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A Labeling Algorithm for Distance Domination on Block Graphs

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Abstract. The k-distance domination problem is to find a minimum vertex set D of a graph such that every vertex of the graph is either in D or within distance k from some vertex of D, where k is a positive integer. In the present paper, by using labeling method, a linear-time algorithm for k-distance domination problem on block graphs is designed.

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1. Introduction

All graphs considered in this paper are simple and connected graphs. For terminology and notation not given here, the reader is referred to [10]. Let $G = (V,E)$ be a graph with *vertex set* V and *edge set* E. For any $v \in V$, The *neighborhood* $N(v)$ of v is the set of vertices adjacent to v, the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *distance* $d_G(u,v)$ between two vertices u and v is the length of a shortest uv-path in G. Let k be a positive integer. For any vertex $v \in V$, the *k-distance neighborhood* of v is $N_k(v) = \{u \mid 0 < d_G(v,u) \leq k\}$. The *closed k-distance neighborhood* of v is $N_k[v] = N_k(v) \cup \{v\}$.

Given a graph $G = (V,E)$, we say that a vertex $v \in V$ *dominates* all vertices in its closed neighborhood $N[v]$. Recall that a subset $D \subseteq V$ is called a *dominating set* of G if every vertex in G is dominated by a vertex in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G.

Domination and some domination-related parameters have been extensively studied (see, for example, [1, 2, 11, 17]). Among these parameters, k-distance domination has received more and more attention in recent years.

We say that a vertex $v \in V$ *k-distance dominates* all vertices in its closed k-distance neighborhood $N_k[v]$. A subset $D \subseteq V$ is called a *k-distance dominating set* of G if every vertex in G is k-distance dominated by a vertex in D. The *k-distance domination number*

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$\gamma_k(G)$ of G is the minimum cardinality among all k-distance dominating sets of G. A k-distance dominating set with cardinality $\gamma_k(G)$ is also called a $\gamma_k(G)$-set. The *k-distance domination problem* is to find a minimum k-distance dominating set of G. It is clear that a dominating set is a 1-distance dominating set, and thus $\gamma(G) = \gamma_1(G)$.

There are many applications of the above generalizations. An interpretation in terms of communication networks is presented by Slater [13] as follows. If V represents a collection of cities and an edge represents a communication link, then one may be interested in selecting a minimum number of cities as sites for transmitting stations so that every city either contains a transmitter or can receive messages from at least one of the transmitting stations through the links. If only direct transmissions are acceptable, then one wishes to find a minimum 1-distance dominating set or 1-step dominating set. If communication over paths of k links (but not of $k+1$ links) is adequate in quality and rapidity, the problem becomes that of determining a minimum k-distance dominating set.

k-distance domination is introduced by Boland, Haynes, and Lawson [3]. k-distance domination problem is NP-complete for general graphs, chord graphs, and bipartite graphs [3,7]. To obtain an algorithm for trees, Slater introduced the concept of "R-domination" and obtained a linear-time algorithm for R-domination on trees. A restriction of R-domination turns to be k-distance domination. As a generalization of Slater's algorithm, a linear-time algorithm for R-domination on block graphs was provided in [6]. In the present paper, we provide a labeling algorithm for the k-distance domination problem on block graphs. We shall show that why our algorithm has different idea from that in [6, 13]. Studies of some other distance or distance-related domination can be seen in, for example, [4, 8, 12, 14–16].

2. A linear-time algorithm for block graphs

In a graph G, a vertex v is a *cut-vertex* if $G-v$ (deleting v together with all edges incident to it) is disconnected. A *block* of G is a maximal connected subgraph without a cut-vertex. If G has no cut-vertex, G itself is a block. The intersection of two blocks contains at most one vertex and a vertex is a cut-vertex if and only if it is the intersection of two or more blocks. In general, the blocks of a connected graph fit together in a treelike structure. A block B of G is called an *end block* if B contains at most one cut-vertex of G. A *block graph* is a graph whose blocks are complete graphs. This name arises because a graph G is the intersection graph of the blocks of some graph if and only if every block of G is complete [9].

