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Some Characterizations of 0-Distributive Semilattices

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Abstract. In this paper we discuss prime down-sets of a semilattice. We give a characterization of prime down-sets of a semilattice. We also give some characterizations of 0-distributive semilattices and a characterization of minimal prime ideals containing an ideal of a 0-distributive semilattice. Finally, we give a characterization of minimal prime ideals of a pseudocomplemented semilattice.

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1. Introduction

Semilattices have been studied by many authors. The class of distributive semilattices is an important subclass of semilattices. We refer the readers to [4, 9, 10] for distributive semilattices. We also refer the monograph [5] for the background of distributive semilattices. The class of 0-distributive semilattices is a nice extension of the class of distributive semilattices. This extension is useful for the study of pseudocomplemented semilattices. For pseudocomplemented semilattices we refer the readers to [2, 3, 5, 6]. We also refer the readers to [7, 8] for 0-distributive semilattices (see [1, 11] for 0-distributive lattices). In this paper we study 0-distributive semilattices. By semilattice we mean meet-semilattice.

A semilattice **S** with 0 is called 0-**distributive** if for any $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ implies $a \wedge d = 0$ for some $d \geqslant b,c$. The **pentagonal lattice** $\mathscr{P}_5$ (see Figure 1) as a semilattice is 0-distributive but the **diamond lattice** $\mathscr{M}_3$ (see Figure 1) as a semilattice is not 0-distributive. A semilattice **S** is called **directed above** if for all $x,y \in S$ there exists $z \in S$ such that $z \geqslant x,y$. Every 0-distributive semilattice is directed above.

Minimal prime ideals and maximal filters play an important role in semilattices. In Section 2, we introduce a notion of minimal prime down-set and maximal filters in semilattices. Here we give a characterization of minimal prime down-sets and maximal filters in semilattices.

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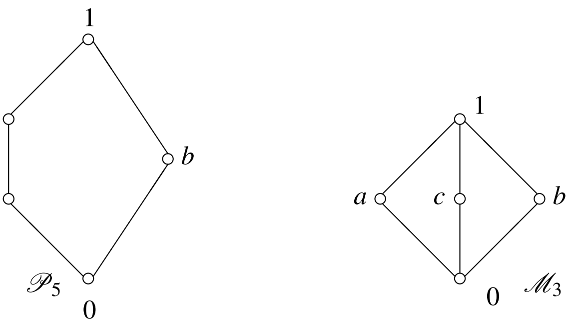

Figure 1. 0-distributive and non-0-distributive

Like as a distributive semilattice (or distributive lattice) Stone's version separation theorem is not true for 0-distributive semilattice. For example, if we consider the pentagonal lattice $\mathscr{P}_5$ (see Figure 1) as a 0-distributive semilattice, then $F = [c)$ is a filter and $I = (a]$, is an ideal such that $F \cap I = \emptyset$ but there is no prime filter containing F and disjoint from I. In Section 3 we discuss Stone's version separation theorem for 0-distributive semilattices. In this section we give some characterizations of 0-distributive semilattices.

In Section 4 we discuss the pseudocomplementation in semilattices. We close the paper with a characterization of a minimal prime ideals of a pseudocomplemented 0-distributive semilattice.

2. Prime down-sets and maximal filters

Let **S** be a semilattice. A non-empty subset D of S is called a **down-set** if $a \in D, b \in S$ with $b \leqslant a$ implies that $b \in D$. A down-set D of **S** is called a **proper down-set** if $D \neq S$. A **prime down-set** is a proper down-set P of **S** such that $a \wedge b \in P$ implies $a \in P$ or $b \in P$. A prime down-set P is called **minimal** if there is a prime down-set Q such that $Q \subseteq P$, then $P = Q$.

