BULLETIN *of the*
MALAYSIAN MATHEMATICAL
SCIENCES SOCIETY
http://math.usm.my/bulletin

Bull. Malays. Math. Sci. Soc. (2) **37**(4) (2014), 1111–1121

Existence of Solutions for a Coupled System of Fractional Differential Equations

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Abstract. In this paper, by using the coincidence degree theory, we consider the following Neumann boundary value problem for a coupled system of fractional differential equations

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t,v(t),v'(t)), & t \in (0,1), \\ D^{\beta}_{0+}v(t) = g(t,u(t),u'(t)), & t \in (0,1), \\ u'(0) = u'(1) = 0, \; v'(0) = v'(1) = 0, \end{cases}$$

where $D^{\alpha}_{0+}, D^{\beta}_{0+}$ are the standard Caputo fractional derivative, $1<\alpha\le 2$, $1<\beta\le 2$. A new result on the existence of solutions for above fractional boundary value problem is obtained.

2010 Mathematics Subject Classification: 34B15

Keywords and phrases: Fractional differential equation, Neumann boundary conditions, coincidence degree theory.

1. Introduction

In recent years, the fractional differential equations have received more and more attention. The fractional derivative has been occurring in many physical applications such as a non-Markovian diffusion process with memory [22], charge transport in amorphous semiconductors [24], propagations of mechanical waves in viscoelastic media [18], etc. Phenomena in electromagnetics, acoustics, viscoelasticity, electrochemistry and material science are also described by differential equations of fractional order (see [6–8, 19, 21, 23]).

Recently, boundary value problems for fractional differential equations have been studied in many papers (see [1, 5, 9–11, 13, 14, 16, 17, 29, 31]). Since Bai *et al.* considered existence of a positive solution for a singular coupled system of nonlinear fractional differential equations in 2004 (see [4]), the coupled system of fractional differential equations have been studied in many papers (see [2, 3, 12, 25–28, 30, 32]). In 2011, Zhang *et al.* (see [32]) considered existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance by using the coincidence degree theory due to Mawhin. In 2012, by using the coincidence degree theory, a coupled system of nonlinear fractional 2m-point boundary value problems at resonance have been studied by W. Jiang (see [12]).

Communicated by Norhashidah Mohd. Ali.
Received: March 13, 2012; *Revised:* August 6, 2012.

To the best of our knowledge, the existence of solutions for a coupled system of Neumann boundary value problems has rarely been studied. Motivated by the work above, in this paper, we consider the following Neumann boundary value problem (NBVP for short) for a coupled system of fractional differential equations given by

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t,v(t),v'(t)), \;\; t\in(0,1), \\ D_{0+}^{\beta}v(t) = g(t,u(t),u'(t)), \;\; t\in(0,1), \\ u'(0)=u'(1)=0, \; v'(0)=v'(1)=0, \end{cases} \tag{1.1}$$

where D_{0+}^{α}, D_{0+}^{β} are the standard Caputo fractional derivative, $1<\alpha\le 2$, $1<\beta\le 2$ and $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous.

The rest of this paper is organized as follows. Section 2 contains some necessary notations, definitions and lemmas. In Section 3, we establish a theorem on existence of solutions for NBVP (1.1) under nonlinear growth restriction of f and g , basing on the coincidence degree theory due to Mawhin (see [20]). Finally, in Section 4, an example is given to illustrate the main result.

2. Preliminaries

In this section, we will introduce some notations, definitions and preliminary facts which are used throughout this paper.

Let X and Y be real Banach spaces and let $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator with index zero, and $P:X\to X$, $Q:Y\to Y$ be projectors such that

$$\operatorname{Im}P=\operatorname{Ker}L, \;\; \operatorname{Ker}Q=\operatorname{Im}L,$$
$$X=\operatorname{Ker}L\oplus\operatorname{Ker}P, \;\; Y=\operatorname{Im}L\oplus\operatorname{Im}Q.$$

It follows that

$$L|_{\operatorname{dom}L\cap\operatorname{Ker}P}:\operatorname{dom}L\cap\operatorname{Ker}P\to\operatorname{Im}L$$

is invertible. We denote the inverse by K_P.

