Several Inequalities for the Volume of the Unit Ball in $\mathbb{R}^n$

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Abstract. In the paper, the author establishes several new inequalities involving the volume of the unit ball in $\mathbb{R}^n$.

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1. Introduction

In the recent past, inequalities about the Euler gamma function $\Gamma(x)$ have attracted the attention of many authors. In particular, several researchers established interesting properties of the volume of the unit ball in $\mathbb{R}^n$,

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)}, n = 1, 2, \ldots.$$  \hspace{1cm} (1.1)

In the paper [5], it was proved that the sequence $\{\Omega_n\}_{n \geq 1}$ attains its maximum at $n = 5$. In the paper [4], the sequence $\{(\Omega_n)^{1/n}\}_{n \geq 1}$ is proved to be monotonically decreased to zero. Other results have been established by Anderson and Qiu [3], and Klain and Rota [9] who proved that the sequence $\{(\Omega_n)^{1/\ln n}\}_{n \geq 1}$ decreases to $e^{-1/2}$, and the sequence $\{n\Omega_n/\Omega_{n-1}\}_{n \geq 1}$ is increasing, respectively. Motivated by the following inequalities

$$\Omega_{n+1}^{n/(n+1)} < \Omega_n, n = 1, 2, \ldots$$  \hspace{1cm} (1.2)

and

$$1 < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < 1 + \frac{1}{n}$$  \hspace{1cm} (1.3)

stated in [4] and [9], Alzer proved in [1] that for all $n \geq 1$,

$$a(\Omega_{n+1})^{n/(n+1)} \leq \Omega_n \leq b(\Omega_{n+1})^{n/(n+1)}$$  \hspace{1cm} (1.4)

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with the best possible constants \( a = 2/\sqrt{\pi} = 1.1283 \cdots \) and \( b = \sqrt{e} = 1.6487 \cdots \). An improvement of the double inequality (1.4) was given in [11]: for \( n \geq 4 \),

\[
(1.5) \quad \frac{k}{\sqrt{2\pi}} \leq \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} \leq \frac{\sqrt{e}}{\sqrt{2\pi}}
\]

where \( k = (64 \cdot 720^{11/12} \cdot 2^{1/22})/(10395\pi^{5/11}) = 1.5714 \cdots \). Equality in the left-hand side of (1.5) occurs if and only if \( n = 11 \).

The following class of inequalities

\[
(1.6) \quad \sqrt{\frac{n+a}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b}{2\pi}}
\]

was studied by Alzer [1] and Qiu [14] where \( a, b \) are real parameters. Later, the inequality (1.6) was recovered in [6]. Furthermore, Mortici established the following new sharp bounds

\[
(1.7) \quad \sqrt{\frac{n+\frac{1}{2}}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+\frac{1}{2}}{2\pi} + \frac{1}{16n\pi}}
\]

which improves the previous results of Alzer et al. in [11]. Therefore, Alzer proved in [1] that for \( n \geq 1 \),

\[
(1.8) \quad \left(1 + \frac{1}{n}\right)^{\alpha} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{n}\right)^{\beta}
\]

in which the best possible constants \( \alpha = 2 - \log_2 \pi \) and \( \beta = 1/2 \). Later, in [11], Mortici showed that for every \( n \geq 4 \),

\[
(1.9) \quad \left(1 + \frac{1}{n}\right)^{1/2 - 1/4n} \leq \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} \leq \left(1 + \frac{1}{n}\right)^{1/2}.
\]

Related references see [7, 8, 13, 15].

The aim of this paper is to establish some new inequalities involving the volume of the unit ball in \( \mathbb{R}^n \).

2. Lemmas

In order to prove the main results, following lemmas are useful.

**Lemma 2.1.** [10, p. 390] Let \( x_i \in \mathbb{R}^+ \), \( i = 1, 2, \ldots, n \) and \( \sum_{i=1}^n x_i = nx \), then

\[
(2.1) \quad \prod_{i=1}^n \Gamma(x_i) \geq (\Gamma(x))^n.
\]

**Lemma 2.2.** [2, Legendre] For every \( z \neq -1, -2, \ldots \), then

\[
(2.2) \quad 2^{2z-1}\Gamma(z)\Gamma(z+1/2) = \pi^{1/2}\Gamma(2z).
\]

**Lemma 2.3.** [4, p. 131] For every integer \( n \geq 1 \), the sequence \( \{ (\Omega_n)^{1/n} \}_{n \geq 1} \) is monotonically decreasing to zero.

