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Some Representations for the Generalized Drazin Inverse of Block Matrices in Banach Algebras

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Abstract. We give explicit representations of the generalized Drazin inverse of a block matrix having generalized Schur complement generalized Drazin invertible in Banach algebras. Also we give equivalent conditions under which the group inverse of a block matrix exists and a formula for its computation. The provided results extend earlier works given in the literature.

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1. Introduction

Let $\mathscr{A}$ be a complex unital Banach algebra with unit 1. For $a \in \mathscr{A}$, the symbols $\sigma(a)$ and $\rho(a)$ will denote the spectrum and the resolvent set of a, respectively. We use $\mathscr{A}^{nil}$ and $\mathscr{A}^{qnil}$, respectively, to denote the sets of all nilpotent and quasinilpotent elements ($\sigma(a) = \{0\}$) of $\mathscr{A}$.

The concept of the generalized Drazin inverse in Banach algebras was introduced by Koliha (see [7]). For $a \in \mathscr{A}$, if there exists an element $b \in \mathscr{A}$ which satisfies

$$bab = b, \qquad ab = ba, \qquad a - a^2b \in \mathscr{A}^{qnil},$$

then b is called the generalized Drazin inverse of a (or Koliha–Drazin inverse of a), and a is generalized Drazin invertible. If the generalized Drazin inverse of a exists, it is unique and denoted by a^d. The set of all generalized Drazin invertible elements of $\mathscr{A}$ is denoted by $\mathscr{A}^d$. If $a \in \mathscr{A}^d$, the spectral idempotent a^π of a corresponding to the set $\{0\}$ is given by $a^\pi = 1 - aa^d$. The Drazin inverse is a special case of the generalized Drazin inverse for which $a - a^2b \in \mathscr{A}^{nil}$. Obviously, if a is Drazin invertible, then it is generalized Drazin invertible. The group inverse is the Drazin inverse for which the condition $a - a^2b \in \mathscr{A}^{nil}$ is replaced with $a = aba$. We use $a^\#$ to denote the group inverse of a, and we use $\mathscr{A}^\#$ to denote the set of all group invertible elements of $\mathscr{A}$. Some interesting result about Cline's formula for the generalized Drazin inverse can be found in [11].

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The next result is proved for matrices [6, Theorem 2.1], for bounded linear operators [4, Theorem 2.3] and for elements of Banach algebra [1].

Lemma 1.1. [1, Example 4.5] *Let $a,b \in \mathscr{A}^d$ and let $ab = 0$. Then*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n (a^d)^{n+1}.$$

If $a \in \mathscr{A}^{qnil}$, then a^d exists and $a^d = 0$. Consequently, by Lemma 1.1, the following lemma, which the part (i) is proved by Castro González and Koliha [1] and part (ii) for bounded linear operators in [4, Theorem 2.2], holds.

Lemma 1.2. *Let $b \in \mathscr{A}^d$ and $a \in \mathscr{A}^{qnil}$.*

(i) [1, Corollary 3.4] *If $ab = 0$, then $a+b \in \mathscr{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n$.*

(ii) *If $ba = 0$, then $a+b \in \mathscr{A}^d$ and $(a+b)^d = \sum_{n=0}^{\infty} a^n (b^d)^{n+1}$.*

Let $p = p^2 \in \mathscr{A}$ be an idempotent. Then we can represent element $a \in \mathscr{A}$ as

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$, $a_{22} = (1-p)a(1-p)$.

The following result is well-known for complex matrices (see [13]) and it is proved for elements of Banach algebra [8].

Lemma 1.3. [8, Lemma 2.2] *Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{A}$ relative to the idempotent $p \in \mathscr{A}$, $a \in (p\mathscr{A}p)^d$ and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathscr{A}p)^d$. If $ca^\pi = 0$, $a^\pi b = 0$ and the generalized Schur complement $s = d - ca^d b$ is equal to 0, then*

(1.1)
$$x^d = \begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix} \begin{bmatrix} [(aw)^d]^2 a & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & a^d b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} [(aw)^d]^2 a & [(aw)^d]^2 b \\ ca^d[(aw)^d]^2 a & ca^d[(aw)^d]^2 b \end{bmatrix}.$$

The Drazin inverse has applications in a number of areas such as control theory, Markov chains, singular differential and difference equations, iterative methods in numerical linear algebra, etc.

Campbell and Meyer [2] proposed the problem of finding an explicit representation for the Drazin inverse of a complex block matrix in terms of its blocks. This problem has not been solved yet without any restrictions upon the blocks. Many authors have considered this problem and presented formulae for the Drazin inverse under specific conditions [3, 5, 10, 15].

Let

$$x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{A} \tag{1.2}$$

relative to the idempotent $p \in \mathscr{A}$, $a \in (p\mathscr{A}p)^d$ and let the generalized Schur complement $s = d - ca^d b \in ((1-p)\mathscr{A}(1-p))^d$. The generalized Schur complement s plays an important role in the representations for x^d in many cases [5, 10, 12, 15].

