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Existence and Properties of Solutions of a Boundary Problem for a Love's Equation

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Abstract. In this paper, we use the Faedo-Galerkin method, compactness method and monotone method in order to study a nonlinear Love's equation with mixed nonhomogeneous conditions. The results obtained here are existence of a weak solution, uniqueness, regularity and asymptotic behavior of solutions.

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1. Introduction

In this paper, we consider the following equation with initial conditions and mixed nonhomogeneous conditions

$$u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda\, |u_t|^{q-2}\, u_t + K\, |u|^{p-2}\, u = F(x,t),\ x \in \Omega = (0,1),\ 0 < t < T, \tag{1.1}$$

$$\varepsilon u_{xtt}(0,t) + u_x(0,t) = hu(0,t) + g(t), \tag{1.2}$$

$$u(1,t) = 0, \tag{1.3}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.4}$$

where $p > 1$, $q > 1$, $\varepsilon > 0$, $\lambda > 0$, $K > 0$, $h \geq 0$ are constants and $\tilde{u}_0$, $\tilde{u}_1$, F, g are given functions satisfying conditions specified later.

When $F = 0$, $\lambda = K = 0$, $\Omega = (0,L)$, Equation (1.1) is related to the Love's equation

$$u_{tt} - \frac{E}{\rho} u_{xx} - 2\mu^2 k^2 u_{xxtt} = 0, \tag{1.5}$$

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presented by V. Radochová in 1978 (see [8]). This equation describes the vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt \int_0^L \left[\frac{1}{2} F\rho \left(u_t^2 + \mu^2 k^2 u_{tx}^2 \right) - \frac{1}{2} F \left(E u_x^2 + \rho \mu^2 k^2 u_x u_{xtt} \right) \right] dx, \tag{1.6}$$

the parameters in (1.6) have the following meanings: u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [8] obtained a classical solution of problem (1.5) associated with initial conditions (1.4) and boundary conditions

$$u(0,t) = u(L,t) = 0, \tag{1.7}$$

or

$$\begin{cases} u(0,t) = 0, \\ \varepsilon u_{xtt}(L,t) + c^2 u_x(L,t) = 0, \end{cases} \tag{1.8}$$

where $c^2 = \frac{E}{\rho}$, $\varepsilon = 2\mu^2 k^2$. On the other hand, the asymptotic behaviour of the solution of problem (1.4), (1.5), (1.7) or (1.8) as $\varepsilon \to 0_+$ are also established by the method of small parameter.

Equations of Love waves or equations for waves of Love types have been studied by many authors, we refer to [3, 4, 7] and references therein.

In [1], Ang and Dinh established a uniqueness and global existence for the problem (1.1)-(1.4) with $\varepsilon = K = h = 0$, $\lambda = 1$, $1 < q < 2$, $F(x,t) = 0$. In this latter case this problem governs the motion of a linear viscoelastic bar.

In this paper, we shall use the Faedo-Galerkin method, compactness method and monotone method in order to study problem (1.1)-(1.4). The results obtained are existence of a weak solution, uniqueness, regularity and asymptotic behavior of solutions.

The paper consists of four sections. Section 2 is devoted to the study of the existence a weak solution for problem (1.1)-(1.4) with $\tilde{u}_0$, $\tilde{u}_1 \in V = \{v \in H^1 : v(1) = 0\}$, $p > 1$, $q > 1$. Here, a energy lemma (as given in Lemma 2.3) is also established in order to pass the limit of a approximate problem and prove the uniqueness in case $p \geq 2$. In Section 3, we consider the regularity of solution for problem (1.1)-(1.4) with $\tilde{u}_0$, $\tilde{u}_1 \in V \cap H^2$, $p \geq 2$, $q \geq 2$ and some other conditions. In case $p = q = 2$, we show that the regularity of solutions depending on the regularity of data. Finally, the asymptotic behavior of solutions as $\varepsilon \to 0_+$ is discussed in Section 4. The results obtained here may be considered as the generalizations of those in [8].

