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Linearization for Systems with Partially Hyperbolic Linear Part

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Abstract. To study the linearization problem of dynamic system on measure chains (time scales), the authors in the previous work assumed that linear system $x^{\Delta} = A(t)x$ should possess exponential dichotomy. In this paper, the assumption is weakened and the setting on the whole linear part $x^{\Delta} = A(t)x$ need not to be hyperbolic. We only need assume partially hyperbolic linear part. More specifically, if system $x^{\Delta} = A(t)x$ is rewritten as two subsystems $\begin{cases} x_1^{\Delta} = A_1(t)x_1, \\ x_2^{\Delta} = A_2(t)x_2 \end{cases}$, it requires that the first subsystem $x_1^{\Delta} = A_1(t)x_1$ has exponential dichotomy, while there is no requirement on the other linear subsystem $x_2^{\Delta} = A_2(t)x_2$. That is, the whole linear system $x^{\Delta} = A(t)x$ need not to possess exponential dichotomy. The previous result is improved in this paper.

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1. Introduction and motivation

In order to unify discrete and continuous dynamic systems, Hilger [11] introduced the calculus on measure chains in 1990. In all subsequent considerations we deal with a measure chain $(\mathbb{T}, \preceq, \mu)$, i.e. a conditionally complete totally ordered set $(\mathbb{T}, \preceq)$ (see [11], Axiom [2]) with growth calibration $\mu : \mathbb{T}^2 \to \mathbb{R}$ (see [11], Axiom [26]). The most intuitive and relevant examples of measure chains are time scales, a *time scale* is an arbitrary nonempty closed subset of the real numbers. Thus, the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ are examples of time scales, as are $[0,1] \cup [2,3]$ and Cantor set, while the rational numbers $\mathbb{Q}$, the complex numbers $\mathbb{C}$, and the open interval between 0 and 1, are *not* time scales. Recently, the dynamic equations on measure chains have been extensively studied. Many basic results were obtained such as oscillation, initial value problem, bounded value problem, exponential dichotomy and stability theory. For more details, e.g., see the monographs [2] and the references [3, 5, 15, 16, 20, 22, 24, 26–31], [1, 4, 8, 17, 23].

Another interesting field of research is the linearization problem. A great contribution to the linearization problem for autonomous differential equations ($\mathbb{T} = \mathbb{R}$) and autonomous difference equations ($\mathbb{T} = \mathbb{Z}$) is the famous Hartman-Grobman theorem (see Hartman [9, 10]

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and Grobman [6,7]). Then Palmer successfully generalized the standard Hartman-Grobman theorem to non-autonomous cases ($\mathbb{T}=\mathbb{R}$, see Palmer [20]; $\mathbb{T}=\mathbb{Z}$, see Kirchgraber and Palmer [14]). To weaken the conditions of Palmer's linearization theorem, some improved linearization results have been reported in [13, 21].

However, there are few papers considering the topological linearization on measure chains. Hilger [12] managed to generalize the Hartman-Grobman theorem to non-autonomous dynamic equations on measure chains. Hilger proved his interesting result in a very general form. In order to shorten Hilger's proofs and obtain some easily verifiable results, a new analytical method is given in Xia *et al.* [25]. By introducing the concept of *topological equivalence on measure chains*, Xia *et al.* [25] obtained some sufficient conditions which guarantee the existence of a equivalent function $H(t,x)$ sending the solutions of nonlinear system $x^\Delta = A(t)x + f(t,x)$ onto those of linear system $x^\Delta = A(t)x$. Recently, Pötzche [19] successfully extended these results and studied the topological decoupling under parameter variation. It should be noted that the results in Xia *et al.* [25] required that linear system $x^\Delta = A(t)x$ should possess exponential dichotomy. A natural question arises: shall we reduce this assumption? More specifically, if system $x^\Delta = A(t)x$ is rewritten as two subsystems $\begin{cases} x_1^\Delta = A_1(t)x_1, \\ x_2^\Delta = A_2(t)x_2 \end{cases}$, is it possible that the first subsystem $x_1^\Delta = A_1(t)x_1$ is required to possess exponential dichotomy, while there is no requirement of the other linear subsystem $x_2^\Delta = A_2(t)x_2$? Based on this conjecture and motivated by above mentioned works, we revisit the *topological conjugacy* problem on measure chains and weaken the conditions in Xia *et al.* [25].

2. Notations, definitions and lemmas

In this section, we shall introduce some notations and basic terminology on measure chains. For further details, see the pioneering paper [11] and the monograph [2]. Let χ be a $\mathbb{K}$-vector space with the norm $\|\cdot\|$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. $\mathscr{L}(\chi_1,\chi_2)$ stands for the linear space of continuous homomorphisms with the norm $\|T\| := \sup_{\|x\|=1}\|Tx\|$ for any $T \in \mathscr{L}(\chi_1,\chi_2)$; I_{χ_1} is the identity mapping on χ_1. Additionally, write $\mathscr{L}(\chi) := \mathscr{L}(\chi,\chi)$ and $\mathscr{N}(T) = T^{-1}(\{0\})$ is the nullspace and $\mathscr{R}(T) := T\chi$ the range of $T \in \mathscr{L}(\chi)$. We also briefly introduce some notions, which are specific for the calculus on measure chains. In particular, $\mathbb{T}_\tau^+$ and $\mathbb{T}_\tau^-$ are the $\mathbb{T}$-intervals $\{t \in \mathbb{T} : \tau \preceq t\}$ and $\{t \in \mathbb{T} : t \preceq \tau\}$, respectively, for any $\tau \in \mathbb{T}$. $\mathscr{C}_{rd}(\mathbb{T},\chi)$ are the *rd*-continuous mappings from $\mathbb{T}$ into χ and $\mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R}) := \{a \in \mathscr{C}_{rd}(\mathbb{T},\mathbb{R}) : 1+\mu^*(t)a(t) > 0 \text{ for } t \in \mathbb{T}\}$ is the linear space of positively regressive functions with the algebraic operations

$$(a\oplus b)(t) := a(t)+b(t)+\mu^*(t)a(t)b(t),\ \ (\alpha\odot a)(t) := \lim_{h\searrow\mu^*(t)} \frac{(1+ha(t))^\alpha - 1}{h} \text{ for } t\in\mathbb{T}$$