Several representations for the Drazin inverse of a 2×2 block matrix under conditions which involve $W = AA^D + A^DBCA^D$ and the generalized Schur complement equals to 0 are presented by Hartwig *et al.* [5]. In [9] Li gave a representation for the Drazin inverse of block matrices with a group invertible generalized Schur complement S and in terms of $W = AA^D + A^DBS^\pi CA^D$, recovering the formula (1.1) for complex matrices [13].

In [14], some representations of the generalized Drazin inverse of a block matrix x in (1.2) with a group invertible generalized Schur complement $s = d - ca^db$ are investigated, under different conditions. The aim of paper [14] was to further weaken the conditions on the elements needed to produce explicit formulae for the generalized Drazin inverse of x compared to those known from the literature.

Under certain conditions, we present some formulae for the generalized Drazin inverse of a block matrix x in (1.2) in terms of $w = aa^d + a^dbs^\pi ca^d$ with generalized Schur complement being generalized Drazin invertible in Banach algebras. Such formulae are very complicated, but the main goal is to establish that x has the generalized Drazin inverse, and the formulae are the means to produce that result. Necessary and sufficient conditions for the existence as well as the expressions for the group inverse of triangular matrices are obtained as a consequence. Recently results [13, 14] are extended to more general settings.

2. Results

Throughout this section when we say that x is defined as in (1.2), we assume that x has a representation as in (1.2) relative to the idempotent $p \in \mathscr{A}$, $a \in (p\mathscr{A}p)^d$ and $s = d - ca^db \in ((1-p)\mathscr{A}(1-p))^d$.

In the beginning of this section we derive new representation of the generalized Drazin inverse of a block matrix x in (1.2) with a generalized Drazin invertible generalized Schur complement in terms of the generalized Drazin inverse of a, s and $a(aa^d + a^dbca^d)$. This representation for the generalized Drazin inverse of x is investigated under some rather cumbersome and complicated conditions but the theorem itself will have useful consequences which will include much simpler conditions.

Theorem 2.1. *Let x be defined as in* (1.2) *and let $w = aa^d + a^dbs^\pi ca^d$ be such that $aw \in (p\mathscr{A}p)^d$. If*

$$a^\pi b = 0, \quad bs^\pi ca^\pi = 0, \quad wbss^d = 0, \quad ss^dca^dbss^d = 0, \quad ss^\pi c = 0, \tag{2.1}$$

then $x \in \mathscr{A}^d$ and

$$\begin{aligned} x^d = {} & \left(1 + \begin{bmatrix} 0 & bs^d \\ 0 & ca^dbs^d \end{bmatrix}\right) \\ & \times \left\{ \begin{bmatrix} 0 & 0 \\ (s^d)^2ca^d(aw)^\pi a - s^dca^d(aw)^da & s^d - s^dca^d(aw)^db + (s^d)^2ca^d(aw)^\pi bs^\pi \end{bmatrix} \right. \\ & + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^{n-1}a^\pi & (s^d)^2ca^d[(aw)^d]^nbs^n - s^dca^d[(aw)^d]^{n+1}bs^n \end{bmatrix} \\ & + \sum_{n=2}^{\infty} \left(\begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^d(aw)^{n-1}(aw)^\pi a & (s^d)^{n+1}ca^d(aw)^{n-1}(aw)^\pi bs^\pi \end{bmatrix} \right. \\ & + \sum_{k=1}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & (s^d)^{n+1}ca^d(aw)^{n-1}[(aw)^d]^kbs^k \end{bmatrix} \end{aligned}$$

$$+\sum_{k=1}^{n-1}\begin{bmatrix} 0 & 0 \\ 0 & (s^d)^{n+1}ca^d(aw)^{k-1}bs^{n-k}s^\pi \end{bmatrix}\Bigg)\Bigg)\Bigg\}$$

(2.2)

$$+\begin{bmatrix} p & -bs^d \\ 0 & (1-p)-ds^d \end{bmatrix}\left(r+\sum_{n=1}^{\infty}\begin{bmatrix} 0 & [(aw)^d]^{n+2}bs^n \\ 0 & ca^d[(aw)^d]^{n+2}bs^n \end{bmatrix}\right),$$

where

$$r=\begin{bmatrix} [(aw)^d]^2a & [(aw)^d]^2bs^\pi \\ ca^d[(aw)^d]^2a & ca^d[(aw)^d]^2bs^\pi \end{bmatrix}.$$

Proof. Since $aa^d+a^\pi=p$ and $ss^d+s^\pi=1-p$, we can write

$$x=\begin{bmatrix} a^2a^d & bs^\pi \\ caa^d & ds^\pi \end{bmatrix}+\begin{bmatrix} aa^\pi & bss^d \\ ca^\pi & dss^d \end{bmatrix}:=y+z.$$

The equalities $a^da^\pi=0$, $s^\pi s^d=0$ and (2.1) imply

$$yz=\begin{bmatrix} bs^\pi ca^\pi & awbss^d \\ ca^dbs^\pi ca^\pi+ss^\pi ca^\pi & cwbss^d+ss^\pi ca^dbss^d \end{bmatrix}=0.$$

