Clearly, every subgroup of an ω-totally weak $p^{\omega\cdot 2+n}$-projective group retains the same property.

The following generalizes Corollary 3.4 from [16]; actually it provides a more simple and direct proof than that given there.

Proposition 2.6. *If G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective, then G is ω-totally weak $p^{\omega\cdot 2+n}$-projective.*

Proof. Since as observed above a subgroup of an ω_1-weakly $p^{\omega\cdot 2+n}$-projective group G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective as well, by what we have already shown all separable subgroups of G should be weakly $p^{\omega\cdot 2+n}$-projective, as required. ∎

The next statement is one of our crucial tools. It follows from the proof of Theorem 4.2 in [4] that if B is a separable group with a $p^{\omega+n}$-projective subgroup T such that B/T is countable, then B is $p^{\omega+n}$-projective too.

Proposition 2.7. *The class of separable weakly $p^{\omega\cdot 2+n}$-projectives is closed under ω_1-bijections.*

Proof. According to Lemma 1.9 of [16], let K be a separable group with a subgroup L such that K/L is countable. As noted above, if K is weakly $p^{\omega\cdot 2+n}$-projective, then so is L as being a subgroup.

Conversely, assume that L is weakly $p^{\omega\cdot 2+n}$-projective. Write L/T is Σ-cyclic for some $p^{\omega+n}$-projective subgroup T. Observe that $K/L \cong K/T/L/T$ is countable and hence one can write $K/T = (A/T) \oplus (B/T)$ where A/T is Σ-cyclic and B/T is countable for some subgroups A and B of K. Since B is separable, it follows in view of the fact from [4], lastly mentioned above, that B is $p^{\omega+n}$-projective. Furthermore, $K/B \cong K/T/B/T \cong A/T$ is Σ-cyclic and so by definition K is finally weakly $p^{\omega\cdot 2+n}$-projective, as claimed. ∎

We point out that we already know from the generalized version of Corollary 2.1 for any $n \geq 0$ that ω_1-weakly $p^{\omega\cdot 2+n}$-projectives are closed under ω_1-bijections, and that for separable groups the ω_1-weakly $p^{\omega\cdot 2+n}$-projectives are exactly the weakly $p^{\omega\cdot 2+n}$-projectives. Nevertheless, the above proof is more conceptual than that we had so far.

And so, we are now able to state and prove the generalization of Theorem 3.3 from [16]. Notice that ω-totally $p^{\omega+n}$-projectives are themselves ω-totally weak $p^{\omega\cdot 2+n}$-projectives, because $p^{\omega+n}$-projective groups are weakly $p^{\omega\cdot 2+n}$-projective.

Theorem 2.3. *The class of ω-totally weak $p^{\omega\cdot 2+n}$-projectives is closed under ω_1-bijections.*

Proof. Referring again to Lemma 1.9 in [16], suppose G is a group with a subgroup H such that G/H is countable. As indicated above, G being ω-totally weak $p^{\omega\cdot 2+n}$-projective implies the same for H.

To show the converse, let X be a separable subgroup of G. Hence $Y = X \cap H$ is a separable subgroup of H and so by hypothesis it is weakly $p^{\omega\cdot 2+n}$-projective. But $X/Y \cong (X+H)/H \subseteq G/H$ is countable. Consequently, it follows from Proposition 2.7 that X is also weakly $p^{\omega\cdot 2+n}$-projective, so that G is ω-totally weak $p^{\omega\cdot 2+n}$-projective, as asserted. ∎

We continue with the following improvement of Theorem 3.6 of [16].

Theorem 2.4. *Suppose G is a group such that $p^\omega G$ is countable. Then the following are equivalent:*

(i) *G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective;*
(ii) *G is ω-totally weak $p^{\omega\cdot 2+n}$-projective;*
(iii) *$G/p^\omega G$ is weakly $p^{\omega\cdot 2+n}$-projective.*

Proof. The implication (i) $\Rightarrow$ (ii) follows from Proposition 2.6. Moreover, the implication (ii) $\Rightarrow$ (iii) follows from Theorem 2.3 because the map $G \to G/p^\omega G$ is ω_1-bijective. The final implication (iii) $\Rightarrow$ (i) follows from the corresponding generalization of Corollary 2.5 to (ω_1-)weakly $p^{\omega\cdot 2+n}$-projectives. ∎

The following three questions are of interest because their positive resolutions will strengthen Corollary 3.5 and Questions 1 and 2, respectively, from [16].