2. Existence and uniqueness of a solution

First, we put $\Omega = (0,1)$; $Q_T = \Omega \times (0,T)$, $T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$ Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u:(0,T)\to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p\,dt\right)^{1/p}<\infty \text{ for } 1\le p<\infty,$$

and

$$\|u\|_{L^\infty(0,T;X)}=\underset{0<t<T}{ess\sup}\|u(t)\|_X \text{ for } p=\infty.$$

Let $u(t),\ u'(t)=u_t(t),\ u''(t)=u_{tt}(t),\ u_x(t),\ u_{xx}(t)$ denote $u(x,t),\ \frac{\partial u}{\partial t}(x,t),\ \frac{\partial^2 u}{\partial t^2}(x,t),\ \frac{\partial u}{\partial x}(x,t),\ \frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we shall use the following norm

$$\|v\|_{H^1}=\left(\|v\|^2+\|v_x\|^2\right)^{1/2}.$$

We put

$$V=\{v\in H^1: v(1)=0\}. \tag{2.1}$$

Then V is a closed subspace of H^1 and on V, $v\longmapsto ||v||_{H^1}$ and $v\longmapsto ||v_x||$ are equivalent norms.

Then the following lemmas are known as a standard one.

Lemma 2.1. *The imbedding $H^1\hookrightarrow C^0([0,1])$ is compact and*

$$||v||_{C^0(\overline{\Omega})}\le\sqrt{2}||v||_{H^1} \text{ for all } v\in H^1. \tag{2.2}$$

Lemma 2.2. *The imbedding $V\hookrightarrow C^0([0,1])$ is compact and*

$$||v||_{C^0(\overline{\Omega})}\le||v_x|| \text{ for all } v\in V. \tag{2.3}$$

We remark that the weak formulation of the initial-boundary value problem (1.1)-(1.4) can be given in the following manner: Find $u\in L^\infty(0,T;V)$, with $u_t\in L^\infty(0,T;V)$, such that u satisfies the following variational equation

$$\left\{\begin{array}{l}\frac{d}{dt}\left[\langle u_t(t),w\rangle+\varepsilon\langle u_{xt}(t),w_x\rangle\right]+\langle u_x(t),w_x\rangle+(hu(0,t)+g(t))\,w(0)\\ \qquad\qquad +\lambda\langle|u_t(t)|^{q-2}u_t(t),w\rangle+K\langle|u|^{p-2}u,w\rangle=\langle F(t),w\rangle,\end{array}\right. \tag{2.4}$$

for all $w\in V$, a.e., $t\in(0,T)$, together with the initial conditions

$$u(0)=\tilde{u}_0,\ u_t(0)=\tilde{u}_1. \tag{2.5}$$

Next, we need the following assumptions:

- (H_1) $p>1,\ q>1,\ \lambda>0,\ K>0,\ \varepsilon>0,\ h\ge 0$;
- (H_2) $\tilde{u}_0,\ \tilde{u}_1\in V$;
- (H_3) $F\in L^1(0,T;L^2)$;
- (H_4) $g\in W^{1,1}(0,T)$.

Then, we have the following theorem.

Theorem 2.1. *Let $T>0$. Suppose that $(H_1)-(H_4)$ hold. Then, there exists a weak solution u of problem (1.1)-(1.4) such that*

$$u\in L^\infty(0,T;V),\ u_t\in L^\infty(0,T;V). \tag{2.6}$$

Furthermore, if $p\ge 2$, the solution is unique.

Proof. The proof is a combination of Galerkin method and compactness arguments, and consits of four steps.

Step 1. *The Faedo-Galerkin approximation* (introduced by Lions [6]). Consider the basis in V

$$w_j(x) = \sqrt{\frac{2}{1+\lambda_j^2}}\cos(\lambda_j x), \quad \lambda_j = (2j-1)\frac{\pi}{2}, \quad j \in \mathbb{N},$$

constructed by the eigenfunctions of the Laplace operator $-\Delta = -\frac{\partial^2}{\partial x^2}$. Put

