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# Topological Solitons and Bifurcation Analysis of the PHI-Four Equation

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**Abstract.** This paper studies the PHI-four equation that arises in Quantum Mechanics. The topological 1-soliton solution or kink solution is obtained by the ansatz method. The bifurcation analysis is then subsequently carried out and several other solutions are retrieved from the analysis. These solutions include the solitary wave solutions, periodic waves and periodic singular waves. The constraint conditions also fall out from the analysis that must exist in order for the soliton solutions to exist. Thus various previous list of solutions for this equation are expanded.

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## 1. Introduction

The PHI-four equation is a very important nonlinear evolution equation (NLEE) in the area of Mathematical Physics, in particular Quantum Mechanics. This equation was studied extensively by several Mathematical Physicists across the globe. It is about time to take a look at this equation from a different perspective in order to extract several other solutions. In order to stay focussed, this paper will concentrate on the ansatz method and the bifurcation analysis to reveal the several other solutions. The integrability studies of these NLEEs is a big deal in this area of Physics and Mathematics [1-25]. However, one must exercise extreme caution in carrying out the integration of these NLEEs as pointed out in 2009 [4]. Without this cautionary approach, the results would be flawed.

The ansatz approach will be first used to carry out the integration of the PHI-four equation. This will reveal a topological 1-solition solution that is also known as the kink solution. This will lead to a couplre of constraint conditions that must remain valid in order for the kink solution to exist. Subsequently, the paper will address the bifurcation analysis of the problem where the phase portraits of this equation of study will be obtained. Additionally by the traveling wave approach, several other solutions will be obtained. They are the

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cnoidal waves, snoidal waves, solitary waves, periodic waves, singular periodic waves and others.

## 2. Topological 1-Soliton Solution Or Kink Solution

The PHI-four equation that is going to be studied in this paper is given by

$$u_{tt} - k^2 u_{xx} = au + bu^3 \tag{2.1}$$

where in (2.1), the dependent variable is $u(x,t)$ while the spatial and temporal independent variables are $x$ and $t$ respectively. The other parameters $k$, $a$ and $b$ are all real-valued constants. In order to extract the topological 1-soliton solution of this equation, it is necessary to bear in mind that the solitons are the outcome of a delicate balance between dispersion and nonlinearity. This leads to the balancing principle that will be applied to obtain the soliton solution. In order to get started, the 1-soliton solution ansatz is taken to be [6–12, 21]:

$$u(x,t) = A \tanh^p \tau \tag{2.2}$$

where

$$\tau = B(x - vt) \tag{2.3}$$

Here in (2.2) and (2.3), the parameters $A$ and $B$ are known as free parameters of the soliton or the kink and $v$ is the velocity of the soliton. The value of the unknown exponent $p$ will fall out during the course of derivation of the soliton solution. Substituting (2.2) into (2.1) and simplifying leads to

$$\begin{aligned} &p(p-1)AB^2\left(v^2-k^2\right)\tanh^{p-2}\tau - 2p^2AB^2\left(v^2-k^2\right)\tanh^p\tau \\ &+ p(p+1)AB^2\left(v^2-k^2\right)\tanh^{p+2}\tau = aA\tanh^p\tau + bA^3\tanh^{3p}\tau \end{aligned} \tag{2.4}$$

By the balancing principle, equation the exponents $3p$ and $p+2$, gives

$$3p = p + 2 \tag{2.5}$$

which gives

$$p = 1 \tag{2.6}$$

This shows that the first term on the left hand side gets knocked off. From the remaining terms, the linearly independent functions are $\tanh^{p+j}\tau$ for $j = 0, 2$. Therefore, setting its respective coefficients to zero, yields

$$A = \sqrt{-\frac{a}{b}} \tag{2.7}$$

and

$$B = \sqrt{-\frac{a}{2\left(v^2-k^2\right)}} \tag{2.8}$$

These values of the free parameters immediately pose the constraint conditions

$$ab < 0 \tag{2.9}$$

and

$$a\left(v^2-k^2\right) < 0 \tag{2.10}$$

respectively. Thus, finally, the 1-soliton solution to the PHI-four equation is given by

$$u(x,t) = A\tanh[B(x-vt)] \tag{2.11}$$

with the free parameters given by (2.7) and (2.8). This kink solution will hold as long as the constraint conditions given by (2.9) and (2.10) remains valid.