Problem 2.2. *Does it follow that a group G is ω-totally weak $p^{\omega\cdot 2+n}$-projective if and only if*

(a) *$p^{\omega\cdot 2+n}G$ is countable and*
(b) *$G/p^{\omega\cdot 2+n}G$ is ω-totally weak $p^{\omega\cdot 2+n}$-projective?*

For the necessity notice that point (a) perhaps might be proved adapting the same technique as in [7], whence point (b) along with the sufficiency follow then easily from Lemma 1.9 (c) of [16] and Theorem 2.3 above.

Problem 2.3. *If G is ω-totally weak $p^{\omega\cdot 2+n}$-projective, does it follow that G is ω_1-weakly $p^{\omega\cdot 2+n}$-projective? That is, does the converse of Proposition 2.6 hold?*

Problem 2.4. *If G is ω-totally weak $p^{\omega\cdot 2+n}$-projective, can we conclude that $G/p^\omega G$ is weakly $p^{\omega\cdot 2+n}$-projective?*

The following notion was stated in [5].

Definition 2.6. *The group G is called n-summable if its p^n-socle $G[p^n]$ is a valuated direct sum of countable valuated subgroups.*

Two major facts which can be found in [5] are that for n-summable groups G we have $\mathrm{length}(G) \leq \omega_1$, and that a group is a dsc group if and only if it is n-summable for all $n < \omega$.

The following concept was originally defined in [14].

Definition 2.7. *The group G is called C_α n-summable for some arbitrary ordinal $\alpha \leq \omega_1$ if, for every $\lambda < \alpha$, each p^λ-high subgroup of G is n-summable.*

In particular, G is a C_α group if all p^λ-high subgroups of G are dsc groups for any $\lambda < \alpha$ (see, e.g., [23]). It follows immediately from Theorem 3.8 of [5] that a group is C_α n-summable for all natural numbers n if and only if it is a C_α group.

Before proving our next basic result, we need two preliminaries. Recall that for any two groups A and B, the symbol $A \bigtriangledown B$ denotes the *torsion product* of A and B, sometime designated by $\mathrm{Tor}(A,B)$.

The following statement was proved in [3], but for readers' convenience we repeat its proof.

Proposition 2.8. *If G is a C_α n-summable group for some ordinal $\alpha \leq \omega_1$ and N is a countable balanced subgroup of G, then G/N is also a C_α n-summable group.*

Proof. It follows from ([14], Corollary 2.4) that a group A is C_α n-summable if and only if $A \bigtriangledown H_\alpha$ is n-summable whenever H_α is the generalized Prüfer group of length α. We shall use this fact below without any concrete further referring.

First, let $\alpha < \omega_1$ whence H_α is countable. There is a short exact (left exact) sequence

$$0 \to N \bigtriangledown H_\alpha \to G \bigtriangledown H_\alpha \to (G/N) \bigtriangledown H_\alpha.$$

Since $N \bigtriangledown H_\alpha$ is countable and nice in $G \bigtriangledown H_\alpha$ (see, e.g., [8] or [21]), it follows that the map $G \bigtriangledown H_\alpha \to (G/N) \bigtriangledown H_\alpha$ has countable nice kernel $N \bigtriangledown H_\alpha$. But $G \bigtriangledown H_\alpha$ is n-summable and thus Theorem 1.1 from [6] applies to show that $(G \bigtriangledown H_\alpha)/(N \bigtriangledown H_\alpha) \cong (G/N) \bigtriangledown H_\alpha$ is n-summable. Therefore, G/N is C_α n-summable, as required.