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{2.7}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear ordinary differential equations

$$\begin{cases} \langle u_m''(t), w_j\rangle + \langle u_{mx}(t) + \varepsilon u_{mx}''(t), w_{jx}\rangle + \lambda \left\langle |u_m'(t)|^{q-2} u_m'(t), w_j \right\rangle \\ \quad +K\left\langle |u_m(t)|^{p-2} u_m(t), w_j\right\rangle + (hu_m(0,t) + g(t))\, w_j(0) = \langle F(t), w_j\rangle, \ 1 \le j \le m, \\ u_m(0) = \tilde{u}_{0m}, \, u_m'(0) = \tilde{u}_{1m}, \end{cases} \tag{2.8}$$

where

$$\begin{cases} \tilde{u}_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to \tilde{u}_0 \text{ strongly in } V, \\ \tilde{u}_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \to \tilde{u}_1 \text{ strongly in } V. \end{cases} \tag{2.9}$$

From the assumptions of Theorem 2.1, system (2.8) has a solution u_m on an interval $[0, T_m] \subset [0, T]$. The following estimates allow one to take $T_m = T$ for all m (see [2]).

Step 2. Multiplying the j^{th} equation of (2.8) by $c_{mj}'(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, after some rearrangements, we get

$$\begin{aligned} S_m(t) &= S_m(0) + 2g(0)\tilde{u}_{0m}(0) + 2\int_0^t \left\langle F(s), u_m'(s)\right\rangle ds \\ &\quad + 2\int_0^t g'(s) u_m(0,s) ds - 2g(t) u_m(0,t) \\ &= S_m(0) + 2g(0)\tilde{u}_{0m}(0) + \sum_{j=1}^{3} I_j, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} S_m(t) &= \left\|u_m'(t)\right\|^2 + \|u_{mx}(t)\|^2 + \varepsilon \left\|u_{mx}'(t)\right\|^2 + h u_m^2(0,t) \\ &\quad + \frac{2K}{p}\|u_m(t)\|_{L^p}^p + 2\lambda \int_0^t \left\|u_m'(s)\right\|_{L^q}^q ds. \end{aligned} \tag{2.11}$$

By (2.9), (2.11) and the imbedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$, there exists a positive constant $\bar{C}_0$ depending only on $\tilde{u}_0$, $\tilde{u}_1$, h, K, p, $g(0)$ and ε, such that

$$\begin{aligned} S_m(0) + 2g(0)\tilde{u}_{0m}(0) + \frac{2K}{p}\|\tilde{u}_{0m}\|_{L^p}^p &= \|\tilde{u}_{1m}\|^2 + \|\tilde{u}_{0mx}\|^2 + \varepsilon \|\tilde{u}_{1mx}\|^2 \\ &\quad + h\tilde{u}_{0m}^2(0) + 2g(0)\tilde{u}_{0m}(0) + \frac{2K}{p}\|\tilde{u}_{0m}\|_{L^p}^p \le \frac{1}{2}\bar{C}_0, \forall\, m. \end{aligned} \tag{2.12}$$

Using (2.3) and the following inequalities

$$2ab \le \beta a^2 + \frac{1}{\beta} b^2, \text{ for all } a,\, b \in \mathbb{R},\ \beta > 0, \tag{2.13}$$

and

$$|u_m(0,t)| \le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \|u_{mx}(t)\| \le \sqrt{S_m(t)}, \tag{2.14}$$

we can estimate all terms in the right-hand side of (2.10) as follows

$$\begin{aligned} I_1 &= 2\int_0^t \left\langle F(s), u_m'(s)\right\rangle ds \le \int_0^t \|F(s)\|\, ds + \int_0^t \|F(s)\| \left\|u_m'(s)\right\|^2 ds \\ &\le C_T + \int_0^t \|F(s)\| S_m(s)\, ds, \end{aligned} \tag{2.15}$$

where C_T indicates a constant depending on T;

$$\begin{aligned} I_2 &= 2\int_0^t g'(s)u_m(0,s)ds \le 2\int_0^t \left|g'(s)\right| \sqrt{S_m(s)}ds + \int_0^t \left|g'(s)\right| S_m(s)ds \\ &\le C_T + \int_0^t \left|g'(s)\right| S_m(s)ds, \end{aligned} \tag{2.16}$$

with $C_T \ge \int_0^T |g'(s)|\, ds$;

$$I_3 = -2g(t)u_m(0,t) \le 2\|g\|_{L^\infty(0,T)} \sqrt{S_m(t)} \le C_T + \frac{1}{2}S_m(t), \tag{2.17}$$

for all $\beta > 0$, $C_T \ge 2\|g\|^2_{L^\infty(0,T)}$.