If Ω is an open bounded subset of X, and $\operatorname{dom}L\cap\overline{\Omega}\neq\varnothing$, the map $N:X\to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N:\overline{\Omega}\to X$ is compact, where I is identity operator.

Lemma 2.1. [20] *Let $L:\operatorname{dom}L\subset X\to Y$ be a Fredholm operator of index zero and $N:X\to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*

(1) $Lx\neq\lambda Nx$ *for every* $(x,\lambda)\in[(\operatorname{dom}L\setminus\operatorname{Ker}L)]\cap\partial\Omega\times(0,1)$;
(2) $Nx\notin\operatorname{Im}L$ *for every* $x\in\operatorname{Ker}L\cap\partial\Omega$;
(3) $\deg(QN|_{\operatorname{Ker}L},\operatorname{Ker}L\cap\Omega,0)\neq 0$, *where $Q:Y\to Y$ is a projection such that* $\operatorname{Im}L=\operatorname{Ker}Q$.

Then the equation $Lx=Nx$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$.

Definition 2.1. [15] *The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function x is given by*

$$I_{0+}^{\alpha}x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)ds,$$

provided that the right side integral is pointwise defined on $(0,+\infty)$.

Definition 2.2. [15] *Assume that $x(t)$ is $(n-1)$-times absolutely continuous function, the Caputo fractional derivative of order $\alpha > 0$ of x is given by*

$$D_{0^+}^{\alpha} x(t) = I_{0^+}^{n-\alpha} \frac{d^n x(t)}{dt^n} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} x^{(n)}(s) ds,$$

where n is the smallest integer greater than or equal to α, provided that the right side integral is pointwise defined on $(0, +\infty)$.

Lemma 2.2. [15] *Let $\alpha > 0$ and $n = -[-\alpha]$. If $x^{(n-1)} \in AC[0,1]$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$$

where $c_i = -\frac{x^{(i)}(0)}{i!} \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, here n is the smallest integer greater than or equal to α.

In this paper, we denote $X = C^1[0,1]$ with the norm $\|x\|_X = \max\{\|x\|_\infty, \|x'\|_\infty\}$ and $Y = C[0,1]$ with the norm $\|y\|_Y = \|y\|_\infty$, where $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$. Then we denote $\overline{X} = X \times X$ with the norm $\|(u,v)\|_{\overline{X}} = \max\{\|u\|_X, \|v\|_X\}$ and $\overline{Y} = Y \times Y$ with the norm $\|(x,y)\|_{\overline{Y}} = \max\{\|x\|_Y, \|y\|_Y\}$ Obviously, both $\overline{X}$ and $\overline{Y}$ are Banach spaces.

Define the operator $L_1 : \operatorname{dom} L \subset X \to Y$ by

$$L_1 u = D_{0^+}^{\alpha} u,$$

where

$$\operatorname{dom} L_1 = \{u \in X | D_{0^+}^{\alpha} u(t) \in Y,\ u'(0) = u'(1) = 0\}.$$

Define the operator $L_2 : \operatorname{dom} L_2 \subset X \to Y$ by

$$L_2 v = D_{0^+}^{\beta} v,$$

where

$$\operatorname{dom} L_2 = \{v \in X | D_{0^+}^{\beta} v(t) \in Y,\ v'(0) = v'(1) = 0\}.$$

Define the operator $L : \operatorname{dom} L \subset \overline{X} \to \overline{Y}$ by

$$L(u,v) = (L_1 u, L_2 v), \tag{2.1}$$

where

$$\operatorname{dom} L = \{(u,v) \in \overline{X} | u \in \operatorname{dom} L_1,\ v \in \operatorname{dom} L_2\}.$$

Let $N : \overline{X} \to \overline{Y}$ be defined as

$$N(u,v) = (N_1 v, N_2 u),$$

where $N_1 : Y \to X$

$$N_1 v(t) = f(t, v(t), v'(t)),$$

and $N_2 : Y \to X$

$$N_2 u(t) = g(t, u(t), u'(t)).$$

Then NBVP (1.1) is equivalent to the operator equation

$$L(u,v) = N(u,v),\ \ (u,v) \in \operatorname{dom} L.$$

3. Main result

In this section, a theorem on existence of solutions for NBVP (1.1) will be given.