Second, if now $\alpha = \omega_1$, then $H_\alpha = \oplus_{\beta<\alpha} H_\beta$ and thus $G \bigtriangledown H_\alpha \cong G \bigtriangledown (\oplus_{\beta<\alpha} H_\beta) \cong \oplus_{\beta<\alpha}(G \bigtriangledown H_\beta)$ is n-summable which implies with the aid of [5] that so is $G \bigtriangledown H_\beta$ for all $\beta < \alpha$. By what we have shown above, $(G/N) \bigtriangledown H_\beta$ is n-summable for all $\beta < \alpha$ which yields in view again of [5] that $(G/N) \bigtriangledown H_\alpha \cong (G/N) \bigtriangledown (\oplus_{\beta<\alpha} H_\beta) \cong \oplus_{\beta<\alpha}((G/N) \bigtriangledown H_\beta)$ is n-summable. This forces that G/N is a C_α n-summable group, indeed, as expected. ∎

The next example, due to Keef and sent to the author in a private communication, manifestly shows that the proof of Proposition 2.8 does not longer work if it is only assumed that N is nice in G. In other words, it is not true that if N is nice in G, then $N \bigtriangledown H$ is nice in $G \bigtriangledown H$ for some group H - notice that such type an example cannot be found in [8].

Example 2.2. Let N be any group such that $p^\omega N \cong \mathbb{Z}(p)$ and M any group such that $p^\omega M \cong \mathbb{Z}(p^2)$. Let $f : p^\omega N \to p^{\omega+1} M$ be some isomorphism and let G be the direct sum of N and M along f, that is,

$$G = (N \oplus M)/\{(x, f(x)), x \in p^\omega N\}.$$

Actually, one can write $G = N + M$ with $N \cap M = p^\omega N = p^{\omega+1} M$. Hence $N \cap p^{\omega+1} M = p^\omega N$ and $M \cap p^\omega N = p^{\omega+1} M$.

Finally, let H be the Prüfer group $H_{\omega+1}$ of length $\omega+1$.

We leave to the reader to check the following:

(1) $p^\omega G = p^\omega N + p^\omega M$;
(2) $p^{\omega+2}G = \{0\}$;
(3) $N \cap p^{\omega+1}G = p^\omega N \neq p^{\omega+1}N = \{0\}$;
(4) N is pure and nice in G but not its isotype subgroup;
(5) $N \bigtriangledown H$ is pure, but not nice in $G \bigtriangledown H$ as well as both $p^\omega N \bigtriangledown H$ and $p^{\omega+1}M \bigtriangledown H$ are not nice in $G \bigtriangledown H$.

This completes the example.

Proposition 2.9. *Suppose that N is a countable nice subgroup of a group G with limit length such that there exists a countable ordinal $\lambda <$ length(G) with the property $N \cap p^\lambda G = \{0\}$ (in particular, N is either finite or length$(G) \geq \omega_1$). If G is C_α n-summable, then G/N is C_α n-summable whenever $\lambda < \alpha \leq \omega_1$.*

Proof. Let H/N be a p^λ-high subgroup of G/N. Hence with the modular law at hand we have

$$(H/N)\cap p^\lambda(G/N) = (H/N)\cap(p^\lambda G+N)/N = [H\cap(p^\lambda G+N)]/N = (H\cap p^\lambda G+N)/N = \{0\}.$$

Therefore, $H \cap p^\lambda G \subseteq N$ and thus we conclude that $H \cap p^\lambda G \subseteq N \cap p^\lambda G = \{0\}$. We claim now that H is p^λ-high in G and so it is isotype in G (see, e.g., [8]). In fact, to show this, assume that $H' \supset H$ and $H' \cap p^\lambda G = \{0\}$ for some $H' \leq G$. Consequently,

$$(H'/N)\cap p^\lambda(G/N) = [H'\cap(p^\lambda G+N)]/N = [(H'\cap p^\lambda G)+N]/N = \{0\}$$

with $H'/N \supset H/N$, which is impossible, and so the claim is sustained.