Combining (2.10), (2.12), (2.15)-(2.17) and choose $\beta = \frac{1}{2}$, the result is

$$S_m(t) \le C_T + \int_0^t d_T^{(1)}(s) S_m(s)\, ds,\ 0 \le t \le T_m, \tag{2.18}$$

where $d_T^{(1)}(s) = 2\left[\|F(s)\| + |g'(s)|\right]$, $d_T^{(1)} \in L^1(0,T)$.

By Gronwall's lemma, we deduce from (2.18) that

$$S_m(t) \le C_T \exp\left[\int_0^T d_T^{(1)}(s)ds\right] \le C_T, \ \text{ for all } t \in [0,T], \tag{2.19}$$

where C_T always indicates a bound depending on T. Thus, we can take constant $T_m = T$ for all m.

On the other hand, we deduce from (2.11) and (2.19) that

$$\begin{cases} \left\| |u_m|^{p-2} u_m \right\|^{p'}_{L^\infty(0,T;L^{p'})} = \|u_m\|^p_{L^\infty(0,T;L^p)} \le \frac{p}{2K} C_T \le C_T, \\ \left\| |u_m'|^{q-2} u_m' \right\|^{q'}_{L^{q'}(Q_T)} = \int_0^T \|u_m'(s)\|^q_{L^q}\, ds \le \frac{1}{2\lambda} C_T \le C_T, \end{cases} \tag{2.20}$$

where C_T always indicates a bound depending on T as above.

Step 3. *Limiting process*. From (2.11), (2.19), (2.20) we deduce the existence of a subsequence of $\{u_m\}$, denoted by the same symbol such that

$$\begin{cases} u_m \to u & \text{in} & L^\infty(0,T;V) & \text{weakly*}, \\ u_m' \to u' & \text{in} & L^\infty(0,T;V) & \text{weakly*}, \\ u_m \to u & \text{in} & L^\infty(0,T;L^p) & \text{weakly*}, \\ u_m' \to u' & \text{in} & L^q(Q_T) & \text{weakly}, \\ |u_m|^{p-2} u_m \to \chi_0 & \text{in} & L^\infty(0,T;L^{p'}) & \text{weakly*}, \\ |u_m'|^{q-2} u_m' \to \chi_1 & \text{in} & L^{q'}(Q_T) & \text{weakly}. \end{cases} \tag{2.21}$$

By the compactness lemma of Lions ([6], p. 57), from $(2.21)_{1,2}$, there exists a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, such that

$$u_m \to u \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{2.22}$$

By means of the continuity of function $x \longmapsto |x|^{p-2}x$, we have

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \text{ a.e. in } Q_T. \tag{2.23}$$

Using Lions's Lemma ([6], Lemma 1.3, p.12), it follows from $(2.20)_1$ and (2.23) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \text{ in } L^{p'}(Q_T) \text{ weakly.} \tag{2.24}$$

By $(2.21)_5$ and (2.24), we deduce that

$$\chi_0 = |u|^{p-2}u. \tag{2.25}$$

Passing to the limit in (2.8) by (2.9), (2.21), (2.24) and (2.25), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\left[\langle u'(t),v\rangle + \varepsilon\langle u_x'(t),v_x\rangle\right] + \langle u_x(t),v_x\rangle + \lambda\langle \chi_1(t),v\rangle + K\left\langle |u(t)|^{p-2}u(t),v\right\rangle \\ \qquad\qquad + (hu(0,t)+g(t))\,v(0) = \langle F(t),v\rangle, \text{ for all } v\in V, \\ u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1. \end{cases} \tag{2.26}$$

It remains to prove that $\chi_1 = |u'|^{q-2}u'$. We need the following lemmas.