## 3. Bifurcation analysis

In this section, the Phi-four equation will be rewritten as

$$u_{tt} - \alpha u_{xx} - \lambda u + \beta u^3 = 0. \tag{3.1}$$

In this section, the aim is to study the traveling wave solutions and their relations for Eq. (3.1) by using the bifurcation method and qualitative theory of dynamical systems [15–20]. Through some special phase orbits, we obtain many smooth periodic wave solutions and periodic blow-up solutions. Their limits contain kink profile solitary wave solutions, unbounded wave solutions, periodic blow-up solutions and solitary wave solutions.

### 3.1. Phase portraits and qualitative analysis

We assume that the traveling wave solutions of (3.1) is of the form

$$u(x,t) = \varphi(\xi), \quad \xi = x - ct, \tag{3.2}$$

we have

$$(c^2-\alpha)\varphi'' - \lambda\varphi + \beta\varphi^3 = 0. \tag{3.3}$$

To relate conveniently, let

$$\eta = \frac{\beta}{c^2-\alpha}, \tag{3.4}$$

and

$$\mu = \frac{\lambda}{c^2-\alpha}. \tag{3.5}$$

Letting $\varphi' = y$, then we get the following planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = -\eta\varphi^3 + \mu\varphi. \end{cases} \tag{3.6}$$

Obviously, the above system (3.6) is a Hamiltonian system with Hamiltonian function

$$H(\varphi, y) = y^2 + \frac{1}{2}\eta\varphi^4 - \mu\varphi^2. \tag{3.7}$$

In order to investigate the phase portrait of (3.6), set

$$f(\varphi) = -\eta\varphi^3 + \mu\varphi. \tag{3.8}$$

Obviously, $f(\varphi)$ has three zero points, $\varphi_-$, $\varphi_0$ and $\varphi_+$, which are given as follows

$$\varphi_- = -\sqrt{\frac{\mu}{\eta}}, \qquad \varphi_0 = 0, \qquad \varphi_+ = \sqrt{\frac{\mu}{\eta}}. \tag{3.9}$$

Letting $(\varphi_i, 0)$ be one of the singular points of system (3.6), then the characteristic values of the linearized system of system (3.6) at the singular points $(\varphi_i, 0)$ are

$$\lambda_\pm = \pm\sqrt{f'(\varphi_i)}. \tag{3.10}$$

From the qualitative theory of dynamical systems, we know that

(1) If $f'(\varphi_i) > 0$, $(\varphi_i, 0)$ is a saddle point.
(2) If $f'(\varphi_i) < 0$, $(\varphi_i, 0)$ is a center point.
(3) If $f'(\varphi_i) = 0$, $(\varphi_i, 0)$ is a degenerate saddle point.

Therefore, we obtain the phase portraits of system (3.6) in Figure 1.

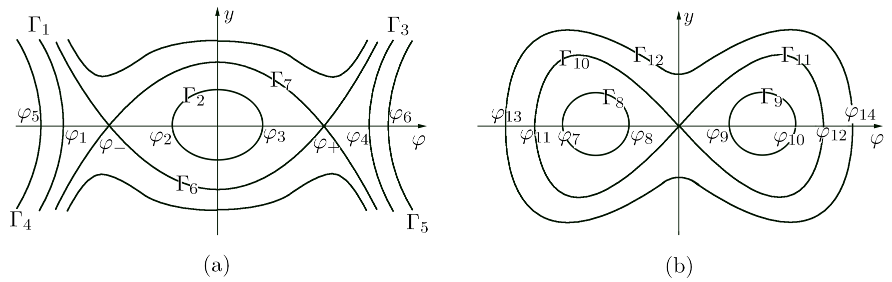



Figure 1. The phase portraits of system (3.6), (a) $\eta < 0$, $\mu < 0$, (b) $\eta > 0$, $\mu > 0$

Let

$$H(\varphi, y) = h, \tag{3.11}$$

where $h$ is Hamiltonian.
Next, we consider the relations between the orbits of (3.6) and the Hamiltonian $h$.
Set

$$h^* = |H(\varphi_+, 0)| = |H(\varphi_-, 0)| = \frac{\mu^2}{2|\eta|}. \tag{3.12}$$

According to Figure 1, we get the following propositions.