Furthermore, H is by assumption n-summable and N being nice in G is nice in H as well. Now, employing [6], H/N should also be n-summable, whence G/N is C_α n-summable invoking [14], as asserted. ∎

Lemma 2.1. *Suppose $\omega+n \leq \alpha \leq \omega_1$ where $n \geq 1$. Then the group G is $p^{\omega+n}$-projective and C_α n-summable if and only if G is a dsc group of length not exceeding $\omega+n$.*

Proof. Observe foremost that G is p^α-projective C_α n-summable, and hence we may apply ([14], Proposition 2.5) to deduce that G is n-summable. Furthermore, we may employ [1] to infer that G is a dsc group of length no more than $\omega+n$, as asserted. ∎

We are now have all the ingredients to prove the following criterion for a group to be a dsc group. Since both C_α groups and n-summable groups are themselves C_α n-summable, the next theorem somewhat significantly generalizes Theorem 3.9 from [16]. (Compare also with [3].)

Theorem 2.5. *Let $\omega+n \leq \alpha \leq \omega_1$ and let G be a balanceable ω_1-$p^{\omega+n}$-projective group or an ω-$p^{\omega+n}$-projective group of limit length for some $n \geq 1$. Then G is a C_α n-summable group if and only if G is a dsc group.*

Proof. Assume firstly that G is balanceable ω_1-$p^{\omega+n}$-projective. The sufficiency is self-evident, so that we concentrate on the necessity. Let $N \leq G$ be a countable nice subgroup such that G/N is $p^{\omega+n}$-projective. By Proposition 2.8, G/N is also a C_α n-summable group. We therefore see that Proposition 2.1 is applicable to derive that G/N is a dsc group. Thus Corollary 2.2 from [6] implies that G is a dsc group, too, as desired.

Secondly, assume that G is finitely ω_1-$p^{\omega+n}$-projective (= ω-$p^{\omega+n}$-projective) of limit length. Hence there is a finite subgroup F of G such that G/F is $p^{\omega+n}$-projective. On the other hand, Proposition 2.9 applies to get that G/F is C_α n-summable. Thus Lemma 2.1 gives that G/F is a dsc group, which implies that so is G, as wanted. ∎

Problem 2.5. *Decide whether or not ω_1-$p^{\omega+n}$-projective C_α n-summable groups are dsc groups.*

We will now considerably enlarge the developed above theory of ω_1-weakly $p^{\omega\cdot 2+n}$-projective groups to ω_1-weakly p^α-projective groups where $\alpha \leq \omega_1$ is an arbitrary ordinal; nevertheless the preceding case when $\alpha = \omega \cdot 2 + n$ was more attractive.

First we state the original notion of a weakly p^α-projective group mainly due to Hill-Megibben [11]; actually, Hill and Megibben considered weak p^α-projectivity for a group G as the existence of a p^α-projective group H containing G. However, as Keef observed in [17], the existence of a p^α-projective group can be replaced by the existence of a p^α-bounded totally projective group. We will restrict out attention only to ordinals $\alpha \leq \omega_1$, so that the containing group should be of necessity a dsc group.

And so, imitating [18], we state the following.

Definition 2.8. *The group G is said to be weakly p^α-projective if there exists a dsc group H of length α such that $G \subseteq H$.*

By analogy with Definition 2.2 we state:

Definition 2.9. *The group G is said to be ω_1-weakly p^α-projective if there exists a countable subgroup C such that G/C is a weakly p^α-projective group.*

We begin with some characteristic properties of these groups. The next statement parallels an assertion due to Nunke (see cf. [20] and [15]) which says that if λ and δ are ordinals such that $G/p^\lambda G$ is p^λ-projective and $p^\lambda G$ is p^δ-projective, then G is $p^{\lambda+\delta}$-projective.