Theorem 2.1. *Any prime down-set of a semilattice contains a minimal prime down-set.*

Proof. Let **S** be a semilattice with 0. Let P be a prime down-set of **S** and let $\mathscr{P}$ be the set of all prime down-sets contained in P. Then $\mathscr{P}$ is non-empty since $P \in \mathscr{P}$. Let $\mathscr{C}$ be a chain in $\mathscr{P}$ and let

$$M := \bigcap \{X \mid X \in \mathscr{C}\}.$$

We claim that M is a prime down-set. Clearly M is non-empty as $0 \in M$. Let $a \in M$ and $b \leqslant a$. Then $a \in X$ for all $X \in \mathscr{C}$. Hence $b \in X$ for all $X \in \mathscr{C}$ as X is a down-set. Thus $b \in M$. Now let $x \wedge y \in M$ for some $x, y \in S$. Then $x \wedge y \in X$ for all $X \in \mathscr{C}$. Since X is a prime down-set for all $X \in \mathscr{C}$, we have either $x \in X$ or $y \in X$ for all $X \in \mathscr{C}$. This implies that either $x \in M$ or $y \in M$. Hence M is a prime down-set.

Thus by applying the dual form of Zorn's Lemma to $\mathscr{P}$, there is a minimal member of $\mathscr{P}$. ∎

Let **S** be a semilattice. A non-empty subset F of S is called a **filter** if

(i) $a, b \in F$ implies $a \wedge b \in F$

(ii) $a \in S$, $b \in F$ with $a \geqslant b$ implies $a \in F$.

A filter F of a semilattice $\mathbf{S}$ is called **proper filter** if $F \neq S$. A **maximal filter** F of $\mathbf{S}$ is a proper filter which is not contained in any other proper filter, that is, if there is a proper filter G such that $F \subseteq G$, then $F = G$.

Following result is due to [8].

Lemma 2.1. *Let M be a proper filter of* $\mathbf{S}$ *with* 0*. Then M is maximal if and only if for all $a \in S \setminus M$, there is some $b \in M$ such that $a \wedge b = 0$.*

Now we have the following result.

Theorem 2.2. *Let F be a non-empty proper subset of a semilattice* $\mathbf{S}$*. Then F is a filter if and only if $S \setminus F$ is a prime down-set.*

Proof. Let F be a filter of a semilattice $\mathbf{S}$. Let $x \in S \setminus F$ and $y \leqslant x$. Then $x \notin F$ and hence $y \notin F$ as F is a filter. This implies $y \in S \setminus F$. Thus $S \setminus F$ is a down-set. Since F is a filter $S \setminus F \neq S$. Thus $S \setminus F$ is a proper down-set. To prove $S \setminus F$ is a prime down-set, let $a, b \in S$ such that $a \wedge b \in S \setminus F$. Then $a \wedge b \notin F$ and hence either $a \notin F$ or $b \notin F$ as F is filter. This implies either $a \in S \setminus F$ or $b \in S \setminus F$. Therefore, $S \setminus F$ is a prime down-set.

Conversely, let $S \setminus F$ be a prime down-set and $x, y \in F$. Then clearly, $x, y \notin S \setminus F$ and hence $x \wedge y \notin S \setminus F$ as $S \setminus F$ is a prime down-set. Thus $x \wedge y \in F$. Suppose $x \in F$ and $x \leqslant y$. Then $x \notin S \setminus F$. Since $S \setminus F$ is a down-set, we have $y \notin S \setminus F$. Hence $y \in F$. This implies F is a filter. ∎

Theorem 2.3. *Let F be a non-empty subset of a semilattice* $\mathbf{S}$*. Then F is a maximal filter if and only if $S \setminus F$ is a minimal prime down-set.*

Proof. Let F be a maximal filter and $S \setminus F$ is not a minimal prime down-set. Then there exists a prime down-set I such that $I \subseteq S \setminus F$ which implies $F \subseteq S \setminus I$ which contradict to the maximality of F. Hence $S \setminus F$ is minimal prime down-set.