for $a,b \in \mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R})$ and reals $\alpha \in \mathbb{R}$. With fixed $\tau \in \mathbb{T}$ and $c,d \in \mathscr{C}_{rd}^+\mathscr{R}(\mathbb{T},\mathbb{R})$ we define the three linear space

$$\mathscr{B}_{\tau,c}^+(\chi) := \Big\{\lambda \in \mathscr{C}_{rd}(\mathbb{T}_\tau^+,\chi) :\ \sup_{\tau\preceq t}\|\lambda(t)\|e_{\ominus c}(t,\tau) < \infty\Big\},$$

$$\mathscr{B}_{\tau,d}^-(\chi) := \Big\{\lambda \in \mathscr{C}_{rd}(\mathbb{T}_\tau^-,\chi) :\ \sup_{t\preceq\tau}\|\lambda(t)\|e_{\ominus d}(t,\tau) < \infty\Big\},$$

$$\mathscr{B}^{\pm}_{\tau,c,d}(\chi) = \left\{ \lambda \in \mathscr{C}_{rd}(\mathbb{T},\chi) \middle| \exists \tau \in \mathbb{T}: \begin{array}{l} \sup_{\tau \preceq t} \|\lambda(t)\| e_{\ominus c}(t,\tau) < \infty \\ \sup_{t \preceq \tau} \|\lambda(t)\| e_{\ominus d}(t,\tau) < \infty \end{array} \right\}$$

of so-called c^+*-quasibounded* and d^-*-quasibounded* mappings, which are immediately seen to be Banach spaces with regard to the norms

$$\|\lambda\|^{+}_{\tau,c} := \sup_{\tau \preceq t} \|\lambda(t)\| e_{\ominus c}(t,\tau), \ \|\lambda\|^{-}_{\tau,d} := \sup_{t \preceq \tau} \|\lambda(t)\| e_{\ominus d}(t,\tau),$$

$$\|\lambda\|^{\pm}_{\tau,c,d} = \max\{ \|\lambda|_{\mathbb{T}^+_\tau}\|^{+}_{\tau,c}, \ \|\lambda|_{\mathbb{T}^-_\tau}\|^{-}_{\tau,d} \}$$

respectively, where $e_c(t,\tau)$ is the real exponential function on $\mathbb{T}$. Throughout this paper, we use the abbreviation $\lfloor b-a \rfloor := \inf_{t\in\mathbb{T}} (b(t)-a(t))$ and introduce the notations $a \lhd b :\Leftrightarrow 0 < \lfloor b-a \rfloor$, $a \unlhd b :\Leftrightarrow 0 \le \lfloor b-a \rfloor$, where two positively regressive functions $a,b \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$ are denoted as *growth rates*, if $\sup_{t\in\mathbb{T}} \mu^*(t)a(t) < \infty$ and $\sup_{t\in\mathbb{T}} \mu^*(t)b(t) < \infty$, respectively. Then we obtain the limits

$$\lim_{t\to+\infty} e_{a\ominus b}(t,\tau) = 0, \quad \lim_{t\to-\infty} e_{b\ominus a}(t,\tau) = 0,$$

for growth rates $a \lhd b$ and on a measure chain, which is unbounded above resp. below.

Definition 2.1. *(see [11]) A mapping $\phi : \mathbb{T} \to \chi$ is said to be differentiable (in a point $t_0 \in \mathbb{T}$), if there exists a unique derivative $\phi^\Delta(t_0) \in \chi$, such that for any $\varepsilon > 0$ the estimate*

$$\|\phi(\sigma(t_0)) - \phi(t) - \mu(\sigma(t_0),t)\phi^\Delta(t_0)\| \le \varepsilon|\mu(\sigma(t_0),t)| \ \text{ for } t \in U$$

holds in a $\mathbb{T}$-neighborhood U of t_0. The Cauchy integral of ϕ is denoted as $\int_\tau^t \phi(s)\Delta s$ for $\tau,t \in \mathbb{T}$, provided it exists.

Now consider the following system

$$x^\Delta = A(t)x, \tag{2.1}$$

where $A \in \mathscr{C}_{rd}(\mathbb{T},\mathscr{L}(\chi))$, ($\mathscr{L}$. Let $\Phi_A(t,\tau) \in \mathscr{L}(\chi)$ denotes the transition operator of (2.1), i.e. the solution of the corresponding operator-valued initial value problem $X^\Delta = A(t)X, X(\tau) = I$ for $\tau,t \in \mathbb{T}$, $t \succeq \tau$. A projection-valued mapping $P : \mathbb{T} \to \mathscr{L}(\chi)$ is an *invariant projector* of (2.1), if $P(t)\overline{\Phi}_A(t,s) = \overline{\Phi}_A(t,s)P(s)$ is fulfilled for $s \preceq t$, $s,t \in \mathbb{T}$. An invariant projector P is denoted as *regular*, if

$$I_\chi + \mu^*(t)A(t)|_{\mathscr{R}(P(t))}\mathscr{R}(P(t)) \to \mathscr{R}(P(\rho_+(t))) \ \text{ is bijective for all } \ t \in \mathbb{T}.$$

This regularity condition enables us to deal with noninvertible equations. For this reason, Pötzsche [20] defined an *extended transition operator* $\overline{\Phi}_A(t,s) \in \mathscr{L}(\chi)$ as follows:

$$\overline{\Phi}_A(t,s) = \begin{cases} [\Phi_A(s,t)]^{-1}, \text{ for } t \prec s, \\ \Phi_A(t,s)], \text{ for } t \succeq s. \end{cases}$$

Clearly, $[I-P(t)]\overline{\Phi}_A(t,s) = \overline{\Phi}_A(t,s)[I-P(s)]$ and $\overline{\Phi}_A(t,s)^{-1} = \overline{\Phi}_A(s,t)$. Moreover, $\overline{\Phi}_A(t,\tau)\xi$ solves the initial value problem $x^\Delta = A(t)x$, $x(\tau) = \xi$ for $\tau,t \in \mathbb{T}$, $\xi \in \chi$.