Proposition 2.10. *If α and β are any two ordinals, then G is a weakly $p^{\alpha+\beta}$-projective group if and only if it has a weakly p^β-projective subgroup B such that G/B is a weakly p^α-projective group.*

Proof. Follows combining the cited above Nunke's result along with the method described in Proposition 2.14 of [11]. ∎

Remark 2.1. Utilizing the last proposition, in particular, we can describe weakly $p^{\omega\cdot 2+n}$-projective groups as the extension of p^n-bounded groups by weakly $p^{\omega\cdot 2}$-projective factors, or as the extension of $p^{\omega+n}$-projective groups by $p^{\omega+m}$-projective factors; note that the value of the natural m here is irrelevant and thus we can consider Σ-cyclic factors. (Compare also with Theorem 2.2.)

The following strengthens Corollary 2.1.

Theorem 2.6. *The class of ω_1-weakly p^α-projectives forms the smallest class which contains the class of weakly p^α-projectives and which is closed under ω_1-bijections.*

Proof. Utilizing [16], it suffices to show that if A is a p^α-bounded group which contains a subgroup G such that A/G is countable, then A is weakly p^α-projective if and only if G is weakly p^α-projective.

As noted above, the necessity follows easily. In order to treat the sufficiency, write $A = G+C$ where C is countable and where $G \subseteq H$ with H a dsc group of length not exceeding α. We will show that $H+C$ is a dsc group of length no more than α. In fact, write $H = H' \oplus H_1$ where H_1 is countable with the property that $H \cap C \subseteq H_1 \subseteq A$. Letting $B = H+C$, one may infer that $B = H' \oplus (H_1 + C)$. Indeed, given $x \in H' \cap (H_1 + C)$, hence $x = h_1 + c$ where $h_1 \in H_1$ and $c \in C$. Thus $x - h_1 \in C \cap H \subseteq H_1$. Finally $x \in H' \cap H_1 = \{0\}$, whence $x = 0$, as required.

Observe that $H_1 + C$ is countable and a subgroup of A. And since H' is a p^α-bounded dsc group, it follows at once that B is a p^α-bounded dsc group as well, as required. Thus $A \subseteq B$ and hence one may conclude that A is weakly p^α-projective. ∎

In [18], the next two statements are proved using a slightly different but an equivalent formulation (compare with Theorem 3.7 and Corollary 3.8 from [18], respectively).

Theorem 2.7. *Suppose $\alpha < \omega_1$ and G is a C_α group. Then G is weakly p^α-projective if and only if it is a p^α-bounded dsc group.*

The last assertion can be refined to the following one.

Theorem 2.8. *Let $\alpha < \omega_1$ and let G be a C_α group. Then G is an ω_1-weakly p^α-projective group if and only if G is a dsc group and $p^\alpha G$ is countable.*

Before giving up some significant improvements of the listed above results, we need one more notion from [22] - see [13], [14] and [15] too: For a group G a functor-group $\mathrm{H}(G)$ is defined, called an *n-cover* of G, such that $\mathrm{H}(G)$ has a nice subgroup V (isometric to $G[p^n]$) and such that $\mathrm{H}(G)/V$ is simply presented. We shall further identify V with $G[p^n]$.

The following assertion extends Theorem 2.7 of Keef in terms of the defined above C_α n-summable groups; it also strengthens our Theorem 2.5 alluded to above and Proposition 2.5 in [14].

Theorem 2.9. *Suppose $\alpha < \omega_1$ and G is a weakly p^α-projective group. Then G is a C_α n-summable group if and only if it is a p^α-bounded n-summable group.*

Proof. First of all, we recollect some useful facts:

(1) G is n-summable $\iff$ $\mathrm{H}(G)$ is a dsc group; Theorem 4.1, cf. [15].

(2) G is C_α n-summable $\iff$ $\mathrm{H}(G)$ is a C_α group; Theorem 2.1, cf. [14].

(3) G is weakly p^α-projective $\Rightarrow$ $\mathrm{H}(G)$ is weakly p^α-projective.