In order to verify that $y\in\mathscr{A}^d$, observe that

$$y=\begin{bmatrix} a^2a^d & bs^\pi \\ caa^d & ca^dbs^\pi \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & ss^\pi \end{bmatrix}:=y_1+y_2.$$

If $A_{y_1}\equiv a^2a^d$, $B_{y_1}\equiv bs^\pi$, $C_{y_1}\equiv caa^d$ and $D_{y_1}\equiv ca^dbs^\pi$, by $(a^2a^d)^\#=a^d$, $A_{y_1}\in(p\mathscr{A}p)^\#$, $S_{y_1}\equiv D_{y_1}-C_{y_1}A_{y_1}^\#B_{y_1}=0$ and $W_{y_1}=A_{y_1}A_{y_1}^\#+A_{y_1}^\#B_{y_1}C_{y_1}A_{y_1}^\#=w$. From $A_{y_1}^\pi B_{y_1}=a^\pi bs^\pi=0$, $C_{y_1}A_{y_1}^\pi=0$ and Lemma 1.3, we have that $y_1\in\mathscr{A}^d$ and

$$y_1^d=\begin{bmatrix} p & 0 \\ ca^d & 0 \end{bmatrix}\begin{bmatrix} [(aw)^d]^2a & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & a^dbs^\pi \\ 0 & 0 \end{bmatrix}=r.$$

Recall that, for $u=\begin{bmatrix} m & t \\ 0 & n \end{bmatrix}$,

$$\lambda\in\rho_{p\mathscr{A}p}(m)\cap\rho_{(1-p)\mathscr{A}(1-p)}(n)\ \Rightarrow\ \lambda\in\rho(u),$$

i.e.

$$\sigma(u)\subseteq\sigma_{p\mathscr{A}p}(m)\cup\sigma_{(1-p)\mathscr{A}(1-p)}(n).$$

Thus, $ss^\pi\in((1-p)\mathscr{A}(1-p))^{qnil}$ gives $y_2\in\mathscr{A}^{qnil}$. Using Lemma 1.2(i), by $y_2y_1=0$, we deduce that $y\in\mathscr{A}^d$ and

$$y^d=\sum_{n=0}^{\infty}(y_1^d)^{n+1}y_2^n=\sum_{n=0}^{\infty}r^{n+1}\begin{bmatrix} 0 & 0 \\ 0 & ss^\pi \end{bmatrix}^n.$$

To prove that $z\in\mathscr{A}^d$, consider

$$z=\begin{bmatrix} aa^\pi & 0 \\ ca^\pi & 0 \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & s^2s^d \end{bmatrix}+\begin{bmatrix} 0 & bss^d \\ 0 & ca^dbss^d \end{bmatrix}:=z_1+z_2+z_3.$$

Because $aa^\pi\in(p\mathscr{A}p)^{qnil}$ and $(s^2s^d)^\#=s^d$, then $z_1\in\mathscr{A}^{qnil}$, $z_2\in\mathscr{A}^\#$ and $z_2^\#=\begin{bmatrix} 0 & 0 \\ 0 & s^d \end{bmatrix}$.

From $z_1z_2=0$ and Lemma 1.2(i), $z_1+z_2\in\mathscr{A}^d$ and $(z_1+z_2)^d=\sum\limits_{n=0}^{\infty}(z_2^\#)^{n+1}z_1^n$. Also, $z_3\in$

$\mathscr{A}^{nil}$, by $z_3^2=0$. Now, by $(z_1+z_2)z_3=0$ and Lemma 1.2(ii), we conclude that $z\in\mathscr{A}^d$ and $z^d=(z_1+z_2)^d+z_3[(z_1+z_2)^d]^2$.

Applying Lemma 1.1, we obtain that $x\in\mathscr{A}^d$ and

$$\begin{aligned} x^d &= \sum_{n=0}^{\infty}(z^d)^{n+1}y^ny^\pi+\sum_{n=0}^{\infty}z^\pi z^n(y^d)^{n+1} \\ &= \sum_{n=0}^{\infty}(1+z_3(z_1+z_2)^d)[(z_1+z_2)^d]^{n+1}y^ny^\pi+\sum_{n=0}^{\infty}z^\pi z^n(y^d)^{n+1}:=X_1+X_2. \end{aligned} \tag{2.3}$$

From $z_1y=0$, we get $(z_1+z_2)^dy=z_2^\#y$ and

$$\begin{aligned} \sum_{n=0}^{\infty}[(z_1+z_2)^d]^{n+1}y^n &= (z_1+z_2)^d+\sum_{n=1}^{\infty}[(z_1+z_2)^d]^n z_2^\# y^n \\ &= (z_1+z_2)^d+\sum_{n=1}^{\infty}\left(z_2^\#+\sum_{k=1}^{\infty}(z_2^\#)^{k+1}z_1^k\right)^n z_2^\# y^n \\ &= (z_1+z_2)^d+\sum_{n=1}^{\infty}\left((z_2^\#)^n+(z_2^\#)^{n-1}\sum_{k=1}^{\infty}(z_2^\#)^{k+1}z_1^k\right)z_2^\# y^n \\ &= \sum_{n=0}^{\infty}(z_2^\#)^{n+1}z_1^n+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}y^n \end{aligned}$$