Definition 2.2. *(see [20]) Let $P : \mathbb{T} \to \mathscr{L}(\chi)$ be an* regular invariant projector *of (2.1). Then system (2.1) is said to possess an* exponential dichotomy, *if the estimates*

$$\begin{cases} \|\Phi_A(t,s)P(s)\| \le K_1 e_a(t,s) \ \text{ for } s \preceq t, \ s,t \in \mathbb{T}, \\ \|\overline{\Phi}_A(t,s)[I-P(s)]\| \le K_2 e_b(t,s) \ \text{ for } t \preceq s, \ s,t \in \mathbb{T} \end{cases} \tag{2.2}$$

hold for real constants $K_1, K_2 \geq 1$ *and growth rates* $a, b \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$, $a \lhd b$.

Now we introduce the concept of *topological equivalence* on measure chains from [25]. Considering the two dynamic systems

$$x^{\Delta} = f(t,x), \tag{2.3}$$

$$y^{\Delta} = g(t,y). \tag{2.4}$$

Definition 2.3. *(see [25]) A continuous function* $H : \mathbb{T} \times \chi \to \chi$ *is said to be* topologically equivalent *between (2.3) and (2.4), if the following conditions hold:*

(i) *for each fixed* t, $H(t,\cdot)$ *is a homeomorphism of* χ *into* χ;
(ii) $H(t,x) - x$ *is* (c,d)*-quasibounded, uniformly with respect to* t;
(iii) *assume that* $G(t,\cdot) = H^{-1}(t,\cdot)$ *has property ii) also;*
(iv) *if* $x(t)$ *is a solution of system (2.3), then* $H(t,x(t))$ *is a solution of system (2.4); if* $y(t)$ *is a solution of system (2.4), then* $G(t,y(t))$ *is a solution of system (2.3).*

If such a mapping H *exists, then system (2.3) and (2.4) are called* topologically conjugated.

Lemma 2.1. *(see [20, 25]) For* $\tau, t, t_1, t_2 \in \mathbb{T}$, $t_1 \preceq t_2$ *and* $a, b \in \mathscr{C}^+_{rd}\mathscr{R}(\mathbb{T},\mathbb{R})$,

$$\int_{t_1}^{t_2} e_a(t,\rho_+(s)) e_b(s,\tau)\Delta s \leq \begin{cases} \frac{e_a(t,\tau)}{\lfloor b-a \rfloor}[e_{b\ominus a}(t_2,\tau) - e_{b\ominus a}(t_1,\tau)], & \text{if } a \lhd b, \\ \frac{e_a(t,\tau)}{\lfloor a-b \rfloor}[e_{b\ominus a}(t_1,\tau) - e_{b\ominus a}(t_2,\tau)], & \text{if } b \lhd a. \end{cases}$$

Consider the inhomogeneous equation

$$x^{\Delta} = A(t)x + r(t). \tag{2.5}$$

Lemma 2.2. *(see [20, 25]) If linear system (2.1) possesses an exponential dichotomy with the estimates (2.2), then for* $r \in \mathscr{B}^{\pm}_{c,d}(\chi)$, *system (2.5) exists exactly one solution* $\lambda_* \in \mathscr{B}^{\pm}_{c,d}(\chi)$, *which can be written as follows*

$$\lambda_*(t) = \int_{-\infty}^{t} \Phi_A(t,\rho_+(s))P(\rho_+(s))r(s)\Delta s - \int_{t}^{+\infty} \overline{\Phi}_A(t,\rho_+(s))[I_\chi - P(\rho_+(s))]r(s)\Delta s.$$

3. Statement of the main result

Consider the nonautonomous linear system

$$\begin{cases} x_1^{\Delta} = A_1(t)x_1, \\ x_2^{\Delta} = A_2(t)x_2, \end{cases} \tag{3.1}$$

and the nonlinear system

$$\begin{cases} x_1^{\Delta} = A_1(t)x_1 + f(t,x_1,x_2), \\ x_2^{\Delta} = A_2(t)x_2, \end{cases} \tag{3.2}$$

where $x_1 \in \chi^n$, $x_2 \in \chi^m$, $A_1 \in \mathscr{C}_{rd}(\mathbb{T},\mathscr{L}(\chi^n))$, $A_2 \in \mathscr{C}_{rd}(\mathbb{T},\mathscr{L}(\chi^m))$, where n is the dimension of vector space χ^n. Now we are ready to state our main results.

Theorem 3.1. For any $\tau \in \mathbb{T}$, $c,d \in \mathscr{C}_{rd}^{+}(\mathbb{T},\mathbb{R})$, $a \lhd c \lhd b$, $a \lhd d \lhd b$, suppose the following conditions hold:

(H$_1$) The linear subsystem $x_1^{\Delta} = A_1(t)x_1$ has exponential dichotomy on $\mathbb{T}$, that is, $x_1^{\Delta} = A_1(t)x_1$ has a fundamental matrix $\Phi_{A_1}(t)$ satisfying (2.2);

(H$_2$) $f : \mathbb{T} \times \chi^n \times \chi^m \to \chi^{n+m}$ is rd-continuous and there exist positive constants κ and γ such that:

$$\|f(t,x_1,x_2)\|_{\tau,c,d}^{\pm} \le \kappa, \ \ \|f(t,x_1,x_2) - f(t,y_1,y_2)\| \le \gamma[\|x_1 - y_1\| + \|x_2 - y_2\|] \ \text{ for } t \in \mathbb{T};$$

(H$_3$) $\gamma C_2(c,d) < 1$, where $C_2(c,d) = \max\left\{C_1(c) + \frac{K_1}{\lfloor d-a \rfloor},\ C_1(d) + \frac{K_2}{\lfloor b-c \rfloor}\right\} > 0$, $C_1(c) = \frac{K_1}{\lfloor c-a \rfloor} + \frac{K_2}{\lfloor b-c \rfloor} > 0$, $C_1(d) = \frac{K_2}{\lfloor d-a \rfloor} + \frac{K_1}{\lfloor b-d \rfloor} > 0$.