In fact, to verify the last point, we may write $G \subseteq A$ where A is totally projective of countable length at most α, whence A is a dsc group of length $\leq \alpha$. Therefore, it readily follows that we can take the n-covers $\mathrm{H}(G)$ and $\mathrm{H}(A)$ such that the inclusion $\mathrm{H}(G) \subseteq \mathrm{H}(A)$ holds, where $\mathrm{H}(A)$ is a dsc group of countable length as well. Indeed, A being a dsc group forces that it is n-summable for each natural n. So, point (1) assures that $\mathrm{H}(A)$ is a dsc group. That length($\mathrm{H}(A)$)$< \omega_1$ follows directly from the definition of the functor-group $\mathrm{H}(A)$. This substantiates the validity of point (3).

To show now the truthfulness of the theorem, let G be a weakly p^α-projective C_α n-summable group. Consequently, utilizing both points (2) and (3), the group $\mathrm{H}(G)$ has to be both weakly p^α-projective and C_α. Thus Theorem 2.7 applies to get that $\mathrm{H}(G)$ is a dsc group. Hence, point (1) is applicable to obtain that G is n-summable, as asserted.

The converse is trivial, and thereby we are finished. ∎

Definition 2.10. *The group G is said to be nicely ω_1-weak p^α-projective if there exists a countable nice subgroup C such that G/C is a weakly p^α-projective group. If C is also isotype in G, that is, C is balanced in G, the group G is called balanceable ω_1-weak p^α-projective. If C is finite, then G is called finitely ω_1-weak p^α-projective or in a more appropriate form ω-weakly p^α-projective.*

We will next enlarge the last statement to the following one:

Theorem 2.10. *Let $\alpha < \omega_1$ and let G be a balanceable ω_1-weak p^α-projective group. Then G is a C_α n-summable group if and only if G is an n-summable group whose $p^\alpha G$ is countable.*

Proof. The sufficiency being obviously elementary, we will treat the necessity so that suppose G is both balanceable ω_1-weak p^α-projective and C_α n-summable. With Proposition 2.8 at hand, we deduce that G/N is simultaneously C_α n-summable and weakly p^α-projective for some countable balanced subgroup N of G. Whence Theorem 2.9 directly implies that G/N is n-summable. Furthermore, we employ [6] to conclude that G is n-summable, as claimed. On the other hand, length$(G/N) \leq \alpha$, and hence $p^\alpha G \subseteq N$ is certainly countable, as stated. ∎

Theorem 2.11. *Let $\alpha < \omega_1$ and let G be an ω-weakly p^α-projective group of limit length. Then G is a C_α n-summable group if and only if G is an n-summable group whose $p^\alpha G$ is finite.*

Proof. Follows in the same manner as Theorem 2.10 bearing in mind Proposition 2.9 instead of Proposition 2.8. ∎

We ask whether or not the "balanced" restriction in Theorem 2.10 is necessary, i.e., whether or not it can be replaced only by "niceness", is unknown yet. Also, whether or not in Theorem 2.11 "ω-weakly p^α-projective" can be replaced by "nicely ω_1-weak p^α-projective" appears to be unsettled too.

However, the following example, firstly exhibited in ([18], Example 3.9), illustrates that Theorems 2.7, 2.8 and their generalizations, namely Theorems 2.9, 2.10 and 2.11, fail for $\alpha = \omega_1$.

Example 2.3. Let M be an ω_1-elementary S-group; so M is a p^α-pure subgroup of a p^{ω_1}-bounded dsc group H with $H/M \cong \mathbb{Z}(p^\infty)$. Therefore, M is a weakly p^{ω_1}-projective C_{ω_1} group which fails to be a dsc group, as asserted. Even more, appealing to [10], M is not a summable group as well.

This suggests us to state the following intriguing problem.

Problem 2.6. Decide whether or not summable weakly p^{ω_1}-projective C_{ω_1} groups are dsc groups.

Remark 2.2. It is worthwhile noticing that if this problem holds in the affirmative, it will significantly improve the classical theorem of P. Hill [10] that summable IT groups are dsc groups, despite of summable C_{ω_1} groups which are not. Note that IT groups are C_{ω_1} groups whereas the converse is not fulfilled.