Theorem 3.1. *Let $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous. Assume that*

(H_1) *there exist nonnegative functions $p_i, q_i, r_i \in C[0,1], (i=1,2)$ with*

$$\frac{\Gamma(\alpha)\Gamma(\beta)-(Q_1+R_1)(Q_2+R_2)}{\Gamma(\alpha)\Gamma(\beta)} > 0$$

such that for all $(u,v)\in\mathbb{R}^2, t\in[0,1]$,

$$|f(t,u,v)| \le p_1(t)+q_1(t)|u|+r_1(t)|v|$$

and

$$|g(t,u,v)| \le p_2(t)+q_2(t)|u|+r_2(t)|v|,$$

where $P_i = \|p_i\|_\infty,\ Q_i=\|q_i\|_\infty,\ R_i=\|r_i\|_\infty, (i=1,2)$;

(H_2) *there exists a constant $B>0$ such that for all $t\in[0,1],\ |u|>B,\ v\in\mathbb{R}$ either*

$$uf(t,u,v)>0,\ ug(t,u,v)>0$$

or

$$uf(t,u,v)<0,\ ug(t,u,v)<0;$$

(H_3) *there exists a constant $D>0$ such that for every $c_1,c_2\in\mathbb{R}$ satisfying* $\min\{c_1,c_2\}>$ *D either*

$$c_1N_1(c_2)>0,\ c_2N_2(c_1)>0$$

or

$$c_1N_1(c_2)<0,\ c_2N_2(c_1)<0.$$

Then NBVP (1.1) has at least one solution.

Now, we begin with some lemmas below.

Lemma 3.1. *Let L be defined by (2.1). Then*

(3.1) $$\operatorname{Ker}L = (\operatorname{Ker}L_1, \operatorname{Ker}L_2) = \{(u,v)\in\overline{X} | (u,v)=(u(0),v(0))\},$$

(3.2)
$$\operatorname{Im}L = (\operatorname{Im}L_1,\operatorname{Im}L_2) = \{(x,y)\in\overline{Y} | \int_0^1 (1-s)^{\alpha-2}x(s)ds=0, \int_0^1 (1-s)^{\beta-2}y(s)ds=0\}.$$

Proof. By Lemma 2.2, $L_1u = D_{0+}^{\alpha}u(t)=0$ has solution

$$u(t)=u(0)+u'(0)t.$$

Combining with the boundary value conditions of NBVP (1.1), one has

$$\operatorname{Ker}L_1 = \{u\in X | u=u(0)\}.$$

For $x\in\operatorname{Im}L_1$, there exists $u\in\operatorname{dom}L_1$ such that $x=L_1u\in Y$. By Lemma 2.2, we have

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)ds+u(0)+u'(0)t.$$

Then, we have

$$u'(t)=\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2}x(s)ds+u'(0).$$

By conditions of NBVP (1.1), we can get that x satisfies

$$\int_0^1 (1-s)^{\alpha-2}x(s)ds = 0.$$

On the other hand, suppose $x \in Y$ and satisfies $\int_0^1 (1-s)^{\alpha-2}x(s)ds = 0$. Let $u(t) = I_{0^+}^{\alpha}x(t)$, then $u \in \operatorname{dom} L_1$ and $D_{0^+}^{\alpha}u(t) = x(t)$. So that, $x \in \operatorname{Im} L_1$. Then we have

$$\operatorname{Im} L_1 = \{x \in Y | \int_0^1 (1-s)^{\alpha-2}x(s)ds = 0\}.$$

Similarly, we can get

$$\operatorname{Ker} L_2 = \{v \in X | v = v(0)\},$$
$$\operatorname{Im} L_2 = \{y \in Y | \int_0^1 (1-s)^{\beta-2}y(s)ds = 0\}.$$

Then, the proof is complete. ∎

Lemma 3.2. *Let L be defined by (2.1). Then L is a Fredholm operator of index zero, and the linear continuous projector operators $P : \overline{X} \to \overline{X}$ and $Q : \overline{Y} \to \overline{Y}$ can be defined as*

$$P(u,v) = (P_1u, P_2v) = (u(0), v(0)),$$
$$Q(x,y) = (Q_1x, Q_2y) = ((\alpha-1)\int_0^1 (1-s)^{\alpha-2}x(s)ds, (\beta-1)\int_0^1 (1-s)^{\beta-2}y(s)ds).$$