Then we have the following conclusions:

(I) system (3.2) and its linear part (3.1) are topologically conjugated;

(II) the equivalent function $H(t,x)$ $(x = (x_1,x_2)^T)$ satisfies $\|H(t,x) - x\|_{\tau,c,d}^{\pm} \le \kappa C_2(c,d)$, uniformly respect to t;

(III) letting $G(t,\cdot) = H^{-1}(t,\cdot)$, then $G(t,\cdot)$ also has the property (II).

Remark 3.1. Condition (H$_1$) only requires that the first linear subsystem $x_1^{\Delta} = A_1(t)x_1$ has exponential dichotomy. However, there is no requirement of the other linear subsystem $x_2^{\Delta} = A_2(t)x_2$. It is assumed that the whole linear system $x^{\Delta} = A(t)x$ possesses exponential dichotomy in [25]. So we improve the main results in [25].

4. Preliminary results for proof

The proof of Theorem 3.1 is long and complicated. So we divide the proof into seven preliminary results. To prove Theorem 3.1, it is imperative to prove seven propositions in this section. Throughout this section, we always assume that the assumptions in Theorem 3.1 are satisfied.

Assume that $\Phi_{A_1}(t,\tau)$ denotes a fundamental matrix of $x_1^{\Delta} = A_1(t)x$, $\begin{bmatrix} X_1(t,\tau,x_{10},x_{20}) \\ X_2(t,\tau,x_{10},x_{20}) \end{bmatrix}$ is a solution of system (3.2) satisfying the initial condition $\begin{bmatrix} X_1(\tau) \\ X_2(\tau) \end{bmatrix} = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}$, and $\begin{bmatrix} Y_1(t,\tau,y_{10},y_{20}) \\ Y_2(t,\tau,y_{10},y_{20}) \end{bmatrix}$ is a solution of (3.1) satisfying initial condition $\begin{bmatrix} Y_1(\tau) \\ Y_2(\tau) \end{bmatrix} = \begin{bmatrix} y_{10} \\ y_{20} \end{bmatrix}$.

Proposition 4.1. *For any fixed (σ,ξ,η), it follows that system*

$$z^{\Delta} = A_1(t)z - f\big(t,X_1(t,\sigma,\xi,\eta),X_2(t,\sigma,\xi,\eta)\big) \tag{4.1}$$

has a unique (c,d)-quasibounded solution $h(t,(\sigma,\xi,\eta))$ and it satisfies

$$\|h(\cdot,(\sigma,\xi,\eta))\|_{\tau,c,d}^{\pm} \le \kappa C_2(c,d),$$

where $C_2(c,d)$ defined in Theorem 3.1.

Proof. For any fixed (σ,ξ,η), define

$$\begin{aligned} h(t,(\sigma,\xi,\eta)) = & -\int_{-\infty}^{t} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\Delta s \\ & + \int_{t}^{+\infty} \overline{\Phi}_{A_1}(t,\rho_+(s))[I_\chi - P(\rho_+(s))]f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\Delta s. \end{aligned} \tag{4.2}$$

By direct differentiation, it is easily shown that $h(t,(\sigma,\xi,\eta))$ is a solution of (4.1). It follows from (4.2) and condition (H_2) that

$$\begin{aligned}\|h(t,(\sigma,\xi,\eta))\| &\le K_1\int_{-\infty}^{t} e_a(t,\rho_+(s))\|f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\|\Delta s\\ &\quad + K_2\int_{t}^{+\infty} e_b(t,\rho_+(s))\|f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\|\Delta s.\end{aligned} \tag{4.3}$$

For fixed $\tau\in\mathbb{T}$, without losing of generality, first we consider (4.3) on $\mathbb{T}_\tau^+$, we have

$$\begin{aligned}\|h(t,(\sigma,\xi,\eta))\| &\le K_1\int_{-\infty}^{\tau} e_a(t,\rho_+(s))\|f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\|\Delta s\\ &\quad + K_1\int_{\tau}^{t} e_a(t,\rho_+(s))\|f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\|\Delta s\\ &\quad + K_2\int_{t}^{-\infty} e_b(t,\rho_+(s))\|f\big(s,X_1(s,\sigma,\xi,\eta),X_2(s,\sigma,\xi,\eta)\big)\|\Delta s\\ &\le K_1\int_{+\infty}^{\tau} e_a(t,\rho_+(s))e_d(s,\tau)\Delta s\|f\big(\cdot,X_1(\cdot,\sigma,\xi,\eta),X_2(\cdot,\sigma,\xi,\eta)\big)\|_{\tau,d}^-\\ &\quad + K_1\int_{\tau}^{t} e_a(t,\rho_+(s))e_c(s,\tau)\Delta s\|f\big(\cdot,X_1(\cdot,\sigma,\xi,\eta),X_2(\cdot,\sigma,\xi,\eta)\big)\|_{\tau,c}^+\\ &\quad + K_2\int_{t}^{+\infty} e_b(t,\rho_+(s))e_c(s,\tau)\Delta s\|f\big(\cdot,X_1(\cdot,\sigma,\xi,\eta),X_2(\cdot,\sigma,\xi,\eta)\big)\|_{\tau,c}^+\\ &\le \Big[K_1\int_{-\infty}^{\tau} e_a(t,\rho_+(s))e_d(s,\tau)\Delta s + K_1\int_{\tau}^{t} e_a(t,\rho_+(s))e_c(s,\tau)\Delta s\\ &\quad + K_2\int_{t}^{+\infty} e_b(t,\rho_+(s))e_c(s,\tau)\Delta s\Big]\|f\big(\cdot,X_1(\cdot,\sigma,\xi,\eta),X_2(\cdot,\sigma,\xi,\eta)\big)\|_{\tau,c,d}^{\pm}\\ &\le \Big[\Big(\frac{K_1}{\lfloor d-a\rfloor}-\frac{K_2}{\lfloor c-a\rfloor}\Big)e_a(t,\tau)+C_1(c)e_c(t,\tau)\Big]\kappa\\ &\le \Big[\Big(C_1(c)+\frac{K_1}{\lfloor d-a\rfloor}\Big)e_c(t,\tau)-\frac{K_1}{\lfloor c-a\rfloor}e_a(t,\tau)\Big]\kappa \text{ for all } \tau\preceq t,\end{aligned} \tag{4.4}$$

which implies

$$\begin{aligned}\|h(t,(\sigma,\xi,\eta))\|e_{\ominus c}(t,\tau) &\le \Big[\Big(C_1(c)+\frac{K_1}{\lfloor d-a\rfloor}\Big)-\frac{K_2}{\lfloor c-a\rfloor}e_{a\ominus c}(t,\tau)\Big]\kappa\\ &\le \kappa C_2(c,d)\ \text{ for all }\ \tau\preceq t.\end{aligned} \tag{4.5}$$