Likewise, in case that Problem 2.2 holds positively, it is rather usual to ask whether or not the more general statement is valid: summable ω_1-weakly p^{ω_1}-projective C_{ω_1} groups are dsc groups as well. This appears to be difficult; however the following affirmation will (eventually) hold: Summable nicely ω_1-weak p^{ω_1}-projective C_{ω_1} groups will (eventually) be dsc groups. Indeed, for any countable nice subgroup C of G, the quotient G/C is both a summable C_{ω_1} group (see [4] and [6]) and a weakly p^{ω_1}-projective group, so that by hypothesis it is a dsc group. Finally, [6] is applicable to establish the claim.

We can now slightly refine Theorem 2.9 in the following manner, but first we need one more pivotal technicality:

Proposition 2.11. *Let α be a countable ordinal, V and W valuated p^n-socles, V n-summable with $V(\alpha) = \{0\}$ and $\varphi : W \to V$ an injective valuated homomorphism. Then W is n-summable if and only if it is C_α n-summable.*

Proof. Letting W is C_α n-summable, we find p^α-bounded n-covers K and L of W and V, respectively. Note that these n-covers can easily be constructed such that there is an embedding $K \to L$. With Theorem 2.9 from [14] at hand, we deduce that L is a dsc group. Next, the utilization of Theorem 2.11 again from [14] gives that K is a C_α group. Finally, Theorem 2.7 is applicable to derive that K is a dsc group, as required in Theorem 4.1 of [15] in order W to be n-summable. ∎

As a direct consequence, we yield:

Theorem 2.12. *Suppose H is an n-summable group of countable length α and G is a subgroup of H. Then G is n-summable if and only if it is C_α n-summable.*

Remark 2.3. It was aforementioned that a group A is weakly p^α-projective for some $\alpha \leq \omega_1$ if it is a subgroup of a dsc group of length $\leq \alpha$. Thus A is also a subgroup of an n-summable group of length $\leq \alpha$, so that Theorem 2.12 is really a reminiscent of both Theorems 2.7 and 2.9, as promised.

It is also worth noticing that the above two assertions (namely, Proposition 2.11 and Theorem 2.12) were suggested to the present author by Prof. Patrick Keef. So, the author is rather appreciated to him for the valuable communication and for the encouragement during the preparation of this exploration.

And so, another direction of generalizing some of the results presented above is the following:

Definition 2.11. *Let $\alpha \leq \omega_1$. A group G is called weakly n-p^α-projective if there exists an n-summable group H of length at most α such that $G \subseteq H$.*

Definition 2.12. *Let $\alpha \leq \omega_1$. A group G is called ω_1-weakly n-p^α-projective if there exists a countable subgroup C such that G/C is a weakly n-p^α-projective group.*

In conjunction with our Definition 2.11, Theorem 2.12 can be rewritten as follows:

Theorem 2.13. *Suppose $\alpha < \omega_1$. Then weakly n-p^α-projective C_α n-summable groups are n-summable, and conversely.*

In the spirit of Definition 2.10, we can also state:

Definition 2.13. *The group G is said to be nicely ω_1-weak n-p^α-projective if there is a countable nice subgroup N of G such that G/N is a weakly n-p^α-projective group. If N is balanced in G, then G is called balanceable ω_1-weak n-p^α-projective, while if N is finite G is called finitely ω_1-weak n-p^α-projective or equivalently ω-weakly n-p^α-projective.*

The following strengthening of Theorems 2.10 and 2.11 are of some interest.

Theorem 2.14. *Let $\alpha < \omega_1$ and G be a balanceable ω_1-weak n-p^{α}-projective group. Then G is a C_{α} n-summable group if and only if G is an n-summable group with countable $p^{\alpha}G$.*

Proof. Follows via a simple parallel with the proof of Theorem 2.10, accomplished with Proposition 2.8 and Theorem 2.13. ∎

Theorem 2.15. *Let $\alpha < \omega_1$ and G be an ω-weakly n-p^{α}-projective group of limit length. Then G is a C_{α} n-summable group if and only if G is an n-summable group with finite $p^{\alpha}G$.*

Proof. Follows by a plain parallel with the proof of Theorem 2.11, accomplished with Proposition 2.9 and Theorem 2.13. ∎

3. Concluding discussion

We close the work with two non-trivial extensions of the groups examined in the above section as well as of these investigated in [13].