**Proposition 3.1.** *Suppose that $\eta < 0$ and $\mu < 0$, we have*

(1) *When $h < 0$ or $h > h^*$, system (3.6) does not any closed orbit.*
(2) *When $0 < h < h^*$, system (3.6) has three periodic orbits $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.*
(3) *When $h = 0$, system (3.6) has two periodic orbits $\Gamma_4$ and $\Gamma_5$.*
(4) *When $h = h^*$, system (3.6) has two heteroclonic orbits $\Gamma_6$ and $\Gamma_7$.*

**Proposition 3.2.** *Suppose that $\eta > 0$ and $\mu > 0$, we have*

(1) *When $h \leqslant -h^*$, system (3.6) does not any closed orbit.*
(2) *When $-h^* < h < 0$, system (3.6) has two periodic orbits $\Gamma_8$ and $\Gamma_9$.*
(3) *When $h = 0$, system (3.6) has two homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$.*
(4) *When $h > 0$, system (3.6) has a periodic orbit $\Gamma_{12}$.*

From the qualitative theory of dynamical systems, we know that a smooth solitary wave solution of a partial differential system corresponds to a smooth homoclinic orbit of a traveling wave equation. A smooth kink wave solution or a unbounded wave solution corresponds to a smooth heteroclinic orbit of a traveling wave equation. Similarly, a periodic orbit of a traveling wave equation corresponds to a periodic traveling wave solution of a partial differential system. According to above analysis, we have the following propositions.

**Proposition 3.3.** *If $\eta < 0$ and $\mu < 0$, we have*

(1) *When $0 < h < h^*$, (3.1) has two periodic wave solutions (corresponding to the periodic orbit $\Gamma_2$ in Fig. 1) and two periodic blow-up wave solutions(corresponding to the periodic orbits $\Gamma_1$ and $\Gamma_3$ in Figure 1).*

(2) *When $h = 0$, (3.1) has periodic blow-up wave solutions(corresponding to the periodic orbits $\Gamma_4$ and $\Gamma_5$ in Figure 1).*

(3) *When $h = h^*$, (3.1) has two kink profile solitary wave solutions and two unbounded wave solutions (corresponding to the heteroclinic orbits $\Gamma_6$ and $\Gamma_7$ in Figure 1).*

**Proposition 3.4.** *If $\eta > 0$ and $\mu > 0$, we have*

(1) *When $-h^* < h < 0$, (3.1) has two periodic wave solutions(corresponding to the periodic orbits $\Gamma_8$ and $\Gamma_9$ in Figure 1).*

(2) *When $h = 0$, (3.1) has two solitary wave solutions(corresponding to the homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$ in Figure 1).*

(3) *When $h > 0$, (3.1) has two periodic wave solutions(corresponding to the periodic orbit $\Gamma_{12}$ in Figure 1).*

### 3.2. Traveling wave solutions and their relations

Firstly, we will obtain the explicit expressions of traveling wave solutions for the (3.1) when $\eta < 0$ and $\mu < 0$.

(1) From the phase portrait, we note that there are three periodic orbits $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ passing the points $(\varphi_1, 0)$,$(\varphi_2, 0)$, $(\varphi_3, 0)$ and $(\varphi_4, 0)$. In $(\varphi, y)$-plane the expressions of the orbits are given as

$$y = \pm\sqrt{-\frac{\eta}{2}}\sqrt{(\varphi-\varphi_1)(\varphi-\varphi_2)(\varphi-\varphi_3)(\varphi-\varphi_4)}, \tag{3.13}$$

where $\varphi_1 = -\sqrt{\frac{\mu-\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_2 = -\sqrt{\frac{\mu+\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_3 = \sqrt{\frac{\mu+\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_4 = \sqrt{\frac{\mu-\sqrt{\mu^2+2\eta h}}{\eta}}$ and $0 < h < h^*$.