**Lemma 2.4.** [12, p. 612] For every \( x \in [1, \infty) \), we have

\[
(2.3) \quad \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x+\alpha} < \Gamma(x+1) < \sqrt{\pi} \left(\frac{x}{e}\right)^x \sqrt{2x+\beta}
\]

where \( \alpha = 1/3 \) and \( \beta = \sqrt{391/30} - 2 = 0.3533 \cdots \).
3. Main results

In what follows, we always suppose \( \beta = \sqrt{391/30} - 2 = 0.3533 \ldots \).

**Theorem 3.1.** For all natural number \( n \), we have

\[
(3.1) \quad \Omega_n \leq \left( \Omega_1 \Omega_2 \cdots \Omega_{n-1} \right)^{1/(n-1)}.
\]

If \( n \) is odd integer, then

\[
(3.2) \quad (\Omega_1 \Omega_2 \cdots \Omega_n)^{1/n} \leq \Omega_{(n+1)/2}.
\]

**Proof.** Using Lemma 2.3, we easily prove inequality (3.1). Next, we only prove inequality (3.2). By virtue of Lemma 2.1, we get

\[
(3.3) \quad (\Omega_1 \Omega_2 \cdots \Omega_n)^{1/n} = \left( \frac{\pi^{1/2} \pi^{2/2} \cdots \pi^{n/2}}{(\Gamma(1/2 + 1) \Gamma(2/2 + 1) \cdots \Gamma(n/2 + 1))} \frac{\pi^{(n+1)/4}}{\pi^{(n+1)/4}} \right)^{1/n} \leq \frac{\Gamma((n+1)/4 + 1)}{\Gamma((n+1)/4)} = \Omega_{(n+1)/2}.
\]

**Theorem 3.2.** For every integer \( n > 1 \), we have

\[
(3.4) \quad \frac{(n+1) (n+1/2)}{(n+\beta)^2} < \frac{\Omega^2_n}{\Omega_{n-1} \Omega_{n+1}} < \frac{(n+1) (n+\beta)}{(n+1/2)^2}.
\]

**Proof.** Easy computation and simplification yield

\[
(3.5) \quad 2^n \Gamma((n+1)/2) \Gamma((n+2)/2) = \pi^{1/2} n!.
\]

Setting \( z = (n+1)/2 \) and \( z = (n+3)/2 \) in (2.2) of Lemma 2.2, we obtain

\[
2^n \Gamma((n+1)/2) \Gamma((n+2)/2) = \pi^{1/2} \Gamma(n+3) = \pi^{1/2} (n+2)!.
\]

Combining (3.4), (3.5) and (3.6) leads to

\[
(3.7) \quad \frac{\Omega^2_n}{\Omega_{n-1} \Omega_{n+1}} = \frac{\sqrt{\pi} (n+2)! \sqrt{\pi n!}}{2^{n} (n+4)/2 (\Gamma((n+2)/2))^3} = \frac{\pi(n+1)! n!}{2^{n+2} (n+2)! (\Gamma(n/2 + 1))^4}
\]

where we apply \( \Gamma((n+4)/2) = (n+2)/2 \Gamma((n+2)/2) \).