Conversely, let $S \setminus F$ be a minimal prime down-set and F is not a maximal filter. Thus there exists a proper filter G such that $F \subseteq G$ which implies $S \setminus G \subseteq S \setminus F$ which contradict the minimality of $S \setminus F$. Hence F is a maximal filter. ∎

3. Minimal prime ideals

Let $\mathbf{S}$ be a semilattice. A down-set I of S is called an **ideal** if $a, b \in I$ implies the existence of $c \in I$ such that $a, b \leqslant c$. The set of all ideals of S is denoted by $\mathscr{I}(S)$. An ideal I of $\mathbf{S}$ is called a **proper ideal** if $I \neq S$. A **prime ideal** P is a proper ideal of $\mathbf{S}$ such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. A prime ideal P is called **minimal** if there is a prime ideal Q such that $Q \subseteq P$, then $P = Q$. A filter F of $\mathbf{S}$ is called a **prime filter** if $F \neq S$ and $S \setminus F$ is a prime ideal.

We shall often use the following lemma in this paper.

Lemma 3.1. *Let* $\mathbf{S}$ *be a directed above semilattice with* 0*. If* $\mathbf{S}$ *is not* 0*-distributive, then the set*

$$F := \{x \in S \mid x \geqslant a \wedge y \neq 0 \textit{ for all } y \geqslant b, c\},$$

where $a, b, c \in S$ such that $a \wedge b = a \wedge c = 0$, is a proper filter.

Proof. Since $\mathbf{S}$ is not 0-distributive, there are $p, q, r \in S$ such that $p \wedge q = p \wedge r = 0$ and $p \wedge d \neq 0$ for all $d \geqslant q, r$. Now we have $p \geqslant p \wedge d$. Thus $p \in F$. Hence F is non-empty. Clearly $0 \notin F$. It is enough to show that F is a filter. Let $x \in F$ and $z \geqslant x$. Then $x \geqslant a \wedge y$ for all $y \geqslant b,c$ and by transitivity $z \geqslant a \wedge y$ for all $y \geqslant b,c$. Hence $z \in F$. Again let $x,z \in F$. Then $x \geqslant a \wedge y$ and $z \geqslant a \wedge y$ for all $y \geqslant b,c$. Thus $x \wedge z \geqslant a \wedge y$ for all $y \geqslant b,c$. Hence $x \wedge z \in F$. This implies F is a filter. ∎

Now we have the following result.

Theorem 3.1. *Every maximal filter of a 0-distributive semilattice is a prime filter.*

Proof. Let $\mathbf{S}$ be a 0-distributive semilattice. Again let Q be a maximal filter of S. We shall show that Q is prime. It is sufficient to show that $S \setminus Q$ is a prime ideal. By Theorem 2.3 we have $S \setminus Q$ is a minimal prime down-set. Now let $x,y \in S \setminus Q$. Then by Lemma 2.1 we have $a \wedge x = 0 = b \wedge y$ for some $a,b \in Q$. Let $c = a \wedge b$. Clearly $c \wedge x = 0 = c \wedge y$ and $c \in Q$. Hence by the 0-distributivity of $\mathbf{S}$ there exists $z \in S$ such that $z \geqslant x,y$ and $c \wedge z = 0$. Hence $z \in S \setminus Q$. Thus $S \setminus Q$ is a prime ideal which implies Q is prime. ∎

Let A be non-empty subset of a semilattice $\mathbf{S}$ with 0. Set

$$A^{\perp} := \{x \in S \mid a \wedge x = 0 \text{ for all } a \in A\}.$$

Then $A^{\perp}$ is called the **annihilator** of A. If $A = S$ then $A^{\perp} = S^{\perp} = (0]$. For $a \in S$, the annihilator of $\{a\}$ is simply denoted by $a^{\perp}$ and hence $a^{\perp} = \{x \in S \mid a \wedge x = 0\}$. We can easily show that

$$A^{\perp} = \bigcap_{a \in A} a^{\perp}.$$

Let $\mathbf{S}$ be a semilattice with 0. An ideal I of $\mathbf{S}$ is called an **annihilator ideal** if $I = A^{\perp}$ for some non-empty subset A of S.

Our aim is to prove a Stone's version separation theorem for 0-distributive semilattices. The following result due to [8, Theorem 7].