Substituting (3.13) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, we have

$$\pm\int_{\varphi}^{\infty}\frac{1}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)(s-\varphi_4)}}\mathrm{d}s = \sqrt{-\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s, \tag{3.14}$$

$$\pm\int_{0}^{\varphi}\frac{1}{\sqrt{(s-\varphi_1)(s-\varphi_2)(s-\varphi_3)(s-\varphi_4)}}\mathrm{d}s = \sqrt{-\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.15}$$

Completing above integrals we obtain

$$\varphi = \pm\frac{\varphi_4}{\mathrm{sn}\left(\varphi_4\sqrt{-\frac{\eta}{2}}\xi, \frac{\varphi_3}{\varphi_4}\right)}, \tag{3.16}$$

$$\varphi = \pm\varphi_3\mathrm{sn}\left(\varphi_4\sqrt{-\frac{\eta}{2}}\xi, \frac{\varphi_3}{\varphi_4}\right). \tag{3.17}$$

Noting that (3.2), we get the following periodic wave solutions

$$u_1(x,t) = \pm\frac{\varphi_4}{\mathrm{sn}\left(\varphi_4\sqrt{-\frac{\eta}{2}}(x-ct), \frac{\varphi_3}{\varphi_4}\right)}, \tag{3.18}$$

and

$$u_2(x,t) = \pm\varphi_3 \mathrm{sn}\left(\varphi_4\sqrt{-\frac{\eta}{2}}(x-ct), \frac{\varphi_3}{\varphi_4}\right). \tag{3.19}$$

(2) From the phase portrait, we note that there are two special orbits $\Gamma_4$ and $\Gamma_5$, which have the same hamiltonian with that of the center point $(0,0)$. In $(\varphi, y)$-plane the expressions of the orbits are given as

$$y = \pm\sqrt{-\frac{\eta}{2}}\varphi\sqrt{(\varphi-\varphi_5)(\varphi-\varphi_6)}, \tag{3.20}$$

where $\varphi_5 = -\sqrt{2\mu/\eta}$ and $\varphi_6 = \sqrt{2\eta/\mu}$.

Substituting (3.20) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the two orbits $\Gamma_4$ and $\Gamma_5$, it follows that

$$\pm\int_{\varphi}^{+\infty}\frac{1}{s\sqrt{(s-\varphi_5)(s-\varphi_6)}}\mathrm{d}s = \sqrt{-\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.21}$$

Completing above integrals we obtain

$$\varphi = \pm\sqrt{\frac{2\mu}{\eta}}\csc\sqrt{-\mu}\xi. \tag{3.22}$$

Noting that (3.2), we get the following periodic blow-up wave solutions

$$u_3(x,t) = \pm\sqrt{\frac{2\mu}{\eta}}\csc\sqrt{-\mu}(x-ct). \tag{3.23}$$

(3) From the phase portrait, we see that there are two heterclinic orbits $\Gamma_6$ and $\Gamma_7$ connected at saddle points $(\varphi_-, 0)$ and$(\varphi_+, 0)$. In $(\varphi, y)$-plane the expressions of the heterclinic orbits are given as

$$y = \pm\sqrt{-\frac{\eta}{2}}\sqrt{(\varphi-\varphi_-)^2(\varphi-\varphi_+)^2}. \tag{3.24}$$

Substituting (3.24) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the heterclinic orbits $\Gamma_6$ and $\Gamma_7$, it follows that

$$\pm\int_0^{\varphi}\frac{1}{(s-\varphi_-)(\varphi_+-s)}\mathrm{d}s = \sqrt{-\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s, \tag{3.25}$$

$$\pm\int_{\varphi}^{+\infty}\frac{1}{(s-\varphi_-)(s-\varphi_+)}\mathrm{d}s = \sqrt{-\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.26}$$

Completing above integrals we obtain

$$\varphi = \pm\sqrt{\frac{\mu}{\eta}}\tanh\sqrt{-\frac{\mu}{2}}\xi, \tag{3.27}$$

$$\varphi = \pm\sqrt{\frac{\mu}{\eta}}\coth\sqrt{-\frac{\mu}{2}}\xi. \tag{3.28}$$

Noting that (3.2), we get the following kink profile solitary wave solutions

$$u_4(x,t) = \pm\sqrt{\frac{\mu}{\eta}}\tanh\sqrt{-\frac{\mu}{2}}(x-ct), \tag{3.29}$$

and unbounded wave solutions

$$u_5(x,t) = \pm\sqrt{\frac{\mu}{\eta}}\coth\sqrt{-\frac{\mu}{2}}(x-ct). \tag{3.30}$$