Now we consider (4.3) on $\mathbb{T}_\tau^-$, similar to the above discussion, we have

$$\|h(t,(\sigma,\xi,\eta))\| \le \kappa\Big[\Big(C_1(d)+\frac{K_2}{\lfloor b-c\rfloor}\Big)-\frac{K_2}{\lfloor b-d\rfloor}e_b(t,\tau)\Big] \text{ for all } t\in\mathbb{T}_\tau^-$$

which implies

$$\begin{aligned}\|h(t,(\sigma,\xi,\eta))\|e_{\ominus d}(t,\tau) &\le \kappa\Big[\Big(C_1(d)+\frac{K_2}{\lfloor b-c\rfloor}\Big)-\frac{K_2}{\lfloor b-d\rfloor}e_{b\ominus d}(t,\tau)\Big]\kappa\\ &\le \kappa C_2(c,d)\ \text{ for all }\ t\preceq\tau.\end{aligned} \tag{4.6}$$

Take the supremum, it follows from (4.5) and (4.6) that

$$\|h(\cdot,(\sigma,\xi,\eta))\|_{\tau,c,d}^{\pm} \le \kappa C_2(c,d)\ \text{ for all } t\in\mathbb{T}.$$

This shows that $h(t,(\sigma,\xi,\eta))$ is a (c,d)-quasibounded solution of (4.1). ∎

Proposition 4.2. *For any fixed (σ,ξ,η), system*

$$z^{\Delta}=A_1(t)z+f\big(t,Y_1(t,\sigma,\xi,\eta)+z,Y_2(t,\sigma,\xi,\eta)\big) \tag{4.7}$$

has a unique (c,d)-quasibounded solution $g(t,(\sigma,\xi,\eta))$ with $\|g(\cdot,(\sigma,\xi,\eta))\|^{\pm}_{\tau,c,d}\le\kappa C_2(c,d)$.

Proof. For convenience, in what follows, $f\big(s,Y_1(s,\sigma,\xi,\eta)+z(s),Y_2(s,\sigma,\xi,\eta)\big)$ is briefly denoted by $f^z_Y(s)$. Let

$B=\{z(t)|\ z(t)$ be a (c,d)-quasibounded function with $\|z\|^{\pm}_{\tau,c,d}\le\kappa C_2(c,d)\}$.

For any $z\in B$, we define map $\mathscr{T}$ as follows

$$\mathscr{T}z(t)=\int_{-\infty}^{t}\Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f^z_Y(s)-\int_t^{+\infty}\overline{\Phi}_{A_1}(t,\rho_+(s))[I_\chi-P(\rho_+(s))]f^z_Y(s)\Delta s. \tag{4.8}$$

Similar to the computation in Proposition 4.1, it is easy to derive that $\|\mathscr{T}z\|^{\pm}_{\tau,c,d}\le\kappa C_2(c,d)$ for all $t\in\mathbb{T}$. Therefore, $\mathscr{T}$ is a self mapping, i.e., $\mathscr{T}:B\to B$.

Now we shall show that $\mathscr{T}$ is a contraction mapping. In fact, for any $z(t)$, $\widetilde{z}(t)\in B$, it follows from (4.8) and condition (H_2) that

$$\begin{aligned}
&\|\mathscr{T}z(t)-\mathscr{T}\widetilde{z}(t)\|\\
&\le K_1\int_{-\infty}^{t}e_a(t,\rho_+(s))\|f^z_Y(s)-f^{\widetilde{z}}_Y(s)\|\Delta s+K_2\int_t^{+\infty}e_b(t,\rho_+(s))\|f^z_Y(s)-f^{\widetilde{z}}_Y(s)\|\Delta s\\
&\le K_1\int_{-\infty}^{t}e_a(t,\rho_+(s))\gamma\|z(s)-\widetilde{z}(s)\|\Delta s+K_2\int_t^{+\infty}e_b(t,\rho_+(s))\gamma\|z(s)-\widetilde{z}(s)\|\Delta s.
\end{aligned} \tag{4.9}$$

Similar to the calculation of (4.5) and (4.6), it follows from (4.9) that

$$\|\mathscr{T}z-\mathscr{T}\widetilde{z}\|^{\pm}_{\tau,c,d}\le\gamma C_2(c,d)\|z-\widetilde{z}\|^{\pm}_{\tau,c,d}\ \text{ for all }\ t\in\mathbb{T}.$$

Condition (H_3) implies $\gamma C_2(c,d)<1$. Thus the map $\mathscr{T}$ has a unique fixed point $z_0(t)$, that is, $z_0(t)$ satisfies the following

$$\begin{aligned}
z_0(t)=&\int_{-\infty}^{t}\Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f^{z_0}_Y(s)\Delta s\\
&-\int_t^{+\infty}\overline{\Phi}_{A_1}(t,\rho_+(s))[I_\chi-P(\rho_+(s))]f^{z_0}_Y(s)\Delta s.
\end{aligned} \tag{4.10}$$

By direct differentiating on (4.10), it is not difficult to show that $z_0(t)$ is a solution of (4.7). Furthermore, $z_0(t)$ is (c,d)-quasibounded solution of (4.7) with $\|z_0\|^{\pm}_{\tau,c,d}\le\kappa C_2(c,d)$.