Theorem 3.2. *Let* $\mathbf{S}$ *be a semilattice with* 0*. Then* $\mathbf{S}$ *is* 0*-distributive if and only if for any filter F of S such that $F \cap x^{\perp} = \emptyset \, (x \in S)$, there exists a prime filter containing F and disjoint from $x^{\perp}$.*

Our conjecture is:

Conjecture 3.1. *Let* $\mathbf{S}$ *be a directed above semilattice with* 0*. Then* $\mathbf{S}$ *is* 0*-distributive if and only if for any filter F and any annihilator ideal I of S such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint from I.*

The necessary conditions of a directed above semilattice to be 0-distributive is given below, but unfortunately, we could not prove or disprove the condition is sufficient or not.

Theorem 3.3. *Let* $\mathbf{S}$ *be a directed above semilattice with* 0*. If for any filter F and any annihilator ideal I of S such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint from I, then* $\mathbf{S}$ *is* 0*-distributive.*

Proof. Suppose the condition holds. If $\mathbf{S}$ is not 0-distributive, then there are $a,b,c \in S$ such that $a \wedge b = 0 = a \wedge c$ and $a \wedge d \neq 0$ for all $d \geqslant b,c$ (such d exists as $\mathbf{S}$ is directed above). Let

$$F := \{x \in S \mid x \geqslant a \wedge y \text{ for all } y \geqslant b,c\}.$$

Then by Lemma 3.1, we have F is a proper filter.

Let I be an annihilator ideal such that $a \wedge d \notin I$ (such annihilator exists as $a \wedge d \notin S^{\perp}$). We shall show that $I \cap F = \emptyset$. If $x \in I \cap F$, then $x \geqslant a \wedge y$ for all $y \geqslant b,c$ which implies $a \wedge d \in I$, which is a contradiction. Hence $I \cap F = \emptyset$. Thus by the assumption, there is a prime filter Q such that $F \subseteq Q$ and $I \cap Q = \emptyset$. This implies $a \in Q$ and $y \in Q$ for all $y \geqslant b,c$. We shall show that either $b \in Q$ or $c \in Q$. If $b,c \notin Q$ then $b,c \in S \setminus Q$. Since Q is a prime filter, $S \setminus Q$ is a prime ideal. So, there is $e \in S \setminus Q$ such that $e \geqslant b,c$ which is a contradiction. Hence either $b \in Q$ or $c \in Q$. This implies, either $a \wedge b \in Q$ or $a \wedge c \in Q$. Hence $0 \in Q$, which contradicts the fact that Q is a prime filter. Therefore, $a \wedge d = 0$ for some $d \geqslant b,c$ and hence **S** is 0-distributive. ∎

Let **S** be a semilattice. For $a \in S$, the ideal $(a]$ is called the **ideal generated by** a. It can be easily seen that $(a]^{\perp} = a^{\perp}$ for any $a \in S$. An ideal I of S is called an **α-ideal** if $(i^{\perp})^{\perp} \subseteq I$ for any $i \in I$.

Now we shall give some characterizations of 0-distributive semilattice. The following lemma is due to [1].

Lemma 3.2. *Every proper filter of a semilattice with* 0 *is contained in a maximal filter.*

We have the following result which is a generalization of [1, Theorem 3.1].

Theorem 3.4. *Let* **S** *be a semilattice with* 0. *Then the following statements* (i)–(iv) *are equivalent and any one of them implies* (v) *and* (vi).

(i) **S** *is* 0*-distributive;*
(ii) *every maximal filter of S is prime;*
(iii) *every minimal prime down-set of S is a minimal prime ideal;*
(iv) *every proper filter of S is disjoint from a minimal prime ideal;*
(v) *for each element $a \in S$ such that $a \neq 0$, there is a minimal prime ideal not containing a;*
(vi) *each element $a \in S$ such that $a \neq 0$ is contained in a prime filter.*

Proof. (i)⇒(ii). This follows by the Lemma 3.1.

(ii)⇒(iii). Let N be a minimal prime down-set. Then by Lemma 2.3 we have $S \setminus N$ is a maximal filter. Hence by (ii) $S \setminus N$ is a prime filter. Thus N is a prime ideal.