implying

$$\begin{aligned} X_1 &= \left(1+z_3\sum_{k=0}^{\infty}(z_2^\#)^{k+1}z_1^k\right)\left(\sum_{n=0}^{\infty}(z_2^\#)^{n+1}z_1^n+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}y^n\right)y^\pi \\ &= \left(1+\begin{bmatrix}0 & b\\ 0 & ca^db\end{bmatrix}\left[z_2^\#+\sum_{k=1}^{\infty}(z_2^\#)^{k+1}z_1^k\right]\right)z_2^\#\left(\sum_{n=0}^{\infty}(z_2^\#)^n z_1^n+\sum_{n=1}^{\infty}(z_2^\#)^n y^n\right)y^\pi \\ &= \left(1+\begin{bmatrix}0 & bs^d\\ 0 & ca^dbs^d\end{bmatrix}\right)\left(z_2^\#y^\pi+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}z_1^ny^\pi+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}y^ny^\pi\right) \\ &= \left(1+\begin{bmatrix}0 & bs^d\\ 0 & ca^dbs^d\end{bmatrix}\right)\left(z_2^\#y^\pi+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}z_1^n+\sum_{n=1}^{\infty}(z_2^\#)^{n+1}y^ny^\pi\right). \end{aligned} \tag{2.4}$$

It can be check that $aa^d(aw)=aw=(aw)aa^d$,

$$y^\pi=y_1^\pi-\sum_{n=1}^{\infty}(y_1^d)^ny_2^n,\qquad yy^\pi=y_1y_1^\pi-y_1\sum_{n=1}^{\infty}(y_1^d)^ny_2^n+y_2$$

and

$$y^ny^\pi=y_1^ny_1^\pi-y_1^n\sum_{k=1}^{\infty}(y_1^d)^ky_2^k+\sum_{k=1}^{n-1}y_1^ky_2^{n-k}+y_2^n\qquad(n=2,3,\dots).$$

Further, note that

$$(y_1^d)^ny_2^n=\begin{bmatrix}0 & [(aw)^d]^{n+1}bs^ns^\pi\\ 0 & ca^d[(aw)^d]^{n+1}bs^ns^\pi\end{bmatrix}\qquad(n=1,2,\dots),$$

$$y_1^ky_2^{n-k}=\begin{bmatrix}0 & (aw)^{k-1}bs^{n-k}s^\pi\\ 0 & ca^d(aw)^{k-1}bs^{n-k}s^\pi\end{bmatrix}\qquad(n=1,2,\dots;\ k=1,\dots,n-1).$$

Also, we can show that

$$y_1^\pi = 1 - y_1 y_1^d = \begin{bmatrix} p-(aw)^d a & -(aw)^d bs^\pi \\ -ca^d(aw)^d a & (1-p)-ca^d(aw)^d bs^\pi \end{bmatrix}$$

and

$$y_1^n y_1^\pi = \begin{bmatrix} aa^d(aw)^{n-1}(aw)^\pi a & (aw)^{n-1}(aw)^\pi bs^\pi \\ ca^d(aw)^{n-1}(aw)^\pi a & ca^d(aw)^{n-1}(aw)^\pi bs^\pi \end{bmatrix} \qquad (n=1,2,\dots).$$

Therefore, by these equalities, (2.4) and, for $n = 1,2,\dots$ and $k = 1,2,\dots,$

$$(z^\#)^{n+1} y_1^k = \begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^d(aw)^{k-1}a & (s^d)^{n+1}ca^d(aw)^{k-1}bs^\pi \end{bmatrix},$$

we obtain

$$\begin{aligned} X_1 = &\left(1+\begin{bmatrix} 0 & bs^d \\ 0 & ca^d bs^d \end{bmatrix}\right)\left\{\begin{bmatrix} 0 & 0 \\ -s^d ca^d(aw)^d a & s^d - s^d ca^d(aw)^d b \end{bmatrix}\right. \\ &-\sum_{n=1}^\infty \begin{bmatrix} 0 & 0 \\ 0 & s^d ca^d[(aw)^d]^{n+1}bs^n \end{bmatrix} + \sum_{n=1}^\infty \begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^{n-1}a^\pi & 0 \end{bmatrix} \\ &+\begin{bmatrix} 0 & 0 \\ (s^d)^2 ca^d(aw)^\pi a & (s^d)^2 ca^d(aw)^\pi bs^\pi \end{bmatrix} + \sum_{n=1}^\infty \begin{bmatrix} 0 & 0 \\ 0 & (s^d)^2 ca^d[(aw)^d]^n bs^n \end{bmatrix} \\ &+\sum_{n=2}^\infty \left(\begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^d(aw)^{n-1}(aw)^\pi a & (s^d)^{n+1}ca^d(aw)^{n-1}(aw)^\pi bs^\pi \end{bmatrix}\right. \\ &+\sum_{k=1}^\infty \begin{bmatrix} 0 & 0 \\ 0 & (s^d)^{n+1}ca^d(aw)^{n-1}[(aw)^d]^k bs^k \end{bmatrix} \\ &\left.\left.+\sum_{k=1}^{n-1} \begin{bmatrix} 0 & 0 \\ 0 & (s^d)^{n+1}ca^d(aw)^{k-1}bs^{n-k}s^\pi \end{bmatrix}\right)\right\} \end{aligned} \tag{2.5}$$