Furthermore, the operator $K_P : \operatorname{Im} L \to \operatorname{dom} L \cap \operatorname{Ker} P$ can be written by

$$K_P(x,y) = (I_{0^+}^{\alpha}x(t),\ I_{0^+}^{\beta}y(t)).$$

Proof. Obviously, $\operatorname{Im} P = \operatorname{Ker} L$ and $P^2(u,v) = P(u,v)$. It follows from $(u,v) = ((u,v) - P(u,v)) + P(u,v)$ that $\overline{X} = \operatorname{Ker} P + \operatorname{Ker} L$. By simple calculation, we can get that $\operatorname{Ker} L \cap \operatorname{Ker} P = \{(0,0)\}$. Then we get

$$\overline{X} = \operatorname{Ker} L \oplus \operatorname{Ker} P.$$

For $(x,y) \in \overline{Y}$, we have

$$Q^2(x,y) = Q(Q_1x, Q_2y)) = (Q_1^2x, Q_2^2y).$$

By the definition of Q_1, we can get

$$Q_1^2x = Q_1 \cdot x(\alpha-1)\int_0^1 (1-s)^{\alpha-2}x(s)ds = Q_1x.$$

Similar proof can show that $Q_2^2y = Q_2y$. Thus, we have $Q^2(x,y) = Q(x,y)$.

Let $(x,y) = ((x,y) - Q(x,y)) + Q(x,y)$, where $(x,y) - Q(x,y) \in \operatorname{Ker} Q = \operatorname{Im} L$, $Q(x,y) \in \operatorname{Im} Q$. It follows from $\operatorname{Ker} Q = \operatorname{Im} L$ and $Q^2(x,y) = Q(x,y)$ that $\operatorname{Im} Q \cap \operatorname{Im} L = \{(0,0)\}$. Then, we have

$$\overline{Y} = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

Thus

$$\dim \operatorname{Ker} L = \dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L.$$

This means that L is a Fredholm operator of index zero.

Now, we will prove that K_P is the inverse of $L|_{\operatorname{dom} L\cap \operatorname{Ker} P}$. In fact, for $(x,y)\in \operatorname{Im} L$, we have

$$LK_P(x,y) = (D^{\alpha}_{0+}(I^{\alpha}_{0+}x), D^{\beta}_{0+}(I^{\beta}_{0+}y)) = (x,y). \tag{3.3}$$

Moreover, for $(u,v)\in \operatorname{dom} L\cap \operatorname{Ker} P$, we have $u(0)=0,\ v(0)=0$ and

$$\begin{aligned} K_PL(u,v) &= (I^{\alpha}_{0+}D^{\alpha}_{0+}u(t),\ I^{\beta}_{0+}D^{\beta}_{0+}v(t)) \\ &= (u(t)+u(0)+u'(0)t,\ v(t)+v(0)+v'(0)t), \end{aligned}$$

which together with $u'(0)=v'(0)=0$ yields that

$$K_PL(u,v) = (u,v). \tag{3.4}$$

Combining (3.3) with (3.4), we know that $K_P = (L|_{\operatorname{dom} L\cap \operatorname{Ker} P})^{-1}$. The proof is now complete. ∎

Lemma 3.3. *Assume $\Omega\subset \overline{X}$ is an open bounded subset such that* $\operatorname{dom} L\cap\overline{\Omega}\neq\varnothing$*, then N is L-compact on $\overline{\Omega}$.*

Proof. By the continuity of f and g, we can get that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzelà-Ascoli theorem, we need only prove that $K_P(I-Q)N(\overline{\Omega})\subset\overline{X}$ is equicontinuous.