(iii)⇒(iv). Let F be a proper filter of S. By Lemma 3.2 there is a maximal filter M such that $F \subseteq M$. Hence by Lemma 2.3 we have $S \setminus M$ is a minimal prime down-set. Thus by (iii) $S \setminus M$ is a minimal prime ideal. Clearly, $F \cap (S \setminus M) = \emptyset$.

(iv)⇒(i). Suppose **S** is not 0-distributive. Then there are $a,b,c \in S$ such that $a \wedge b = a \wedge c = 0$ and $a \wedge d \neq 0$ for all $d \geqslant b,c$. Now set

$$F = \{x \in S \mid x \geqslant a \wedge y \text{ for all } y \geqslant b,c\}.$$

Then by Lemma 3.1, we have F is a proper filter and hence by (iv) there exists a prime ideal Q such that $F \cap Q = \emptyset$. Thus $a \wedge p \notin Q$ for any $p \geqslant b,c$. This implies $a,p \notin Q$ for any $p \geqslant b,c$. Now $a \notin Q$ implies $b,c \in Q$. Then there is $m \geqslant b,c$ such that $m \in Q$ which is a contradiction. Therefore, $a \wedge d = 0$ for some $d \geqslant b,c$ and hence **S** is 0-distributive.

(iv)⇒(v). Let $a \in S$ such that $a \neq 0$. Then $[a)$ is a proper filter. Then by (iv) $[a)$ is disjoint from a minimal prime ideal N of S. Thus $a \notin N$.

(v)⇒(vi). Let $a \in S$ such that $a \neq 0$. Then by (v) there is a minimal prime ideal P such that $a \notin P$ which implies $a \in S \setminus P$. By the definition of prime filter we have $S \setminus P$ is a prime filter. ∎

Now we have following result which is a generalization of [1, Lemma 1.8].

Lemma 3.3. *Let A be a non-empty subset of a semilattice* $\mathbf{S}$ *with* 0. *Then $A^{\perp}$ is the intersection of all the minimal prime down-set not containing A.*

Proof. Let $\mathbf{S}$ be a semilattice with 0 and $\emptyset \neq A \subseteq S$. Suppose

$$X := \bigcap \{P \mid A \nsubseteq P \text{ and } P \text{ is a minimal prime down-set}\}$$

Let $x \in A^{\perp}$. Then $x \wedge y = 0$ for all $y \in A$. This implies there is $z \notin P$ such that $x \wedge z = 0 \in P$. As P is prime, we have $x \in P$. Hence $x \in X$.

Conversely let, $x \in X$. If $x \notin A^{\perp}$. Then $x \wedge q \neq 0$ for some $q \in A$. Let $D = [x \wedge q)$. Then $0 \notin D$. Hence, $D \neq S$. Then by Lemma 3.2 we have $D \subseteq M$ for some maximal filter M. Hence by Lemma 2.3 we have $S \setminus M$ is a minimal prime down-set. Now $x \notin S \setminus M$ as $x \in D$ implies $x \in M$. Moreover $A \nsubseteq S \setminus M$ as $q \in A$ but $q \in M$ implies $q \notin S \setminus M$, which is a contradiction to $x \in X$. Hence $x \in A^{\perp}$. Thus the lemma is proved. ∎

Theorem 3.5. *Let* $\mathbf{S}$ *be a 0-distributive semilattice. If A is a non-empty subset of S and F is a proper filer intersecting A, there is a minimal prime ideal containing $A^{\perp}$ and disjoint from F.*

Proof. Let $\mathbf{S}$ be a directed above semilattice with 0. Again let A be a non-empty subset of $\mathbf{S}$ and F be a proper filter such that $F \cap A \neq \emptyset$. Then Lemma 2.2 $S \setminus F$ is a prime down-set and by Lemma 2.1 $N \subseteq S \setminus F$ for some minimal prime down-set N. Clearly, $N \cap F = \emptyset$. Also $A \nsubseteq S \setminus F$ and so $A \nsubseteq N$. By Lemma 3.3 $A^{\perp} \subseteq N$. Since $\mathbf{S}$ is 0-distributive, by theorem 3.4(iv) N is a minimal prime ideal. ∎