Definition 3.1. *The group G is called α-simply presented for some ordinal α if there exists a simply presented subgroup A of length $\leq \alpha$ such that G/A is simply presented.*

There are two important consequences of the last notion. Taking $\alpha = n$ to be a natural, we will obtain the so-called *n-simply presented groups* from [13] that are groups G with p^n-bounded subgroups P such that G/P is simply presented. Observe that by Theorem 1.2 (a1) of [16] every ω_1-$p^{\omega+n}$-projective group is of necessity n-simply presented.

On the other hand, we defined in [15] the class of n-totally projective groups like this:

Definition 3.2. *The group G is said to be n-totally projective if for each limit ordinal μ, $G/p^{\mu}G$ is $p^{\mu+n}$-projective. If $G/p^{\mu+n}G$ is $p^{\mu+n}$-projective, the group G is said to be strongly n-totally projective.*

Observe that both n-simply presented groups and strongly n-totally projective groups are themselves n-totally projective (see [15] too).

So, we come to

Problem 3.1 Does it follow that ω-totally $p^{\omega+n}$-projectives are n-totally projective or even n-simply presented? Notice that if the answer is "no" this will settle Question 1 from [16] in the negative, while if it is "yes" the solution will be in the affirmative. This is so, because by [7] the subgroup $p^{\omega+n}G$ has to be countable provided G is ω-totally $p^{\omega+n}$-projective, whereas by Definition 3.2 the quotient $G/p^{\omega}G$ must be $p^{\omega+n}$-projective provided G is n-totally projective. Moreover, the choice $\alpha = \omega$ leads to ω-simply presented groups which are groups G with Σ-cyclic subgroups S such that G/S is simply presented. Clearly weak $p^{\omega\cdot 2}$-projectives are themselves ω-simply presented.

Definition 3.3. *The group G is called strongly α-simply presented for some ordinal α if there exists a nice simply presented subgroup B of length $\leq \alpha$ such that G/B is simply presented.*

Substituting $\alpha = n$ to be a natural, we establish the so-called *strongly n-simply presented groups* from [13] that are groups G with nice p^n-bounded subgroups N such that G/N is simply presented. Besides, weakly $p^{\omega\cdot 2}$-projectives are even strongly ω-simply presented as well as ω_1-$p^{\omega+n}$-projectives need not be strongly n-simply presented - in fact, about the latter claim in ([16], Example 2.3) was constructed a $p^{\omega+n}$-bounded ω_1-$p^{\omega+n}$-projective group which is not $p^{\omega+n}$-projective and thus it is not strongly n-simply presented as asserted (see [13]). Likewise, strongly n-simply presented groups are strongly n-totally projective.

Problem 3.2. Describe the properties of α-simply presented and strongly α-simply presented groups for any fixed ordinal α and, in the particular case, for $\alpha = \omega$.

The following problem arises quite naturally as well.

Problem 3.3 Is the group G weakly n-p^{α}-projective for each natural n if and only if G is weakly p^{α}-projective? Moreover, is the group G ω_1-weakly n-p^{α}-projective for every natural n if and only if G is ω_1-weakly p^{α}-projective?

Definition 3.4. *The group G is said to be δ-$p^{\omega+n}$-projective for some ordinal δ if there exists a subgroup S of length $< \delta$ and power $< |\delta|$ (i.e., $p^{\gamma}S = \{0\}$ for some $\gamma < \delta$ and $|S| < |\delta|$) such that G/S is $p^{\omega+n}$-projective.*

Problem 3.4. Describe some crucial properties of δ-$p^{\omega+n}$-projective groups.

Corrigendum. In the second part in the proof of Corollary 2.4 from [6] the letters "*A*" and "*G*" should be reversed.

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