Lemma 2.3. *Let u be the weak solution of the following problem*

$$\begin{cases} u'' - u_{xx} - \varepsilon u_{xx}'' = \Phi,\ 0<x<1,\ 0<t<T, \\ \varepsilon u_x''(0,t) + u_x(0,t) = G(t),\ u(1,t)=0, \\ u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1, \\ u\in L^\infty(0,T;V),\ \ u'\in L^\infty(0,T;V), \\ \tilde{u}_0,\ \tilde{u}_1\in V,\ G\in L^2(0,T),\ \Phi\in L^1(0,T;L^2). \end{cases} \tag{2.27}$$

Then we have

$$\begin{aligned} &\frac{1}{2}\left\|u'(t)\right\|^2 + \frac{1}{2}\|u_x(t)\|^2 + \frac{\varepsilon}{2}\left\|u_x'(t)\right\|^2 + \int_0^t G(s)u'(0,s)ds \\ &\geq \frac{1}{2}\|\tilde{u}_1\|^2 + \frac{1}{2}\|\tilde{u}_{0x}\|^2 + \frac{\varepsilon}{2}\|\tilde{u}_{1x}\|^2 + \int_0^t \left\langle \Phi(s),u'(s)\right\rangle ds,\ a.e.,\ t\in[0,T]. \end{aligned} \tag{2.28}$$

Furthermore, if $\tilde{u}_0=\tilde{u}_1=0$, there is equality in (2.28).

Proof of Lemma 2.3. The idea of the proof is the same as in ([5], Lemma 2.1, p. 79). Fix t_1, t_2, $0<t_1<t_2<T$ and let $v(x,t)$ be the function defined as follows

$$v(x,t) = \theta_m(t)\left[\left(\theta_m(t)u'(x,t)\right)*\rho_k(t)*\rho_k(t)\right], \tag{2.29}$$

where

(i) θ_m is a continuous, piecewise linear function on $[0,T]$ defined as follows:

$$\theta_m(t) = \begin{cases} 0, & \text{if,}\quad t\in[0,T]\setminus[t_1+1/m,t_2-1/m], \\ 1, & \text{if,}\quad t\in[t_1+2/m,t_2-2/m], \\ m(t-t_1-1/m), & \text{if,}\quad t\in[t_1+1/m,t_1+2/m], \\ -m(t-t_2+1/m), & \text{if,}\quad t\in[t_2-2/m,t_2-1/m]. \end{cases} \tag{2.30}$$

(ii) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k \in C_c^\infty(\mathbb{R}),\ \ \rho_k(t) = \rho_k(-t),\ \ \int_{-\infty}^{+\infty} \rho_k(t)dt = 1,\ \text{supp}\ \rho_k \subset [-1/k, 1/k]. \tag{2.31}$$

(iii) $(*)$ is the convolution product in the time variable, ie.,

$$(u * \rho_k)(x,t) = \int_{-\infty}^{+\infty} u(x, t-s)\rho_k(s)ds. \tag{2.32}$$

We take the scalar product of the function $v(x,t)$ in (2.29) with equation $(2.27)_1$, then integrate with respect to the time variable from 0 to T, and we have

$$X_{mk} + Y_{mk} = Z_{mk}, \tag{2.33}$$

where

$$\begin{cases} X_{mk} = \int_0^T \langle u''(t), v(t)\rangle dt, \\ Y_{mk} = -\int_0^T \langle \frac{\partial}{\partial x}\left(u_x(t) + \varepsilon u_{xtt}(t)\right), v(t)\rangle dt, \\ Z_{mk} = \int_0^T \langle \Phi(t), v(t)\rangle dt. \end{cases} \tag{2.34}$$

By using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$ we can show after some lengthy calculation

$$\begin{cases} \lim\limits_{k\to+\infty} X_{mk} = -\int_0^T \theta_m\theta_m' \left\|u'(t)\right\|^2 dt, \\ \lim\limits_{k\to+\infty} Y_{mk} = -\int_0^T \theta_m\theta_m' \left\|u_x(t)\right\|^2 dt - \varepsilon\int_0^T \theta_m\theta_m' \left\|u_x'(t)\right\|^2 dt + \int_0^T \theta_m^2 G(t) u'(0,t)dt, \\ \lim\limits_{k\to+\infty} Z_{mk} = \int_0^T \theta_m^2 \langle \Phi(t), u'(t)\rangle dt. \end{cases} \tag{2.35}$$