From the continuity of f and g, there exist constants $A_i>0$, $i=1,2$, such that for all $(u,v)\in\overline{\Omega}$

$$\begin{aligned} |(I-Q_1)N_1v| &\le A_1, \\ |(I-Q_2)N_2u| &\le A_2. \end{aligned}$$

Furthermore, for $0\le t_1<t_2\le 1$, $(u,v)\in\overline{\Omega}$, we have

$$\begin{aligned} &|K_P(I-Q)N(u(t_2),v(t_2)) - (K_P(I-Q)N(u(t_1),v(t_1)))| \\ =&\ |\ (I^{\alpha}_{0+}(I-Q_1)N_1v(t_2),\ I^{\beta}_{0+}(I-Q_2)N_2u(t_2)) \\ &-(I^{\alpha}_{0+}(I-Q_1)N_1v(t_1),\ I^{\beta}_{0+}(I-Q_2)N_2u(t_1))\ | \\ =&\ |\ (I^{\alpha}_{0+}(I-Q_1)N_1v(t_2) - I^{\alpha}_{0+}(I-Q_1)N_1v(t_1), \\ &I^{\beta}_{0+}(I-Q_2)N_2u(t_2) - I^{\beta}_{0+}(I-Q_2)N_2u(t_1))\ |\ . \end{aligned}$$

By

$$\begin{aligned} &\left|I^{\alpha}_{0+}(I-Q_1)N_1v(t_2) - I^{\alpha}_{0+}(I-Q_1)N_1v(t_1)\right| \\ \le&\frac{1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-1}(I-Q_1)N_1v(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-1}(I-Q_1)N_1v(s)ds\right| \\ \le&\frac{A_1}{\Gamma(\alpha)}\left[\int_0^{t_1}(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right] \\ =&\frac{A_1}{\Gamma(\alpha+1)}(t_2^{\alpha}-t_1^{\alpha}) \end{aligned}$$

and

$$\begin{aligned} &|(I^{\alpha}_{0+}(I-Q_1)N_1v)'(t_2) - (I^{\alpha}_{0+}(I-Q_1)N_1v)'(t_1)| \\ =&\frac{\alpha-1}{\Gamma(\alpha)}\left|\int_0^{t_2}(t_2-s)^{\alpha-2}(I-Q_1)N_1v(s)ds - \int_0^{t_1}(t_1-s)^{\alpha-2}(I-Q_1)N_1v(s)ds\right| \end{aligned}$$

$$\leq \frac{A_1}{\Gamma(\alpha-1)}\left[\int_0^{t_1}(t_1-s)^{\alpha-2}-(t_2-s)^{\alpha-2}ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}ds\right]$$
$$\leq \frac{A_1}{\Gamma(\alpha)}[t_2^{\alpha-1}-t_1^{\alpha-1}+2(t_2-t_1)^{\alpha-1}].$$

Similar proof can show that

$$\left|I_{0^+}^{\beta}(I-Q_2)N_2u(t_2)-I_{0^+}^{\beta}(I-Q_2)N_2u(t_1)\right| \leq \frac{A_2}{\Gamma(\beta+1)}(t_2^{\beta}-t_1^{\beta}),$$

$$|(I_{0^+}^{\beta}(I-Q_2)N_2u)'(t_2)-(I_{0^+}^{\beta}(I-Q_2)N_2u)'(t_1)| \leq \frac{A_2}{\Gamma(\beta)}[t_2^{\beta-1}-t_1^{\beta-1}+2(t_2-t_1)^{\beta-1}].$$

Since $t^{\alpha}, t^{\alpha-1}, t^{\beta}$ and $t^{\beta-1}$ are uniformly continuous on $[0,1]$, we can get that $K_P(I-Q)N(\overline{\Omega}) \subset \overline{X}$ is equicontinuous.

Thus, we get that $K_P(I-Q)N : \overline{\Omega} \to \overline{X}$ is compact. The proof is complete. ∎

Lemma 3.4. *Suppose* $(H_1),(H_2)$ *hold, then the set*

$$\Omega_1 = \{(u,v) \in \operatorname{dom} L \setminus \operatorname{Ker} L \mid L(u,v) = \lambda N(u,v),\ \lambda \in (0,1)\}$$

is bounded.