Observe that, by

$$\begin{aligned} zz^d y &= [(z_1+z_2)(z_1+z_2)^d + z_3(z_1+z_2)^d]y = (z_1+z_2)z_2^\# y + z_3 z_2^\# y \\ &= (z_2+z_3)z_2^\# y, \end{aligned}$$

we have $zz^d y^d = (z_2+z_3)z_2^\# y^d$, $z^\pi y^d = \begin{bmatrix} p & -bs^d \\ 0 & (1-p)-ds^d \end{bmatrix} y^d$ and $zz^\pi y^d = z_1 y (y^d)^2 = 0$. Hence,

$$\begin{aligned} X_2 &= z^\pi y^d + \sum_{n=1}^\infty z^n z^\pi (y^d)^{n+1} = z^\pi y^d \\ &= \begin{bmatrix} p & -bs^d \\ 0 & (1-p)-ds^d \end{bmatrix}\left(r + \sum_{n=1}^\infty \begin{bmatrix} 0 & [(aw)^d]^{n+2}bs^n \\ 0 & ca^d[(aw)^d]^{n+2}bs^n \end{bmatrix}\right). \end{aligned} \tag{2.6}$$

Thus, from (2.3), (2.5) and (2.6), we get (2.2). ∎

Similarly as Theorem 2.1, we get the following formula for the generalized Drazin inverse of block matrix. For the sake of clarity of presentation, the proof is given.

Theorem 2.2. *Let x be defined as in* (1.2) *and let $w = aa^d + a^d bs^\pi ca^d$ be such that $aw \in (p\mathscr{A}p)^d$. If*

$$ca^\pi = 0, \quad a^\pi bs^\pi c = 0, \quad ss^d cw = 0, \quad ss^d ca^d bss^d = 0, \quad bs^\pi s = 0,$$

then $x \in \mathscr{A}^d$ and

$$\begin{aligned}
x^d = {} & \left(t + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ s^n ca^d[(aw)^d]^{n+2}a & s^n ca^d[(aw)^d]^{n+2}b \end{bmatrix} \right) \begin{bmatrix} p & 0 \\ -s^d c & (1-p) - s^d d \end{bmatrix} \\
& + \left\{ \begin{bmatrix} 0 & aa^d(aw)^\pi b(s^d)^2 - (aw)^d bs^d \\ 0 & s^d - s^\pi ca^d(aw)^d bs^d + s^\pi ca^d(aw)^\pi b(s^d)^2 \end{bmatrix} + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^{n-1}a^\pi b(s^d)^{n+1} \\ 0 & 0 \end{bmatrix} \right. \\
& + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & s^{n+1}ca^d[(aw)^d]^{n+2}bs^d - s^{n+1}s^\pi ca^d[(aw)^d]^{n+1}b(s^d)^2 \end{bmatrix} \\
& + \sum_{n=2}^{\infty} \left(\begin{bmatrix} 0 & (aw)^{n-1}(aw)^\pi b(s^d)^{n+1} \\ 0 & s^\pi ca^d(aw)^{n-1}(aw)^\pi b(s^d)^{n+1} \end{bmatrix} \right. \\
& + \sum_{k=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & s^{k+1}ca^d[(aw)^d]^{k+1}(aw)^{n-1}b(s^d)^{n+1} \end{bmatrix} \\
& \left.\left. + \sum_{k=1}^{n-1} \begin{bmatrix} 0 & 0 \\ 0 & s^{n-k}s^\pi ca^d(aw)^{k-1}b(s^d)^{n+1} \end{bmatrix} \right) \right\} \left(1 + \begin{bmatrix} 0 & 0 \\ s^d c & s^d ca^d b \end{bmatrix} \right),
\end{aligned} \tag{2.7}$$

where

$$t = \begin{bmatrix} [(aw)^d]^2 a & [(aw)^d]^2 b \\ s^\pi ca^d[(aw)^d]^2 a & s^\pi ca^d[(aw)^d]^2 b \end{bmatrix}.$$

Proof. Notice that

$$x = \begin{bmatrix} a^2a^d & aa^d b \\ s^\pi c & s^\pi d \end{bmatrix} + \begin{bmatrix} aa^\pi & a^\pi b \\ ss^d c & ss^d d \end{bmatrix} := y + z$$

and $zy = 0$.

To show that $y \in \mathscr{A}^d$, let

$$y = \begin{bmatrix} a^2a^d & aa^d b \\ s^\pi c & s^\pi ca^d b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s^\pi s \end{bmatrix} := y_1 + y_2.$$

Then, by Lemma 1.3, $y_1 \in \mathscr{A}^d$ and $y_1^d = t$. Since $y_2 \in \mathscr{A}^{qnil}$ and $y_1 y_2 = 0$, by Lemma 1.2(ii), $y \in \mathscr{A}^d$ and $y^d = \sum_{n=0}^{\infty} y_2^n (y_1^d)^{n+1}$.