Secondly, we will obtain the explicit expressions of traveling wave solutions for the (3.1) when $\eta > 0$ and $\mu > 0$.
(1) From the phase portrait, we see that there are two closed orbits $\Gamma_8$ and $\Gamma_9$ passing the points $(\varphi_7,0)$, $(\varphi_8,0)$, $(\varphi_9,0)$ and $(\varphi_{10},0)$. In $(\varphi,y)$-plane the expressions of the closed orbits are given as

$$y = \pm\sqrt{\frac{\eta}{2}}\sqrt{(\varphi-\varphi_7)(\varphi-\varphi_8)(\varphi-\varphi_9)(\varphi_{10}-\varphi)}, \tag{3.31}$$

where $\varphi_7 = -\sqrt{\frac{\mu+\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_8 = -\sqrt{\frac{\mu-\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_9 = \sqrt{\frac{\mu-\sqrt{\mu^2+2\eta h}}{\eta}}$, $\varphi_{10} = \sqrt{\frac{\mu+\sqrt{\mu^2+2\eta h}}{\eta}}$ and $-h^* < h < 0$.

Substituting (3.31) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along $\Gamma_8$ and $\Gamma_9$, we have

$$\pm\int_{\varphi_7}^{\varphi}\frac{1}{\sqrt{(\varphi_{10}-s)(\varphi_9-s)(\varphi_8-s)(s-\varphi_7)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s, \tag{3.32}$$

$$\pm\int_{\varphi_{10}}^{\varphi}\frac{1}{\sqrt{(s-\varphi_7)(s-\varphi_8)(s-\varphi_9)(\varphi_{10}-s)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.33}$$

Completing above integrals we obtain

$$\varphi = \frac{(\varphi_{10}-\varphi_8)\varphi_7 + (\varphi_8-\varphi_7)\varphi_{10}\left(\operatorname{sn}\left(\omega\sqrt{\frac{\eta}{2}}\xi,\kappa\right)\right)^2}{\varphi_{10}-\varphi_8+(\varphi_8-\varphi_7)\left(\operatorname{sn}\left(\omega\sqrt{\frac{\eta}{2}}\xi,\kappa\right)\right)^2}, \tag{3.34}$$

$$\varphi = \sqrt{\varphi_{10}^2 - (\varphi_{10}^2-\varphi_9^2)\left(\operatorname{sn}\left(\varphi_{10}\sqrt{\frac{\eta}{2}}\xi, \frac{\sqrt{\varphi_{10}^2-\varphi_9^2}}{\varphi_{10}}\right)\right)^2}, \tag{3.35}$$

where $\omega = \frac{\sqrt{(\varphi_{10}-\varphi_8)(\varphi_9-\varphi_7)}}{2}$ and $\kappa = \sqrt{\frac{(\varphi_{10}-\varphi_9)(\varphi_8-\varphi_7)}{(\varphi_{10}-\varphi_8)(\varphi_9-\varphi_7)}}$.

Noting that (3.2), we get the following periodic wave solutions

$$u_6(x,t) = \frac{\left((\varphi_{10}-\varphi_8)\varphi_7 + (\varphi_8-\varphi_7)\varphi_{10}\left(\operatorname{sn}\left(\omega\sqrt{\frac{\eta}{2}}(x-ct),\kappa\right)\right)^2\right)}{\varphi_{10}-\varphi_8+(\varphi_8-\varphi_7)\left(\operatorname{sn}\left(\omega\sqrt{\frac{\eta}{2}}(x-ct),\kappa\right)\right)^2}, \tag{3.36}$$

and

$$u_7(x,t) = \sqrt{\varphi_{10}^2 - (\varphi_{10}^2 - \varphi_9^2)\left(\operatorname{sn}\left(\varphi_{10}\sqrt{\frac{\eta}{2}}(x-ct), \frac{\sqrt{\varphi_{10}^2 - \varphi_9^2}}{\varphi_{10}}\right)\right)^2}. \tag{3.37}$$

(2) From the phase portrait, we see that there are two symmetric homoclinic orbits $\Gamma_{10}$ and $\Gamma_{11}$ connected at the saddle point $(0,0)$. In $(\varphi, y)$-plane the expressions of the homoclinic orbits are given as

$$y = \pm\sqrt{\frac{\eta}{2}}\varphi\sqrt{(\varphi - \varphi_{11})(\varphi_{12} - \varphi)}, \tag{3.38}$$

where $\varphi_{11} = -\sqrt{2\mu/\eta}$ and $\varphi_{12} = \sqrt{2\mu/\eta}$.