Proof. Take $(u,v) \in \Omega_1$, then $N(u,v) \in \operatorname{Im} L$. By (3.2), we have

$$\int_0^1 (1-s)^{\alpha-2} f(s,v(s),v'(s))ds = 0,\ \int_0^1 (1-s)^{\beta-2} g(s,u(s),u'(s))ds = 0.$$

Then, by the integral mean value theorem, there exist constants $\xi,\eta \in (0,1)$ such that $f(\xi,v(\xi),v'(\xi)) = 0$ and $g(\eta,u(\eta),u'(\eta)) = 0$. Then we have

$$v(\xi)f(t,v(\xi),v'(\xi)) = 0,\ u(\eta)g(t,u(\eta),u'(\eta)) = 0.$$

So, from (H_2), we get $|v(\xi)| \leq B$ and $|u(\eta)| \leq B$. Hence

$$|u(t)| = \left|u(\eta)+\int_{\eta}^{t} u'(s)ds\right| \leq B+\|u'\|_{\infty}. \tag{3.5}$$

That is

$$\|u\|_{\infty} \leq B+\|u'\|_{\infty}. \tag{3.6}$$

Similar proof can show that

$$\|v\|_{\infty} \leq B+\|v'\|_{\infty}. \tag{3.7}$$

By $L(u,v) = \lambda N(u,v)$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,v(s),v'(s))ds+u(0)$$

and

$$v(t) = \frac{\lambda}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1} g(s,u(s),u'(s))ds+v(0).$$

Then we get

$$u'(t) = \frac{\lambda}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,v(s),v'(s))ds$$

and

$$v'(t) = \frac{\lambda}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} g(s,u(s),u'(s))ds.$$

So, we have

$$\begin{aligned}
\|u'\|_\infty \leq & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f(s,v(s),v'(s))| ds \\
\leq & \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} [p_1(s) + q_1(s)|v(s)| + r_1(s)|v'(s)|] ds \\
\leq & \frac{1}{\Gamma(\alpha-1)} [P_1 + Q_1 B + (Q_1+R_1)\|v'\|_\infty] \int_0^t (t-s)^{\alpha-2} ds \\
\leq & \frac{1}{\Gamma(\alpha)} [P_1 + Q_1 B + (Q_1+R_1)\|v'\|_\infty]. \qquad (3.8)
\end{aligned}$$

Similarly, we can get

$$\|v'\|_\infty \leq \frac{1}{\Gamma(\beta)} [P_2 + Q_2 B + (Q_2+R_2)\|u'\|_\infty]. \qquad (3.9)$$

Together with (3.8) and (3.9), we have

$$\|u'\|_\infty \leq \frac{1}{\Gamma(\alpha)} \{P_1 + Q_1 B + (Q_1+R_1) \frac{1}{\Gamma(\beta)} [P_2 + Q_2 B + (Q_2+R_2)\|u'\|_\infty]\}.$$

Thus, from $\frac{\Gamma(\alpha)\Gamma(\beta)-(Q_1+R_1)(Q_2+R_2)}{\Gamma(\alpha)\Gamma(\beta)} > 0$ and (3.9), we obtain

$$\|u'\|_\infty \leq \frac{\Gamma(\beta)(P_1+Q_1B)+(Q_1+R_1)(P_2+Q_2B)}{\Gamma(\alpha)\Gamma(\beta)-(Q_1+R_1)(Q_2+R_2)} := M_1$$

and

$$\|v'\|_\infty \leq \frac{1}{\Gamma(\beta)} [P_2 + Q_2 B + (Q_2+R_2) M_1] := M_2.$$

Together with (3.6) and (3.7), we get

$$\|(u,v)\|_{\overline{X}} \leq \max\{M_1+B, M_2+B\} := M.$$

So Ω_1 is bounded. The proof is complete. ∎

Lemma 3.5. *Suppose* (H_3) *holds, then the set*

$$\Omega_2 = \{(u,v) | (u,v) \in \operatorname{Ker} L,\ N(u,v) \in \operatorname{Im} L\}$$

is bounded.

Proof. For $(u,v) \in \Omega_2$, we have $(u,v) = (c_1,c_2),\ c_1,c_2 \in \mathbb{R}$. Then from $N(u,v) \in \operatorname{Im} L$, we get

$$\int_0^1 (1-s)^{\alpha-2} f(s,c_2,0)ds = 0, \int_0^1 (1-s)^{\beta-2} g(s,c_1,0)ds = 0,$$

which together with (H_3) imply $|c_1|, |c_2| \leq D$. Thus, we have

$$\|(u,v)\|_{\overline{X}} \leq D.$$

Hence, Ω_2 is bounded. The proof is complete. ∎

Lemma 3.6. *Suppose the first part of (H_3) holds, then the set*

$$\Omega_3 = \{(u,v) \in \operatorname{Ker} L | \lambda(u,v) + (1-\lambda)QN(u,v) = (0,0),\ \lambda \in [0,1]\}$$

is bounded.