4. Pseudocomplementation for 0-distributive semilattices

Let $\mathbf{S}$ be a semilattice with 0. An element $d \in S$ is called the **pseudocomplement** of $x \in S$, if $x \wedge d = 0$ and $y \in S, x \wedge y = 0$ implies $y \leqslant d$. The pseudocomplement of x is denoted by x^{*}. A semilattice $\mathbf{S}$ is called pseudocomplemented if each element of S has a pseudocomplement. The pseudocomplement of 0 is the largest element 1. Thus a pseudocomplemented semilattice contains both the smallest element and the largest element.

Theorem 4.1. *Every pseudocomplemented semilattice is 0-distributive but the converse is not true.*

Proof. Let $\mathbf{S}$ be a pseudocomplemented semilattice. Suppose $a, b, c \in S$ with $a \wedge b = 0 = a \wedge c$. By the definition of pseudocomplemented, $b \leqslant a^{*}$, $c \leqslant a^{*}$ and $a \wedge a^{*} = 0$. Thus $\mathbf{S}$ is a 0-distributive semilattice.

To prove the converse is not true, consider the semilattice, $\mathscr{M}_2$ shown in the Figure 2, which is clearly 0-distributive but not pseudocomplemented as a^{*} does not exist.

∎

Theorem 4.2. *Let* $\mathbf{S}$ *be a pseudocomplemented semilattice and let J be an ideal of S. Then a prime ideal P containing J is a minimal prime ideal containing J if and only if for each $x \in P$ there is $y \in S \setminus P$ such that $x \wedge y \in J$.*

Proof. Let P be a prime ideal of S containing J such that the given condition holds. We shall show that P is a minimal prime ideal containing J. Let K be a prime ideal containing J such that $K \subseteq P$. Let $x \in P$. Then there is $y \in S \setminus P$ such that $x \wedge y \in J$. Hence $x \wedge y \in K$ as K containing J. Since K is prime and $y \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore, P is a minimal prime ideal containing J.

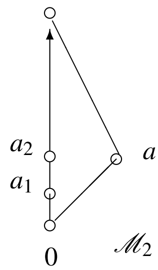

Figure 2. A 0-distributive but not pseudocomplemented semilattice

Conversely, let P be a minimal prime ideal containing J. Let $x \in P$. Suppose for all $y \in S \setminus P$, $x \wedge y \notin J$. Set $D = (S \setminus P) \vee [x)$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \wedge x$ for some $q \in S \setminus P$. Thus, $x \wedge q = 0 \in J$ which is a contradiction. Therefore, $0 \notin D$. Since $(0] = 1^{\perp}$ by Theorem 3.2, there is a prime filter Q such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by the definition of prime filter of a semilattice, M is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$ but $x \in P$. This shows that P is not minimal which is a contradiction. Hence the given condition holds. ∎

We enclose the paper with the following useful characterization of minimal prime ideal.

Theorem 4.3. *Let* **S** *be a pseudocomplemented semilattice and let P be a prime ideal of S. Then the followings are equivalent:*

(i) *P is minimal.*
(ii) *$x \in P$ implies that $x^* \notin P$.*

Proof. (i)⇒(ii). Let P be a minimal prime ideal and let $x^* \in P$ for some $x \in P$. Set $D = (S \setminus P) \vee [x)$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \wedge x$ for some $q \in S \setminus P$, which implies $q \leqslant x^* \in P$ which is a contradiction. Therefore, $0 \notin D$. Since $(0] = 1^{\perp}$ by Theorem 3.2, there is a prime filter Q such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by the definition of prime filter of a semilattice, M is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$ but $x \in P$. This shows that P is not minimal which is a contradiction. Hence (ii) holds.

(ii)⇒(i). Let P be a prime ideal of S such that (ii) holds. We shall show that P is a minimal prime ideal. Let K be a prime ideal satisfying (ii) such that $K \subseteq P$. Let $x \in P$. Then $x \wedge x^* = 0 \in K$. Since K is prime and $x^* \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore, P is a minimal prime ideal. ∎

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