Letting $m \to \infty$, (2.33) – (2.35) yield

$$\begin{aligned} &\frac{1}{2}\left\|u'(t_2)\right\|^2 - \frac{1}{2}\left\|u'(t_1)\right\|^2 + \frac{1}{2}\left\|u_x(t_2)\right\|^2 - \frac{1}{2}\left\|u_x(t_1)\right\|^2 + \frac{\varepsilon}{2}\left\|u_x'(t_2)\right\|^2 - \frac{\varepsilon}{2}\left\|u_x'(t_1)\right\|^2 \\ &\quad + \int_{t_1}^{t_2} G(t)u'(0,t)dt = \int_{t_1}^{t_2} \langle \Phi(t), u'(t)\rangle dt,\ \text{a.e.},\ t_1\ t_2 \in (0,T),\ t_1 < t_2. \end{aligned} \tag{2.36}$$

From (2.36), using the weak lower semicontinuity of the functional $v \longmapsto \|v\|^2$, we obtain (2.28) by taking $t_2 = t$ and passing to the limit as $t_1 \to 0_+$.

In the case of $\tilde{u}_0 = \tilde{u}_1 = 0$, we prolong u, Φ, G by 0 as $t < 0$ and we deduce equality (2.36) is true for almost $t_1 < t_2 < T$. Taking $t_1 < 0$ in (2.36), its right-hand side is 0, we take $t_1 \to 0_-$, we have equality (2.28).

The proof of Lemma 2.3 is completed. ∎

Remark 2.1. Lemma 2.3 is a relative generalization of a lemma of Lions ([6], Lemma 6.1, p. 224).

We now prove that $\chi_1 = |u'|^{q-2}u'$. From (2.10) and (2.11) we deduce

$$\begin{aligned} 2\lambda\int_0^t \left\langle \left|u_m'(s)\right|^{q-2} u_m'(s), u_m'(s)\right\rangle ds &= 2\lambda\int_0^t \left\|u_m'(s)\right\|_{L^q}^q ds \\ &= \|\tilde{u}_{1m}\|^2 + \varepsilon\|\tilde{u}_{1mx}\|^2 + \|\tilde{u}_{0mx}\|^2 + h\tilde{u}_{0m}^2(0) + \frac{2K}{p}\|\tilde{u}_{0m}\|_{L^p}^p \\ &\quad - \left\|u_m'(t)\right\|^2 - \varepsilon\left\|u_{mx}'(t)\right\|^2 - \|u_{mx}(t)\|^2 - hu_m^2(0,t) - \frac{2K}{p}\|u_m(t)\|_{L^p}^p \end{aligned}$$

$$+2\int_0^t \left\langle F(s),u_m'(s)\right\rangle ds-2\int_0^t g(s)u_m'(0,s)ds. \tag{2.37}$$

Using Lemma 2.3, with $\Phi=F-K\,|u|^{p-2}\,u-\lambda\chi_1$, $G(t)=hu(0,t)+g(t)$, it follows from (2.8), (2.9), (2.21), (2.28), (2.37) that