Now we are going to show that the (c,d)-quasibounded solution $z_0(t)$ is unique. For this purpose, we assume that there is another (c,d)-quasibounded solution $z_1(t)$ of (4.7). For any $z_1(\tau)=z_0$, by variation of constants, $z_1(t)$ can be written as follows

$$z_1(t)=\Phi_{A_1}(t,\tau)z_0+\int_\tau^t\Phi_{A_1}(t,\rho_+(s))f^{z_1}_Y(s)\Delta s, \tag{4.11}$$

where $f^{z_1}_Y(s)=f\big(s,Y_1(s,\sigma,\xi,\eta)+z_1(s),Y_2(s,\sigma,\xi,\eta)\big)$. By using $I=P+(I-P)$, it follows from (4.11) that

$$z_1(t)=\Phi_{A_1}(t,\tau)z_0+\int_\tau^t\Phi_{A_1}(t,\rho_+(s))\Big[P(\rho_+(s))+(I-P(\rho_+(s)))\Big]f^{z_1}_Y(s)\Delta s$$

$$
\begin{aligned}
&= \Phi_{A_1}(t,\tau)z_0 + \int_{-\infty}^{t} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s \\
&\quad - \int_{-\infty}^{\tau} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s \\
&\quad + \int_{\tau}^{+\infty} \Phi_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s \\
&\quad - \int_{t}^{+\infty} \Phi_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s \\
&= \Phi_{A_1}(t,\tau)z_0 - \Phi_{A_1}(t,\tau)\int_{-\infty}^{\tau} \Phi_{A_1}(\tau,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s \\
&\quad + \Phi_{A_1}(t,\tau)\int_{\tau}^{+\infty} \Phi_{A_1}(\tau,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s \\
&\quad + \int_{-\infty}^{t} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s \\
&\quad - \int_{t}^{+\infty} \Phi_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s. \qquad (4.12)
\end{aligned}
$$

Set

$$
\begin{aligned}
z_0^1 &:= \int_{-\infty}^{\tau} \Phi_{A_1}(\tau,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s, \\
z_0^2 &:= \int_{\tau}^{+\infty} \Phi_{A_1}(\tau,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s.
\end{aligned}
$$

It is easy to prove that z_0^1 and z_0^2 are convergent. In fact, simple computation shows that

$$
\|z_0^1\|_{\tau,c,d}^{+} \le \frac{\kappa K_1}{\lfloor c-a \rfloor}, \text{ and } \|z_0^2\|_{\tau,c,d}^{+} \le \frac{\kappa K_2}{\lfloor b-d \rfloor}.
$$

Thus, (4.12) can be rewritten as

$$
\begin{aligned}
z_1(t) &= \Phi_{A_1}(t,\tau)[z_0+z_0^1+z_0^2] + \int_{-\infty}^{t} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s \\
&\quad - \int_{t}^{+\infty} \Phi_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s. \qquad (4.13)
\end{aligned}
$$

Now we observe that the left side of (4.13) (i.e. $z_1(t)$) is (c,d)-quasibounded due to our assumption; the sum between the second term and the third term on the right side of (4.13) is also (c,d)-quasibounded by similar computation to $h(t,(\sigma,\xi,\eta))$ in (4.2) (less than $C_2(c,d)$). So a consequent conclusion is that the first term on the right side of (4.13) $\Phi_{A_1}(t,\tau)[z_0+z_0^1+z_0^2]$ is (c,d)-quasibounded. It is easy to see that $\Phi_{A_1}(t,\tau)[z_0+z_0^1+z_0^2]$ is a solution of subsystem $x_1^\Delta = A_1(t)x_1$. Notice that the linear system $x_1^\Delta = A_1(t)x_1$ has no non-trivial (c,d)-quasibounded solution. Therefore, $\Phi_{A_1}(t,\tau)[z_0+z_0^1+z_0^2] \equiv 0$. Consequently, it follows from (4.13) that the another (c,d)-quasibounded solution $z_1(t)$ can be uniquely represented as

(4.14)

$$
z_1(t) = \int_{-\infty}^{t} \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))f_Y^{z_1}(s)\Delta s - \int_{t}^{+\infty} \overline{\Phi}_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))]f_Y^{z_1}(s)\Delta s.
$$

It follows from condition (H$_1$)-(H$_2$), (4.10) and (4.14) that

$$
\|z_1(t)-z_0(t)\| \le K_1 \int_{-\infty}^{t} e_a(t,\rho_+(s))\gamma\|z_1(s)-z_0(s)\|\Delta s
$$

$$+K_2\int_t^{+\infty} e_b(t,\rho_+(s))\gamma\|z_1(s)-z_0(s)\|\Delta s.$$

Similar to the calculation of (4.5) and (4.6), we have

$$\|z_1-z_0\|^{\pm}_{\tau,c,d}\le \gamma C_2(c,d)\|z_1-z_0\|^{\pm}_{\tau,c,d}.$$

Because of $\gamma C_2(c,d)<1$, $z_1(t)\equiv z_0(t)$. This implies that the (c,d)-quasibounded solution of (4.7) is unique, which is of course dependent on (σ,ξ,η). We may name it $g(t,(\sigma,\xi,\eta))$. From the above proof, it is easy to see that $\|g(\cdot,(\sigma,\xi,\eta))\|^{\pm}_{\tau,c,d}\le \kappa C_2(c,d)$. ∎

Proposition 4.3. *Let $x(t)=(x_1(t),x_2(t))^T$ be any solution of system (3.1). Then system*

$$z^{\Delta}=A_1(t)z+f\big(t,x_1(t)+z,x_2(t)\big)-f\big(t,x_1(t),x_2(t)\big), \tag{4.15}$$

has a unique (c,d)-quasibounded solution $z\equiv 0$.

Proof. Obviously, $z\equiv 0$ is a (c,d)-quasibounded solution of (4.15). Now we shall show that the (c,d)-quasibounded solution is unique. Or else, if there is another (c,d)-quasibounded solution $z_1(t)$. Then by similar arguments to (4.14), $z_1(t)$ can be uniquely represented as follows

$$\begin{aligned} z_1(t)=&\int_{-\infty}^t \Phi_{A_1}(t,\rho_+(s))P(\rho_+(s))[f\big(s,x_1(s)+z_1(s),x_2(s)\big)-f\big(s,x_1(s),x_2(s)\big)]\Delta s\\ &-\int_t^{+\infty}\overline{\Phi}_{A_1}(t,\rho_+(s))[I-P(\rho_+(s))][f\big(s,x_1(s)+z_1(s),x_2(s)\big)-f\big(s,x_1(s),x_2(s)\big)]\Delta s. \end{aligned}$$

By condition (H_1)-(H_2), we have

$$\|z_1(t)\|\le K_1\int_{-\infty}^t e_a(t,\rho_+(s))\gamma\|z_1(s)\|\Delta s+K_2\int_t^{+\infty} e_b(t,\rho_+(s))\gamma\|z_1(s)\|\Delta s. \tag{4.16}$$

Similar to the calculation of (4.5) and (4.6), it follows from (4.16) that

$$\|z_1\|^{\pm}_{\tau,c,d}\le \gamma C_2(c,d)\|z_1\|^{\pm}_{\tau,c,d}\ \text{ for all }\ t\in\mathbb{T}.$$

Since $\gamma C_2(c,d)<1$, $z_1(t)\equiv 0$. This completes the proof of Proposition 4.3.