Now, we will check that $z \in \mathscr{A}^d$. If

$$z = \begin{bmatrix} aa^\pi & a^\pi b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s^2s^d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ ss^d c & ss^d ca^d b \end{bmatrix} := z_1 + z_2 + z_3,$$

we have $z_1 \in \mathscr{A}^{qnil}$, $z_2 \in \mathscr{A}^\#$, $z_3^2 = 0$, $z_2 z_1 = 0$ and $z_3(z_1 + z_2) = 0$. Applying Lemma 1.2, first $z_1 + z_2 \in \mathscr{A}^d$ and $(z_1 + z_2)^d = \sum_{n=0}^{\infty} z_1^n (z_2^\#)^{n+1}$; and then $z \in \mathscr{A}^d$ and $z^d = (z_1 + z_2)^d + [(z_1 + z_2)^d]^2 z_3$.

Using Lemma 1.1, we deduce that $x \in \mathscr{A}^d$ and

$$x^d = \sum_{n=0}^{\infty} (y^d)^{n+1} z^n z^\pi + \sum_{n=0}^{\infty} y^\pi y^n (z^d)^{n+1} := X_1 + X_2. \tag{2.8}$$

By $yz_1 = 0$, $y(z_1+z_2)^d = yz_2^\#$, $y^d z^\pi = y^d \begin{bmatrix} p & 0 \\ -s^d c & (1-p) - s^d d \end{bmatrix}$ and $y^d z^\pi z = 0$. So, $X_1 = y^d z^\pi + \sum\limits_{n=1}^{\infty} (y^d)^{n+1} z^\pi z^n = y^d z^\pi$. Since, for $n = 1, 2, \dots$ and $k = 1, 2, \dots$,

$$y_2^n t^k = \begin{bmatrix} 0 & 0 \\ s^n s^\pi c a^d [(aw)^d]^{k+1} a & s^n s^\pi c a^d [(aw)^d]^{k+1} b \end{bmatrix},$$

we get

$$\begin{aligned} X_1 = {} & \left(t + \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 0 \\ s^n c a^d [(aw)^d]^{n+2} a & s^n c a^d [(aw)^d]^{n+2} b \end{bmatrix} \right) \\ & \times \begin{bmatrix} p & 0 \\ -s^d c & (1-p) - s^d d \end{bmatrix}. \end{aligned} \tag{2.9}$$

Furthermore,

$$\begin{aligned} X_2 &= \sum_{n=0}^{\infty} y^\pi y^n (z^d)^{n+1} = \sum_{n=0}^{\infty} y^\pi y^n [(z_1+z_2)^d]^{n+1} (1 + (z_1+z_2)^d z_3) \\ &= \left(y^\pi (z_1+z_2)^d + \sum_{n=1}^{\infty} y^\pi y^n z_2^\# [(z_1+z_2)^d]^n \right) (1 + (z_1+z_2)^d z_3) \\ &= \left[y^\pi \sum_{n=0}^{\infty} z_1^n (z_2^\#)^{n+1} + \sum_{n=1}^{\infty} y^\pi y^n z_2^\# \left((z_2^\#)^n + \sum_{k=1}^{\infty} z_1^k (z_2^\#)^{k+n} \right) \right] \\ &\quad \times \left(1 + z_2^\# z_3 + \sum_{n=1}^{\infty} z_1^n (z_2^\#)^{n+1} z_3 \right) \\ &= \left[y^\pi \sum_{n=0}^{\infty} z_1^n (z_2^\#)^{n+1} + \sum_{n=1}^{\infty} y^\pi y^n (z_2^\#)^{n+1} \right] (1 + z_2^\# z_3) \\ &= \left[y^\pi z_2^\# + \sum_{n=1}^{\infty} z_1^n (z_2^\#)^{n+1} + \sum_{n=1}^{\infty} y^\pi y^n (z_2^\#)^{n+1} \right] (1 + z_2^\# z_3). \end{aligned}$$

We can get

$$y^\pi = y_1^\pi - \sum_{n=0}^{\infty} y_2^{n+1} (y_1^d)^{n+1}, \qquad y^\pi y = y_1^\pi y_1 - \sum_{n=0}^{\infty} y_2^{n+1} (y_1^d)^{n+1} y_1 + y_2,$$

$$y^\pi y^n = y_1^\pi y_1^n - \sum_{n=0}^{\infty} y_2^{n+1} (y_1^d)^{n+1} y_1^n + \sum_{k=1}^{n-1} y_2^{n-k} y_1^k + y_2^n, \quad (n = 2, 3, \dots),$$

and also, for $n = 1, 2, \dots$ and $k = 1, \dots, n-1$,

$$y_1^\pi = \begin{bmatrix} p - (aw)^d a & -(aw)^d b \\ -s^\pi c a^d (aw)^d a & (1-p) - s^\pi c a^d (aw)^d b \end{bmatrix},$$