Substituting (3.38) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbits $\Gamma_{10}$ and $\Gamma_{11}$, we have

$$\pm\int_{\varphi_{11}}^{\varphi} \frac{1}{s\sqrt{(s-\varphi_{11})(\varphi_{12}-s)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s, \tag{3.39}$$

$$\pm\int_{\varphi_{12}}^{\varphi} \frac{1}{s\sqrt{(s-\varphi_{11})(\varphi_{12}-s)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.40}$$

Completing above integrals we obtain

$$\varphi = -\sqrt{\frac{2\mu}{\eta}}\operatorname{sech}\sqrt{\mu}\xi, \tag{3.41}$$

and

$$\varphi = \sqrt{\frac{2\mu}{\eta}}\operatorname{sech}\sqrt{\mu}\xi. \tag{3.42}$$

Noting that (3.2), we get the following solitary wave solutions

$$u_8(x,t) = -\sqrt{\frac{2\mu}{\eta}}\operatorname{sech}\sqrt{\mu}(x-ct), \tag{3.43}$$

and

$$u_9(x,t) = \sqrt{\frac{2\mu}{\eta}}\operatorname{sech}\sqrt{\mu}(x-ct). \tag{3.44}$$

(3) From the phase portrait, we see that there are a closed orbit $\Gamma_{12}$ passing the points $(\varphi_{13}, 0)$ and $(\varphi_{14}, 0)$. In $(\varphi, y)$-plane the expressions of the closed orbits are given as

$$y = \pm\sqrt{\frac{\eta}{2}}\sqrt{(\varphi_{14}-\varphi)(\varphi-\varphi_{13})(\varphi-c_1)(\varphi-\overline{c}_1)}, \tag{3.45}$$

where $\varphi_{13} = -\sqrt{\frac{\mu+\sqrt{\mu^2-2\eta h}}{\eta}}$, $\varphi_{14} = \sqrt{\frac{\mu+\sqrt{\mu^2-2\eta h}}{\eta}}$, $c_1 = i\sqrt{\frac{\mu-\sqrt{\mu^2-2\eta h}}{\eta}}$, $\overline{c}_1 = -i\sqrt{\frac{\mu-\sqrt{\mu^2-2\eta h}}{\eta}}$ and $h > 0$.

Substituting (3.45) into $\mathrm{d}\varphi/\mathrm{d}\xi = y$ and integrating them along the orbit $\Gamma_{12}$, we have

$$\pm\int_{\varphi_{13}}^{\varphi} \frac{1}{\sqrt{(\varphi_{14}-s)(s-\varphi_{13})(s-c_1)(s-\bar{c}_1)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s, \tag{3.46}$$

$$\pm\int_{\varphi}^{\varphi_{14}} \frac{1}{\sqrt{(\varphi_{14}-s)(s-\varphi_{13})(s-c_1)(s-\bar{c}_1)}}\mathrm{d}s = \sqrt{\frac{\eta}{2}}\int_0^{\xi}\mathrm{d}s. \tag{3.47}$$

Completing above integrals we obtain

$$\varphi = \varphi_{13}\mathrm{cn}\left(\sqrt{\mu}\xi, -\varphi_{13}\sqrt{\frac{\eta}{2\mu}}\right), \tag{3.48}$$

and

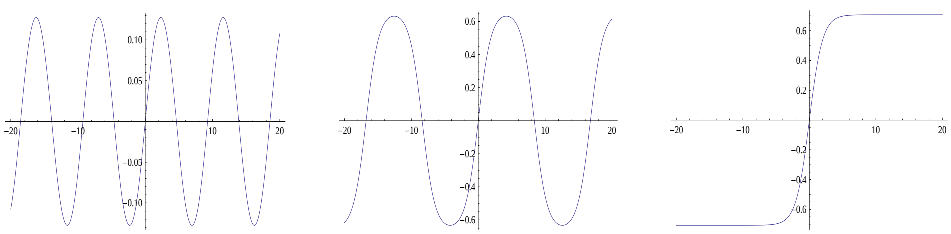
$$\varphi = \varphi_{14}\mathrm{cn}\left(\sqrt{\mu}\xi, \varphi_{14}\sqrt{\frac{\eta}{2\mu}}\right). \tag{3.49}$$

Noting that (3.2), we get the following periodic wave solutions

$$u_{10}(x,t) = \varphi_{13}\mathrm{cn}\left(\sqrt{\mu}(x-ct), -\varphi_{13}\sqrt{\frac{\eta}{2\mu}}\right), \tag{3.50}$$

and

$$u_{11}(x,t) = \varphi_{14}\mathrm{cn}\left(\sqrt{\mu}(x-ct), \varphi_{14}\sqrt{\frac{\eta}{2\mu}}\right). \tag{3.51}$$



(a) (b) (c)

Figure 2. The imaginary part of the periodic wave solution $u_{2_+}(x,t)$ evolute into the kink wave solutions $u_{4_+}(x,t)$ at $t=0$ with the conditions (3.52). (a) $h=0.008$; (b) $h=0.12$; (c) $h=0.125$.