Given a block graph G, since its blocks fit together in a treelike structure, then we may give some similar terminology and definitions to those in a tree. Define the *distance between two blocks* B_1 and B_2 as $d_G(B_1,B_2) = \max\,\{d_G(v_1,v_2)|\ v_1 \in B_1, v_2 \in B_2\} - 1$. Define the *distance between a vertex v and a block B* as $d_G(v,B) = \max\,\{d_G(v,u)|\ u \in B\} - 1$. Now, we assume that the block graph G is rooted at any end block, say B_0, of it. Then the *height* of G is the maximum among the distances between B_0 and all end blocks. If $G = B_0$, then G is a complete graph and the height of G is zero. Let h be the height of G and let the i-th *level* $A_i, 0 \le i \le h$, be the set of blocks of G which are at distance i from B_0.

For a block graph G which is rooted at an end block B_0 and has the height at least one, and for a vertex v with the farthest distance from B_0, we use $F_k(v)$ to denote the unique cut-vertex of G in $N_k(v)$ which has the minimum distance from B_0.

Now, we work on an algorithm for finding a minimum k-distance dominating set of a block graph. In our algorithm, we will use a label $L(v)$ for each vertex v in current block graph G as follows.

$$L(v) = \begin{cases} 0, & \text{if } v \text{ needs not to be } k\text{-distance dominated by any vertex in } G; \\ 1, & \text{if } v \text{ needs to be } k\text{-distance dominated by a vertex in } G; \\ 2, & \text{if } v \text{ is put into the output minimum } k\text{-distance dominating set.} \end{cases}$$

If all vertices of G are labeled as above, we call G a *labeled graph*. Initially, all vertices of G are labeled 1. In every step, our algorithm visits a vertex farthest from the root block B_0, relabel some vertices, deletes this vertex (together with its incident edges) from the current labeled block graph and obtains a new labeled block graph.

In every step, the labels of the vertices of G will be changed, and the vertices with label 2 will be put into the output minimum k-distance dominating set. So we give some definitions for a labeled graph G. An *optional k-distance dominating set* of G is any set $D \subseteq V$ which contains all vertices with label 2, and k-distance dominates all vertices with label 1. Note that a vertex with label 0 needs not to be k-distance dominated by a vertex in D but can be used in D to dominate vertices with label 1, a vertex v with $L(v) = 1$ should be k-distance dominated by a vertex (v or another vertex different from v) of G in D. The *optional k-distance domination number* $\gamma_{ok}(G)$ is the minimum cardinality among all optional k-distance dominating sets of G. an optional k-distance dominating set of G with cardinality $\gamma_{ok}(G)$ is also called a γ_{ok}-set.

Note that the k-distance domination problem is just the optional k-distance domination problem with all vertices being labeled 1. This generalization can be viewed as a labeling algorithm. The idea of a labeling algorithm was first introduced by Cockayne, Goodman, and Hedetniemi for solving the domination problem in trees [5]. It is a natural but powerful tool when we use an induction to treat a treelike structure.

As a k-distance dominating set of a graph $G = (V,E)$ is indeed an optional k-distance dominating set of G when all vertices of G are labeled 1, in order to find a minimum k-distance dominating set of G, we only have to label all vertices of G with label 1 and find a minimum optional k-distance dominating set of G. Now a linear-time algorithm for finding a minimum optional k-distance dominating set of a block graph is shown as follows.

Algorithm OkDDB: optional k-distance domination on block graphs.
Input: a block graph G, rooted at an end block B_0, with all its vertices being labeled 1.
Output: a minimum optional k-distance dominating set D of G, consisting of all vertices with label 2 when the algorithm stops.

Method. In every step, the algorithm visits a non-cut vertex in an end block B, label or relabel some vertices, deletes this vertex from B and G.