Using Lemma 2.4, we have

\[
(3.8) \quad \sqrt{\pi} \left( \frac{n}{2e} \right)^{n/2} \sqrt{n + 1/3} < \Gamma(n/2 + 1) < \sqrt{\pi} \left( \frac{n}{2e} \right)^{n/2} \sqrt{n + \beta}
\]

and

\[
(3.9) \quad \sqrt{\pi} \left( \frac{n}{e} \right)^n \sqrt{2n + 1/3} < n! < \sqrt{\pi} \left( \frac{n}{e} \right)^n \sqrt{2n + \beta}.
\]
Applying (3.8) and (3.9), we have

\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} > \frac{\pi \left( \sqrt{\frac{n}{e}} \right)^n \sqrt{2n + \frac{1}{3}}}{2^{2n+1} \left( \sqrt{\pi \left( \frac{n}{2e} \right)^{n/2} \sqrt{n+\beta}} \right)^4} = \frac{(n+1) \left( n + \frac{1}{6} \right)}{(n+\beta)^2}
\]

and

\[
\frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{\pi \left( \sqrt{\frac{n}{e}} \right)^n \sqrt{2n + \beta}}{2^{2n+1} \left( \sqrt{\pi \left( \frac{n}{2e} \right)^{n/2} \sqrt{n+1/3}} \right)^4} = \frac{(n+1) \left( n + \frac{\beta}{2} \right)}{(n + \frac{1}{3})^2}.
\]

The proof of Theorem 3.2 is complete.

Noting simple inequalities

\[
\frac{(n+1) \left( n + \frac{1}{6} \right)}{(n+\beta)^2} > \frac{n+\frac{1}{6}}{n+\beta}
\]

and

\[
\frac{(n+1) \left( n + \frac{\beta}{2} \right)}{(n + \frac{1}{3})^2} < \frac{n+1}{n + \frac{1}{3}},
\]

we get the Corollary 3.1.

**Corollary 3.1.** For every integer \( n \geq 1 \), it holds

\[
\frac{n + \frac{1}{6}}{n + \beta} < \frac{\Omega_n^2}{\Omega_{n-1}\Omega_{n+1}} < \frac{n+1}{n + \frac{1}{3}}.
\]

**Theorem 3.3.** For every integer \( n \geq 1 \), it holds

\[
\frac{\sqrt{\pi}}{\sqrt{2\pi}} \frac{\left( \sqrt{\frac{n+\beta}{2}} \right)^{(2n+1)/(n+1)}}{\sqrt{(n+1) \left( n + \frac{\beta}{2} \right)}} < \frac{\Omega_n}{(\Omega_n)^{n/(n+1)}} \frac{\Gamma((n+1)/2)\Gamma((n+3)/2)}{\Gamma((n+1)/2)} = \pi^{1/2} \Gamma(n+2) = \pi^{1/2} (n+1)!.}
\]

**Proof.** Setting \( z = (n+2)/2 \) in (2.2) of Lemma 2.2, we get

\[
2^{n+1} \Gamma((n+2)/2)\Gamma((n+3)/2) = \pi^{1/2} \Gamma(n+2) = \pi^{1/2} (n+1)!.}
\]

Easy computation and simplification yield

\[
\frac{\Omega_n}{(\Omega_n)^{n/(n+1)}} = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \frac{\Gamma((n+1)/2+1)}{(\pi^{(n+1)/2})^{n/(n+1)}}
\]

\[
= \frac{2^{n+1} (\Gamma((n+1)/2+1))^{n/(n+1)}}{\sqrt{\pi} (n+1)!}.
\]
Similarly to proof of Theorem 3.2, we have

\[
\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} > \frac{2^{n+1} \left( \sqrt{\pi} \left( \frac{n+1}{2e} \right)^{(n+1)/2} \sqrt{n+1 + \frac{1}{2}} \right)^{(2n+1)/(n+1)}}{\sqrt{\pi} \left( \frac{n+1}{e} \right)^{n+1} \sqrt{2n+2 + \frac{1}{3}}}
\]

\[
= \frac{\sqrt{e}}{2^{n/2} \pi} \frac{\left( \sqrt{n+1 + \frac{1}{3}} \right)^{(2n+1)/(n+1)}}{(n+1) \left( n + 1 + \frac{1}{2} \right)}
\]

and

\[
\frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{2^{n+1} \left( \sqrt{\pi} \left( \frac{n+1}{2e} \right)^{(n+1)/2} \sqrt{n+1 + \beta} \right)^{(2n+1)/(n+1)}}{\sqrt{\pi} \left( \frac{n+1}{e} \right)^{n+1} \sqrt{2n+2 + \frac{1}{3}}}
\]

\[
= \frac{\sqrt{e}}{2^{n/2} \pi} \frac{\left( \sqrt{n+1 + \beta} \right)^{(2n+1)/(n+1)}}{(n+1) \left( n + 1 + \frac{7}{6} \right)}
\]

The proof of Theorem 3.3 is complete.