Thirdly, we will give that relations of the traveling wave solutions.

(1) Letting $h \to h^*-$, it follows that $\varphi_4 \to \sqrt{\mu/\eta}$, $\varphi_3 \to \sqrt{\mu/\eta}$, $\varphi_3/\varphi_4 \to 1$ and $\mathrm{sn}(\sqrt{-\mu}(x-ct),1) = \tanh\sqrt{-\mu}(x-ct)$. Therefore, we obtain $u_1(x,t) \to u_5(x,t)$ and $u_2(x,t) \to u_4(x,t)$.
(2) Letting $h \to 0+$, it follows that $\varphi_4 \to \sqrt{2\mu/\eta}$, $\varphi_3 \to 0$, $\varphi_3/\varphi_4 \to 0$ and $\mathrm{sn}(\sqrt{-\mu}(x-ct),0) = \sin\sqrt{-\mu}(x-ct)$. Therefore, we obtain $u_1(x,t) \to u_3(x,t)$.
(3) Letting $h \to 0-$, it follows that $\varphi_{10} \to \sqrt{2\mu/\eta}$, $\varphi_9 \to 0$, $\varphi_8 \to 0$, $\varphi_7 \to -\sqrt{2\mu/\eta}$, $\omega \to \sqrt{\mu/2\eta}$, $k \to 1$ and $\mathrm{sn}(\sqrt{\mu/2}(x-ct),1) = \tanh\sqrt{\mu/2}(x-ct)$. Therefore, we obtain $u_6(x,t) \to u_8(x,t)$.

(4) Letting $h \to 0-$, it follows that $\varphi_{10} \to \sqrt{2\mu/\eta}$, $\varphi_9 \to 0$, $\varphi_8 \to 0$, $\varphi_7 \to -\sqrt{2\mu/\eta}$, $\frac{\sqrt{\varphi_{10}^2-\varphi_9^2}}{\varphi_{10}} \to 1$ and $\mathrm{sn}(\sqrt{\mu}(x-ct),1) = \tanh\sqrt{\mu}(x-ct)$. Therefore, we obtain $u_7(x,t) \to u_9(x,t)$.

(5) Letting $h \to 0+$, it follows that $\varphi_{14} \to \sqrt{2\mu/\eta}$, $\varphi_{13} \to -\sqrt{2\mu/\eta}$, $-\varphi_{13}\sqrt{\eta/2\mu} \to 1$, $\varphi_{14}\sqrt{\eta/2\mu} \to 1$ and $\mathrm{cn}(\sqrt{\mu}(x-ct),1) = \mathrm{sech}\sqrt{\mu}(x-ct)$. Therefore, we obtain $u_{10}(x,t) \to u_8(x,t)$ and $u_{11}(x,t) \to u_9(x,t)$.

Finally, we will show that the periodic wave solutions $u_{2_+}(x,t)$ evolute into the kink profile solitary wave solutions $u_{4_+}(x,t)$ when the Hamiltonian $h \to h^*-$ (corresponding to the changes of phase orbits of Figure 1 as $h$ varies). We take some suitable choices of the parameters, such as

$$\alpha = 2, \qquad \beta = -2, \qquad c = 2, \qquad \lambda = -1, \tag{3.52}$$

as an illustrative sample and draw their plots (see Figure 2).

## 4. Conclusions

This paper studies the PHI-four equation by the aid of ansatz method and bifurcation analysis. These approaches allowed to reveal several solution of this equation. They are cnoidal waves, snoidal waves, solitary waves, kinks, periodic waves, periodic singular waves and others. The constraint conditions imposes the restrictions on the choice of the parameters and coefficients of the governing equations. There are several other NLEES where, particularly, the bifurcation method, can be applied to obtain these interesting solutions to them. The results of these research will be available in due course of time.

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