$$y_1^n y_1^\pi = \begin{bmatrix} aa^d(aw)^{n-1}(aw)^\pi a & aa^d(aw)^{n-1}(aw)^\pi b \\ s^\pi ca^d(aw)^{n-1}(aw)^\pi a & s^\pi ca^d(aw)^{n-1}(aw)^\pi b \end{bmatrix},$$

$$y_2^{n-k} y_1^k = \begin{bmatrix} 0 & 0 \\ s^{n-k}s^\pi ca^d(aw)^{k-1}a & s^{n-k}s^\pi ca^d(aw)^{k-1}b \end{bmatrix},$$

$$y_1^n (z_2^\#)^{n+1} = \begin{bmatrix} 0 & aa^d(aw)^{n-1}b(s^d)^{n+1} \\ 0 & s^\pi ca^d(aw)^{n-1}b(s^d)^{n+1} \end{bmatrix}.$$

Now we obtain

$$\begin{aligned} X_2 = \Bigg\{ &\begin{bmatrix} 0 & -(aw)^d bs^d \\ 0 & s^d - s^\pi ca^d(aw)^d bs^d \end{bmatrix} + \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & s^{n+1}ca^d[(aw)^d]^{n+2}bs^d \end{bmatrix} \\ &+ \sum_{n=1}^{\infty} \begin{bmatrix} 0 & a^{n-1}a^\pi b(s^d)^{n+1} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & aa^d(aw)^\pi b(s^d)^2 \\ 0 & s^\pi ca^d(aw)^\pi b(s^d)^2 \end{bmatrix} \\ &- \sum_{n=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & s^{n+1}s^\pi ca^d[(aw)^d]^{n+1}b(s^d)^2 \end{bmatrix} + \sum_{n=2}^{\infty} \Bigg(\begin{bmatrix} 0 & (aw)^{n-1}(aw)^\pi b(s^d)^{n+1} \\ 0 & s^\pi ca^d(aw)^{n-1}(aw)^\pi b(s^d)^{n+1} \end{bmatrix} \\ &+ \sum_{k=0}^{\infty} \begin{bmatrix} 0 & 0 \\ 0 & s^{k+1}ca^d[(aw)^d]^{k+1}(aw)^{n-1}b(s^d)^{n+1} \end{bmatrix} \\ &+ \sum_{k=1}^{n-1} \begin{bmatrix} 0 & 0 \\ 0 & s^{n-k}s^\pi ca^d(aw)^{k-1}b(s^d)^{n+1} \end{bmatrix} \Bigg) \Bigg\} \left(1 + \begin{bmatrix} 0 & 0 \\ s^d c & s^d ca^d b \end{bmatrix} \right) \end{aligned} \tag{2.10}$$

The equalities (2.8), (2.9) and (2.10) imply (2.7). ∎

If we assume that the generalized Drazin-Schur complement s is group invertible in Theorem 2.1 and Theorem 2.2, we obtain [14, Theorem 2.1 and Theorem 2.2].

Using Theorem 2.1 and Theorem 2.2, we can get the next result which recovers Lemma 1.3 and the analogy result for matrices [13].

Corollary 2.1. *Let x be defined as in* (1.2) *and let $w = aa^d + a^d bca^d$ be such that $aw \in (p\mathscr{A}p)^d$. If $s = 0$, and if*

$$(a^\pi b = 0 \text{ and } bca^\pi = 0) \text{ or } (ca^\pi = 0 \text{ and } a^\pi bc = 0) \text{ or } (a^\pi b = 0 \text{ and } ca^\pi = 0),$$

then $x \in \mathscr{A}^d$ and x^d is defined as in (1.1).

In the following theorems, we study the group inverse of a triangular block matrix. First, if $b = 0$ in Theorem 2.1, we obtain the equivalent conditions for the existence and representation of the group inverse of x.

Theorem 2.3. *Let $x = \begin{bmatrix} a & 0 \\ c & s \end{bmatrix} \in \mathscr{A}$ relative to the idempotent $p \in \mathscr{A}$, $a \in (p\mathscr{A}p)^d$ and $s \in ((1-p)\mathscr{A}(1-p))^d$. Assume that $ss^\pi c = 0$. Then*

$$x \in \mathscr{A}^\# \text{ if and only if } a \in (p\mathscr{A}p)^\#,\ s \in ((1-p)\mathscr{A}(1-p))^\# \text{ and } s^\pi ca^\pi = 0.$$

Furthermore, if $a \in (p\mathscr{A}p)^\#$, $s \in ((1-p)\mathscr{A}(1-p))^\#$ and $s^\pi ca^\pi = 0$, then

$$x^\# = \begin{bmatrix} a^\# & 0 \\ s^\pi c(a^\#)^2 - s^\# ca^\# + (s^\#)^2 ca^\pi & s^\# \end{bmatrix}.$$

Proof. Using Theorem 2.1 for $b=0$, by $s=d$, $w=aa^d$, $(aw)^{\#}=a^d$ and $a^d(aw)^{\pi}=a^da^{\pi}=0$, we have $x\in\mathscr{A}^d$ and

$$x^d=\begin{bmatrix} a^d & 0 \\ s^{\pi}c(a^d)^2-s^dca^d & s^d \end{bmatrix}+\sum_{n=1}^{\infty}\begin{bmatrix} 0 & 0 \\ (s^d)^{n+1}ca^{n-1}a^{\pi} & 0 \end{bmatrix}.$$