Proof. For $(u,v) \in \Omega_3$, we have $(u,v) = (c_1, c_2)$, $c_1, c_2 \in \mathbb{R}$ and

$$\lambda c_1 + (1-\lambda)(\alpha-1)\int_0^1 (1-s)^{\alpha-2} f(s,c_2,0)ds = 0, \tag{3.10}$$

$$\lambda c_2 + (1-\lambda)(\beta-1)\int_0^1 (1-s)^{\beta-2} g(s,c_1,0)ds = 0. \tag{3.11}$$

If $\lambda = 0$, then $|c_1|, |c_2| \le D$ because of the first part of (H_3). If $\lambda = 1$, then $c_1 = c_2 = 0$. For $\lambda \in (0,1]$, we can obtain $|c_1|, |c_2| \le D$. Otherwise, if $|c_1|$ or $|c_2| > D$, in view of the first part of (H_3), one has

$$\lambda c_1^2 + (1-\lambda)(\alpha-1)\int_0^1 (1-s)^{\alpha-2} c_1 f(s,c_2,0)ds > 0,$$

or

$$\lambda c_2^2 + (1-\lambda)(\beta-1)\int_0^1 (1-s)^{\beta-2} c_2 g(s,c_1,0)ds > 0,$$

which contradict to (3.10) or (3.11). Therefore, Ω_3 is bounded. The proof is complete. ∎

Remark 3.1. If the second part of (H_3) holds, then the set

$$\Omega_3' = \{(u,v) \in \operatorname{Ker} L | -\lambda(u,v) + (1-\lambda)QN(u,v) = (0,0),\ \lambda \in [0,1]\}$$

is bounded.

Proof of Theorem 3.1. Set $\Omega = \{(u,v) \in \overline{X} | \|(u,v)\|_{\overline{X}} < \max\{M,D\}+1\}$. It follows from Lemma 3.2 and Lemma 3.3 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By Lemma 3.4 and Lemma 3.5, we get that the following two conditions are satisfied

(1) $L(u,v) \ne \lambda N(u,v)$ for every $((u,v),\lambda) \in [(\operatorname{dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(2) $Nx \notin \operatorname{Im} L$ for every $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$.

Take

$$H((u,v),\lambda) = \pm\lambda(u,v) + (1-\lambda)QN(u,v).$$

According to Lemma 3.6 (or Remark 3.1), we know that $H((u,v),\lambda) \ne 0$ for $(u,v) \in \operatorname{Ker} L \cap \partial\Omega$. Therefore

$$\begin{aligned}\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, (0,0)) &= \deg(H(\cdot,0), \Omega \cap \operatorname{Ker} L, (0,0)) \\ &= \deg(H(\cdot,1), \Omega \cap \operatorname{Ker} L, (0,0)) \\ &= \deg(\pm I, \Omega \cap \operatorname{Ker} L, (0,0)) \ne 0.\end{aligned}$$

So that, the condition (3) of Lemma 2.1 is satisfied. By Lemma 2.1, we can get that $L(u,v) = N(u,v)$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$. Therefore NBVP (1.1) has at least one solution. The proof is complete.

4. Example

Example 4.1. Consider the following NBVP

$$(4.1)\qquad \begin{cases} D_{0^+}^{\frac{3}{2}} u(t) = \frac{1}{16}[v(t) - 10] + \frac{t^2}{16} e^{-|v'(t)|}, \ t \in [0,1] \\ D_{0^+}^{\frac{5}{4}} v(t) = \frac{1}{12}[u(t) - 8] + \frac{t^3}{12} \sin^2(u'(t)), \ t \in [0,1] \\ u'(0) = u'(1) = 0, \ v'(0) = v'(1) = 0. \end{cases}$$

Choose $p_1(t) = 11/16$, $p_2(t) = 3/4$, $q_1(t) = 1/16$, $q_2(t) = 1/12$, $r_1(t) = r_2(t) = 0$, $B = D = 10$. By simple calculation, we can get that $(H_1), (H_2)$ and the first part of (H_3) hold. By Theorem 3.1, we obtain that the problem NBVP (4.1) has at least one solution.

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