Begin
    $D = \emptyset$.
    While the height of G is at least one **do**
        Let B be an end block of G with the maximum level number;
        For every non-cut vertex $v \in B$ **do**
            If $L(v) = 0$, **then** $B \leftarrow B - v$, $G \leftarrow G - v$;
            If $L(v) = 1$, **then**
                If there exists some vertex $u \in N_k(v)$ such that $L(u) = 2$, **then**

$B \leftarrow B - v$, $G \leftarrow G - v$;
If $L(x) \neq 2$ for every $u \in N_k(v)$, **then**
$L(F_k(v)) \leftarrow 2$, $B \leftarrow B - v$, $G \leftarrow G - v$;
If $L(v) = 2$, **then**
For every vertex $x \in N_k(v)$ **do**
If $L(x) = 2$, **then** do nothing;
If $L(x) \neq 2$, **then** $L(x) \leftarrow 0$;
End for;
$D \leftarrow D \cup \{v\}$, $B \leftarrow B - v$, $G \leftarrow G - v$;
End for;
While the height of G is zero **do**
If there is a vertex $u \in N_k(v)$ such that $L(u) = 2$, **then**
$D \leftarrow D \cup \{x \in G | \ L(x) = 2\}$ and stop;
Else
If $L(x) = 0$ for every $x \in V(G)$, **then** stop;
Else, select an arbitrary vertex u, $L(u) \leftarrow 2$, $D \leftarrow D \cup \{u\}$ and stop.
End

It is easy to see that the running time of the algorithm is $O(n)$, as it merely executes a simple for-loop, all of the statements within which can be executed in at most constant time, with an adequate data structure. The correctness of the algorithm is based on the following theorem.

Theorem 2.1. *Algorithm* OkDDB *produces a minimum k-distance dominating set of a block graph G.*

Proof. It is sufficient to consider a block graph G with the height at least one, since the last step (the second while sentence) in algorithm OkDDB clearly finds a minimum k-distance dominating set of a complete graph. Suppose G is the current labeled block graph rooted at an end block B_0, v is the current vertex in G which has the farthest distance from B_0. Then, the proof of Theorem 2.1 is followed by a series of claims.

Claim 1. If $L(v) = 0$, then $\gamma_{ok}(G) = \gamma_{ok}(G - v)$.

Let D be a γ_{ok}-set of G. If $v \in D$, then $D \setminus \{v\} \cup \{F_k(v)\}$ is an optional k-distance dominating set of G, since $N_k(v) \subseteq N_k(F_k(v))$, that is, all vertices which are k-distance dominated by v can also be k-distance dominated by $F_k(v)$. So assume $v \notin D$. Then clearly D is also an optional k-distance dominating set of $G - v$. Hence $\gamma_{ok}(G - v) \leq \gamma_{ok}(G)$.

Conversely, let D' be a γ_{ok}-set of $G - v$. Since $L(v) = 0$ in G, v needs not to be k-distance dominated by a vertex in G. It follows that D' is also an optional k-distance dominating set of G. Therefore $\gamma_{ok}(G) \leq \gamma_{ok}(G - v)$.

Claim 2. If $L(v) = 1$ and there exists some vertex $u \in N_k(v)$ such that $L(u) = 2$, then $\gamma_{ok}(G) = \gamma_{ok}(G - v)$.

Let D be a γ_{ok}-set of G. Since $L(u) = 2$ in G, we have $u \in D$ by the definition of an optional k-distance dominating set. By the minimality of D, $v \notin D$. It follows that D is also an optional k-distance dominating set of $G - v$. Thus $\gamma_{ok}(G - v) \leq \gamma_{ok}(G)$.

Conversely, let D' be a γ_{ok}-set of $G - v$. Since $L(u) = 2$ in $G - v$, we know $u \in D'$. Then it follows that D' is also an optional k-distance dominating set of G, since v is k-distance dominated by $u \in D'$ in G. Hence, $\gamma_{ok}(G) \leq \gamma_{ok}(G - v)$.

Claim 3. If $L(v)=1$ and there exists no vertex in $N_k(v)$ with label 2, and G' is the block graph which results from G by deleting v and relabeling $F_k(v)$ with 2, then $\gamma_{ok}(G)=\gamma_{ok}(G')$.