Now we introduce two functions as follows

$$H(t,x)=\begin{bmatrix} H_1(t,x_1,x_2)\\ H_2(t,x_1,x_2)\end{bmatrix}=\begin{bmatrix} x_1+h(t,(t,x_1,x_2))\\ x_2\end{bmatrix},\ x=\begin{bmatrix} x_1\\ x_2\end{bmatrix}, \tag{4.17}$$

$$G(t,y)=\begin{bmatrix} G_1(t,y_1,y_2)\\ G_2(t,y_1,y_2)\end{bmatrix}=\begin{bmatrix} y_1+g(t,(t,y_1,y_2))\\ y_2\end{bmatrix},\ y=\begin{bmatrix} y_1\\ y_2\end{bmatrix}. \tag{4.18}$$ ∎

Proposition 4.4. *For any fixed (τ,x_{10},x_{20}),*

$$\begin{bmatrix} H_1\big(t,X_1(t,\tau,x_{10},x_{20}),X_2(t,\tau,x_{10},x_{20})\big)\\ H_2\big(t,X_1(t,\tau,x_{10},x_{20}),X_2(t,\tau,x_{10},x_{20})\big)\end{bmatrix}$$

is a solution of linear system (3.1).

Proof. To prove Proposition 4.4, we substitute $\big(t,X_1(t,\sigma,\xi,\eta),X_2(t,\sigma,\xi,\eta)\big)$ into (σ,ξ,η) of (4.1) in Proposition 4.1. Since system (4.1) is not changed, the (c,d)-quasibounded solution of system (4.1) is unique. Therefore, by the uniqueness, we have

$$h\big(t,(t,X_1(t,\tau,x_{10},x_{20}),X_2(t,\tau,x_{10},x_{20}))\big)=h(t,(\tau,x_{10},x_{20})).$$

Then, it follows from (4.17) that

$$
\begin{aligned}
& H_1\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big) \\
& = X_1(t,\tau,x_{10},x_{20}) + h\Big(t, \big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big)\Big) \\
& = X_1(t,\tau,x_{10},x_{20}) + h(t,(\tau,x_{10},x_{20})). \qquad (4.19)
\end{aligned}
$$

For convenience, $H_1\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big)$ is denoted briefly by $H_1(t)$. Note that $\begin{bmatrix} X_1(t,\tau,x_{10},x_{20}) \\ X_2(t,\tau,x_{10},x_{20}) \end{bmatrix}$, $h\big(t,(\tau,x_{10},x_{20})\big)$ are solutions of system (3.2) and (4.1), respectively. Differentiating on (4.19), we obtain

$$
\begin{aligned}
H_1^{\Delta}(t) &= A_1(t)X_1(t,\tau,x_{10},x_{20}) + f\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big) \\
&\quad + A_1(t)h(t,(\tau,x_{10},x_{20})) - f\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big) \\
&= A_1(t)H_1(t),
\end{aligned}
$$

which implies that $H_1(t)$ is a solution of $x_1^{\Delta}(t) = A_1(t)x_1(t)$.

On the other hand, let $\Phi_{A_2}(t)$ be a fundamental matrix of $x_2^{\Delta} = A_2(t)x_2$, and then

$$
H_2\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big) = X_2(t,\tau,x_{10},x_{20}) = \Phi_{A_2}(t,\tau)x_{20}. \qquad (4.20)
$$

For convenience, $H_2\big(t, X_1(t,\tau,x_{10},x_{20}), X_2(t,\tau,x_{10},x_{20})\big)$ is denoted briefly by $H_2(t)$. Differentiating on (4.20), we deduce that

$$
H_2^{\Delta}(t) = A_2(t)\Phi_{A_2}(t,\tau)x_{20} = A_2(t)H_2(t).
$$

which implies that $H_2(t)$ is a solution of $x_2^{\Delta}(t) = A_2(t)x_2(t)$. Thus, $H(t) = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix}$ is a solution of linear system (3.1) $\begin{cases} x_1^{\Delta} = A_1(t)x_1 \\ x_2^{\Delta} = A_2(t)x_2 \end{cases}$. ∎

Proposition 4.5. *For any fixed* (τ, y_{10}, y_{20}),

$$
\begin{bmatrix} G_1\big(t, Y_1(t,\tau,y_{10},y_{20}), Y_2(t,\tau,y_{10},y_{20})\big) \\ G_2\big(t, Y_1(t,\tau,y_{10},y_{20}), Y_2(t,\tau,y_{10},y_{20})\big) \end{bmatrix}
$$

is a solution of nonlinear system (3.2).

The proof of Proposition 4.5 is similar to that of Proposition 4.4, we omit it here.