Now we get

$$x^2x^d=\begin{bmatrix} a^2a^d & 0 \\ caa^d+\sum\limits_{n=1}^{\infty}s^2(s^d)^{n+1}ca^{n-1}a^{\pi} & s^2s^d \end{bmatrix},$$

which gives that $x^2x^d=x$ is equivalent to $a^2a^d=a$, $s^2s^d=s$ and $\sum\limits_{n=1}^{\infty}s(s^d)^nca^{n-1}a^{\pi}=ca^{\pi}$. Therefore, $x\in\mathscr{A}^{\#}$ if and only if $a\in(p\mathscr{A}p)^{\#}$, $s\in((1-p)\mathscr{A}(1-p))^{\#}$ and $ss^dca^{\pi}=ca^{\pi}$. ∎

By Theorem 2.3, if x is defined as in Theorem 2.3, we can get:

(1) if $s^{\pi}c=0$, then

$$x\in\mathscr{A}^{\#} \text{ if and only if } a\in(p\mathscr{A}p)^{\#} \text{ and } s\in((1-p)\mathscr{A}(1-p))^{\#};$$

(2) if $s\in((1-p)\mathscr{A}(1-p))^{-1}$, then

$$x\in\mathscr{A}^{\#} \text{ if and only if } a\in(p\mathscr{A}p)^{\#}.$$

In addition, if $s\in((1-p)\mathscr{A}(1-p))^{-1}$ and $a\in(p\mathscr{A}p)^{\#}$,

$$x^{\#}=\begin{bmatrix} a^{\#} & 0 \\ -s^{-1}ca^{\#}+s^{-2}ca^{\pi} & s^{-1} \end{bmatrix}.$$

For $c=0$ in Theorem 2.2, we show the next result similarly as Theorem 2.3.

Theorem 2.4. *Let* $x=\begin{bmatrix} a & b \\ 0 & s \end{bmatrix}\in\mathscr{A}$ *relative to the idempotent* $p\in\mathscr{A}$, $a\in(p\mathscr{A}p)^d$ *and* $s\in((1-p)\mathscr{A}(1-p))^d$. *Assume that* $bs^{\pi}s=0$. *Then*

$$x\in\mathscr{A}^{\#} \textit{ if and only if } a\in(p\mathscr{A}p)^{\#},\ s\in((1-p)\mathscr{A}(1-p))^{\#} \textit{ and } a^{\pi}bs^{\pi}=0.$$

Furthermore, if $a\in(p\mathscr{A}p)^{\#}$, $s\in((1-p)\mathscr{A}(1-p))^{\#}$ *and* $a^{\pi}bs^{\pi}=0$, *then*

$$x^{\#}=\begin{bmatrix} a^{\#} & (a^{\#})^2bs^{\pi}-a^{\#}bs^{\#}+a^{\pi}b(s^{\#})^2 \\ 0 & s^{\#} \end{bmatrix}.$$

Notice that, if x is defined as in Theorem 2.4, we have:

(1) if $bs^{\pi}=0$, then

$$x\in\mathscr{A}^{\#} \text{ if and only if } a\in(p\mathscr{A}p)^{\#} \text{ and } s\in((1-p)\mathscr{A}(1-p))^{\#};$$

(2) if $s\in((1-p)\mathscr{A}(1-p))^{-1}$, then

$$x\in\mathscr{A}^{\#} \text{ if and only if } a\in(p\mathscr{A}p)^{\#}.$$

In addition, if $s\in((1-p)\mathscr{A}(1-p))^{-1}$ and $a\in(p\mathscr{A}p)^{\#}$,

$$x^{\#}=\begin{bmatrix} a^{\#} & -a^{\#}bs^{-1}+a^{\pi}bs^{-2} \\ 0 & s^{-1} \end{bmatrix}.$$

Expressions for the group inverses in Theorem 2.3 and Theorem 2.4 are the special cases of [1, Theorem 2.3] for Banach algebra elements and [4, Theorem 2.2] for bounded linear operators. Also these expressions are extensions of formulae in [14, Theorem 2.3 and Theorem 2.4].

In the end of this section, we state an example to illustrate our results.

Example 2.1. In Banach algebra $\mathscr{A}$, if $x = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix} \in \mathscr{A}$ (or $x = \begin{bmatrix} p & 0 \\ c & 0 \end{bmatrix} \in \mathscr{A}$) relative to the idempotent $p \in \mathscr{A}$, then $a^d = a = p$, $a^\pi = 0$, $s = 0 = s^d$, $s^\pi = 1 - p$ and $w = p = aw = (aw)^d$. Using Theorem 2.1 or Theorem 2.2, we get that $x \in \mathscr{A}^d$ and $x^d = \begin{bmatrix} p & b \\ 0 & 0 \end{bmatrix}$ or $x^d = \begin{bmatrix} p & 0 \\ c & 0 \end{bmatrix}$.

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