Let D be a γ_{ok}-set of G. If $v\in D$, then $D\setminus\{v\}\cup\{F_k(v)\}$ is an optional k-distance dominating set of $G-v$, in which $F_k(v)$ is considered as a vertex with label 2. Next assume $v\notin D$. Since $L(v)=1$, there must exist some vertex $x\in N_k(v)\cap D$ to k-distance dominate v. If $x\neq F_k(v)$, then noting the fact that v is the farthest vertex from B_0, it is easy to see that all the vertices which are k-distance dominated by x can also be k-distance dominated by $F_k(v)$. So $D\setminus\{x\}\cup\{F_k(v)\}$ is an optional k-distance dominating set of $G-v$, in which $F_k(v)$ is considered as a vertex with label 2. If $x=F_k(v)$, then D is obviously an optional k-distance dominating set of $G-v$, in which $F_k(v)$ is also considered as a vertex with label 2. In either case, $\gamma_{ok}(G')\leq\gamma_{ok}(G)$.

Conversely, let D' be a γ_{ok}-set of G'. Since $L(F_k(v))=2$ in G', $F_k(v)\in D'$. Then it follows that D' is also an optional k-distance dominating set of G, since v is k-distance dominated by $F_k(v)$ in G. Hence, $\gamma_{ok}(G)\leq\gamma_{ok}(G')$.

Claim 4. If $L(v)=2$ and G' is the block graph which results from G by deleting v and relabeling every vertex $x\in N_k(v)$ such that $L(x)\neq 2$ with label 0, then $\gamma_{ok}(G)=\gamma_{ok}(G')+1$.

Let D be a γ_{ok}-set of G. Since $L(v)=2$, we know $v\in D$. Note that all the vertices in $N_k(v)$ have labels 2 or 0 in G', which means that in G', any vertex in $N_k(v)$ is either in D or needs not to be k-distance dominated by a vertex in G'. So $D\setminus\{v\}$ is an optional k-distance dominating set of G', and thus $\gamma_{ok}(G')\leq\gamma_{ok}(G)-1$.

Conversely, let D' be a γ_{ok}-set of G'. Obviously $D'\cup\{v\}$ is an optional k-distance dominating set of G. This means that $\gamma_{ok}(G)\leq\gamma_{ok}(G')+1$, and completes the proof of Theorem 1. ∎

To obtain an efficient algorithm for k-distance domination problem on trees, Slater introduced the concept of *R-dominating set* as follows [13]. Given a graph G with vertex set $V=\{1,2,\dots,n\}$, suppose one has an ordered n-tuple of ordered pairs of integers, say $R=((a_1,b_1),(a_2,b_2),\dots,(a_n,b_n))$, where $a_i\geq 0$ and $b_i\geq 1$ for $1\leq i\leq n$. Now $B\subseteq V$ will be said to *dominate* $i\in V$ if and only if either (1) there is a vertex b of B such that $d_G(i,b)\leq a_i$, or (2) there is a vertex j of V such that $d_G(i,j)+b_j\leq a_i$. If B dominates every vertex of V, then B will be said to be an *R-dominating set* of G. Note that if let $a_i=k$ and $b_i=1+k$ for every $1\leq i\leq n$, then an R-dominating set of G becomes a k-distance dominating set of G. Thus an algorithm for finding a minimum R-dominating set of G is sufficient to find a minimum k-distance dominating set of G. Slater designed a recursive algorithm for finding a minimum R-dominating set of a tree T, by decreasing a_i and increasing b_j step by step, where, i is an endvertex of T and j is the vertex adjacent to i. A vertex i is put into the minimum R-dominating set only when a_i is, or has been reduced to, zero or when i is, or has become, an isolated vertex with $a_i<b_i$. As a generalization of Slater's algorithm, a linear-time algorithm for R-domination on block graphs was provided in [6].

It is noticeable that, the ideas of our algorithm and the algorithms in [6, 13] are different. Also, if all the vertices of G are labeled arbitrarily in the input of the algorithm, we can obtain an algorithm for optional k-distance domination problem on any labeled block graphs.

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