Proposition 4.6. *For any fixed* $t \in \mathbb{T}$, $y_1 \in \chi^n$, $y_2 \in \chi^m$, *then following equality always holds:*

$$
\begin{bmatrix} H_1\big(t, G_1(t,y_1,y_2), G_2(t,y_1,y_2)\big) \\ H_2\big(t, G_1(t,y_1,y_2), G_2(t,y_1,y_2)\big) \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
$$

Proof. Let $y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$ be any solution of linear system (3.1). From Proposition 4.5, we conclude that $G(t,y(t)) = \begin{bmatrix} G_1\big(t, y_1(t), y_2(t)\big) \\ G_2\big(t, y_1(t), y_2(t)\big) \end{bmatrix}$ is a solution of nonlinear system (3.2). On the other hand, in view of Proposition 4.5, it is easy to see that

$$
H(t, G(t,y(t))) = \begin{bmatrix} H_1\big(t, G_1(t,y_1(t),y_2(t)), G_2(t,y_1(t),y_2(t))\big) \\ H_2\big(t, G_1(t,y_1(t),y_2(t)), G_2(t,y_1(t),y_2(t))\big) \end{bmatrix}
$$

is another solution of linear system (3.1). For the sake of convenience, denote this solution $H(t,G(t,y(t)))$ as $\bar{y}(t)=\begin{bmatrix}\bar{y}_1(t)\\ \bar{y}_2(t)\end{bmatrix}$. Let

$$I(t)=\begin{bmatrix}I_1(t)\\ I_2(t)\end{bmatrix}=\bar{y}(t)-y(t)=\begin{bmatrix}\bar{y}_1(t)-y_1(t)\\ \bar{y}_2(t)-y_2(t)\end{bmatrix}.$$

To prove this proposition, we need show that $I(t)\equiv 0$. To this end, it suffices to prove that $I_1(t)\equiv 0$ and $I_2(t)\equiv 0$. Firstly we show that $I_1(t)=\bar{y}_1(t)-y_1(t)=0$. In fact, differentiating it, we have

$$I_1^{\Delta}(t)=\bar{y}_1^{\Delta}(t)-y_1^{\Delta}(t)=A_1(t)\bar{y}(t)-A_1(t)y(t)=A_1(t)I_1(t),$$

which implies $I_1(t)$ is a solution of linear subsystem $x_1^{\Delta}(t)=A_1(t)x_1(t)$. Moreover, it follows from (4.17) and (4.18) that

$$\begin{aligned}\|I_1(t)\| &= \|\bar{y}_1(t)-y_1(t)\|\\ &= \|H_1\big(t,G_1(t,y_1(t),y_2(t)),G_2(t,y_1(t),y_2(t))\big)-y_1(t)\|\\ &\le \|H_1\big(t,G_1(t,y_1(t),y_2(t)),G_2(t,y_1(t),y_2(t))\big)-G_1\big(t,y_1(t),y_2(t)\big)\|\\ &\quad +\|G_1\big(t,y_1(t),y_2(t)\big)-y_1(t)\|\\ &= \|h\big(t,(t,G_1(t,y_1(t),y_2(t)),G_2(t,y_1(t),y_2(t)))\big)\|\\ &\quad +\|g\big(t,(t,y_1(t),y_2(t))\big)\|.\end{aligned}\tag{4.21}$$

By using Proposition 4.1 and 4.2, from (4.18), we have

$$\begin{aligned}\|I_1\|^{\pm}_{\tau,c,d} &\le \|h\big(\cdot,(\cdot,G_1(\cdot,y_1(\cdot),y_2(\cdot)),G_2(\cdot,y_1(\cdot),y_2(\cdot)))\big)\|^{\pm}_{\tau,c,d}\\ &\quad +\|g\big(\cdot,(\cdot,y_1(\cdot),y_2(\cdot))\big)\|^{\pm}_{\tau,c,d}\\ &\le 2\kappa C_2(c,d)+2\kappa C_2(c,d)=4\kappa C_2(c,d).\end{aligned}$$

This means $I_1(t)$ is a (c,d)-quasibounded solution of linear subsystem $x_1^{\Delta}=A_1(t)x_1$, but $x^{\Delta}=A(t)x$ has not nontrivial (c,d)-quasibounded solution on $\mathbb{T}$. Therefore $I_1(t)\equiv 0$, i.e., $\bar{y}_1(t)=y_1(t)$.

Now we show that $I_2(t)\equiv 0$. In fact, by the second equality of (4.17), we see that

$$\bar{y}_2(t)=H_2\big(t,G_1(t,y_1(t),y_2(t)),G_2(t,y_1(t),y_2(t))\big)=G_2(t,y_1(t),y_2(t))=y_2(t).$$

Thus, $I(t)\equiv 0$, that is,

$$\bar{y}(t)=\begin{bmatrix}\bar{y}_1(t)\\ \bar{y}_2(t)\end{bmatrix}=\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=y(t),\ \text{ or } H(t,G(t,y(t)))\equiv y(t).$$

Since $y(t)$ is an arbitary solution of linear system (3.1), the proof of Proposition 4.6. ∎

Proposition 4.7. *For any fixed $t\in\mathbb{T}$, $x_1\in\chi^n$, $x_2\in\chi^m$, then following equality always holds:*

$$\begin{bmatrix}G_1\big(t,H_1(t,x_1,x_2),H_2(t,x_1,x_2)\big)\\ G_2\big(t,H_1(t,x_1,x_2),H_2(t,x_1,x_2)\big)\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

The proof is similar to Proposition 4.6, we omit here.

5. Proof of main results

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Note that $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $H(t,x) = \begin{bmatrix} H_1(t,x) \\ H_2(t,x) \end{bmatrix}$, $G(t,y) = \begin{bmatrix} G_1(t,y) \\ G_2(t,y) \end{bmatrix}$. To prove $H(t,x)$ is a equivalent function of linear system (3.1) into nonlinear system (3.2), we are going to show that $H(t,\cdot)$ and $G(t,\cdot)$ satisfy the four conditions of Definition 2.3.

Proof of condition (i): For any fixed $t \in \mathbb{T}$, it follows from Propositions 4.6 and 4.7 that $H(t,\cdot)$ is a bijective mapping of χ^{n+m} into itself and $H^{-1}(t,\cdot) = G(t,\cdot)$.

Proof of condition (ii): From (4.17) and Proposition 4.1, it is not difficult to derive that $\|H(t,x)-x\|^{\pm}_{\tau,c,d} = \|h(t,(t,x))\|^{\pm}_{\tau,c,d} \le \kappa C_2(c,d)$, $\kappa C_2(c,d)$ is a constant.

Proof of condition (iii): From (4.18) and Proposition 4.2, it is not difficult to derive that $\|G(t,y)-y\|^{\pm}_{\tau,c,d} = \|g(t,(t,y))\|^{\pm}_{\tau,c,d} \le \kappa C_2(c,d)$.

Using Propositions 4.4 and 4.5, it is easy to show that condition (iv) is also true.

This completes the proof of Theorem 3.1.

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