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\[
\frac{\sqrt{n+1 + \beta}}{(n+1) \left( n + \frac{7}{6} \right)} \leq \frac{(n+1 + \beta)^{(2n+2)/(2n+1)}}{(n+1) \left( n + 1 \right)} = \frac{n+1 + \beta}{n+1}
\]

and

\[
\frac{\sqrt{n+1 + \beta}}{(n+1) \left( n + 1 + \frac{1}{2} \right)} \geq \frac{\sqrt{n+1 + \beta}}{(n+1) \left( n + 1 + \frac{7}{6} \right)} \geq \frac{1}{2^{n/2} \sqrt{n+1 + \frac{1}{2}}}
\]

we easily get the Corollary 3.2.

**Corollary 3.2.** For every integer \( n \geq 1 \), we have

\[
\frac{\sqrt{e}}{2^{n+1/2} \sqrt{2n+2}} \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{\Omega_n}{(\Omega_{n+1})^{n/(n+1)}} < \frac{\sqrt{e}}{2^{n+1/2} \sqrt{2n+2}} \frac{n+1 + \beta}{n+1}
\]

Finally, we give a monotone result related to the volume of the unit ball in \( \mathbb{R}^n \).

**Theorem 3.4.** For every integer \( n \geq 3 \), the sequence \( \{ (\Omega_n)^{1/H_n} \}_{n \geq 3} \) is monotonically decreasing to zero, where \( H_n \) denotes the \( n \)-th harmonic number. Further, the sequence \( \{ (\Omega_n)^{1/H_n} \}_{n \geq 1} \) attains its maximum at \( n = 3 \).

**Proof.** By taking the logarithm, we only prove that

\[
\frac{\ln \Omega_n}{H_n} \geq \frac{\ln \Omega_{n+1}}{H_{n+1}}.
\]
For \( n \geq 5 \), using (1.7), we have
\[
\frac{\ln \Omega_n}{H_n} - \frac{\ln \Omega_{n+1}}{H_{n+1}} > \frac{\ln \sqrt{\frac{n+2}{2\pi}}}{H_{n+1}} > 0.
\]
Direct computation can yield
\[
\frac{\ln \Omega_1}{H_1} < \frac{\ln \Omega_2}{H_2} < \frac{\ln \Omega_3}{H_3} > \frac{\ln \Omega_4}{H_4} > \frac{\ln \Omega_5}{H_5}.
\]
Furthermore, by Stolz’s theorem, we get
\[
\lim_{n \to \infty} \left( \frac{\ln \Omega_n}{H_n} \right)^{\frac{1}{n}} = \exp \left\{ \lim_{n \to \infty} \frac{\ln \Omega_n}{H_n} \right\} = \exp \left\{ \lim_{n \to \infty} \frac{\ln \Omega_n - \ln \Omega_{n-1}}{H_n - H_{n-1}} \right\}
\]
\[
= \exp \left\{ \lim_{n \to \infty} n \ln \frac{\Omega_n}{\Omega_{n-1}} \right\} = 0.
\]
The proof of Theorem 3.4 is complete.

**Remark 3.1.** The sequence \((\Omega_n)^{1/H_n}\) can be rearranged as \(\{ [(\Omega_n)^{1/n}]^{1/H_n} \}\). Since \((\Omega_n)^{1/n}\) is decreasing to 0 and \(n/H_n\) can be easily proved to be increasing to \(\infty\), so \(\lim_{n \to \infty} (\Omega_n)^{1/H_n} = 0\) can be proved easily.

**Remark 3.2.** By the well-known software MATHEMATICA Version 7.0.0, we can show that
1. the double inequality (3.3) is better than (1.9),
2. the double inequality (3.13) and (1.5) are not included each other.

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**References**