$$\begin{aligned}
&2\lambda\limsup_{m\to\infty}\int_0^t \left\langle \left|u_m'(s)\right|^{q-2}u_m'(s),u_m'(s)\right\rangle ds\\
&\le \|\tilde u_1\|^2+\varepsilon\,\|\tilde u_{1x}\|^2+\|\tilde u_{0x}\|^2+h\tilde u_0^2(0)+\frac{2K}{p}\,\|\tilde u_0\|_{L^p}^p\\
&\quad-\liminf_{m\to\infty}\left\|u_m'(t)\right\|^2-\varepsilon\liminf_{m\to\infty}\left\|u_{mx}'(t)\right\|^2-\liminf_{m\to\infty}\left(\|u_{mx}(t)\|^2+hu_m^2(0,t)\right)\\
&\quad-\frac{2K}{p}\liminf_{m\to\infty}\|u_m(t)\|_{L^p}^p+2\int_0^t\left\langle F(s),u'(s)\right\rangle ds-2\int_0^t g(s)u'(0,s)ds\\
&\le \|\tilde u_1\|^2+\varepsilon\,\|\tilde u_{1x}\|^2+\|\tilde u_{0x}\|^2+h\tilde u_0^2(0)+\frac{2K}{p}\,\|\tilde u_0\|_{L^p}^p\\
&\quad-\left\|u'(t)\right\|^2-\varepsilon\left\|u_x'(t)\right\|^2-\|u_x(t)\|^2-hu^2(0,t)\\
&\quad-\frac{2K}{p}\,\|u(t)\|_{L^p}^p+2\int_0^t\left\langle F(s),u'(s)\right\rangle ds-2\int_0^t g(s)u'(0,s)ds\\
&\le \|\tilde u_1\|^2+\|\tilde u_{0x}\|^2+\varepsilon\,\|\tilde u_{1x}\|^2-\left\|u'(t)\right\|^2-\|u_x(t)\|^2-\varepsilon\left\|u_x'(t)\right\|^2\\
&\quad+2\int_0^t\left\langle F(s)-K|u(s)|^{p-2}u(s)-\lambda\chi_1(s),u'(s)\right\rangle ds-2\int_0^t\left(hu(0,s)+g(s)\right)u'(0,s)ds\\
&\quad+2\lambda\int_0^t\left\langle\chi_1(s),u'(s)\right\rangle ds\le 2\lambda\int_0^t\left\langle\chi_1(s),u'(s)\right\rangle ds.
\end{aligned} \tag{2.38}$$

Consider

$$\phi_m(t)=\int_0^t\left\langle \left|u_m'(s)\right|^{q-2}u_m'(s)-|v(s)|^{q-2}v(s),u_m'(s)-v(s)\right\rangle ds\ge 0, \tag{2.39}$$

for all $v\in L^q(Q_T)$.

Combining $(2.21)_{2-6}$, (2.38) and (2.39), we have

$$0\le\limsup_{m\to\infty}\phi_m(t)\le\int_0^t\left\langle\chi_1(s)-|v(s)|^{q-2}v(s),u'(s)-v(s)\right\rangle ds,\ \forall\, v\in L^q(Q_T). \tag{2.40}$$

In (2.40), choose $v(s)=u'(s)-\delta w$, with $\delta>0$ and $w\in L^q(Q_T)$. Apply the argument of Minty and Browder (see Lions [6], p. 172), we obtain $\chi_1=|u'|^{q-2}u'$.

The proof of existence is completed.

Step 4. *Uniqueness of the solution.* Assume now that $p\ge 2$ holds.

Let u, v be two weak solutions of the problem (1.1) – (1.4), such that

$$u,v\in L^\infty(0,T;V)\ \text{ and }\ u',v'\in L^\infty(0,T;V). \tag{2.41}$$

Then $w = u - v$ is the weak solution of the following problem

$$(2.42)\quad \begin{cases} w_{tt} - \varepsilon w_{xxtt} - w_{xx} = -\lambda\left(|u'|^{q-2}u' - |v'|^{q-2}v'\right) - K\left(|u|^{p-2}u - |v|^{p-2}v\right) = 0, \\ \varepsilon w_{xtt}(0,t) + w_x(0,t) = hw(0,t),\ w(1,t) = 0, \\ w(x,0) = w_t(x,0) = 0, \\ w,\ w' \in L^\infty(0,T;V). \end{cases}$$

Using Lemma 2.3 with $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi = -\lambda\left(|u'|^{q-2}u' - |v'|^{q-2}v'\right) - K\left(|u|^{p-2}u - |v|^{p-2}v\right)$, $G(t) = hw(0,t)$, we obtain

$$\begin{aligned} &\sigma(t) + 2\lambda\int_0^t \left\langle \left|u'(s)\right|^{q-2}u'(s) - \left|v'(s)\right|^{q-2}v'(s), u'(s) - v'(s)\right\rangle ds \\ (2.43)\qquad &= -2K\int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds,\ \text{a.e. } t \in [0,T], \end{aligned}$$

where

$$(2.44)\qquad \sigma(t) = \left\|w'(t)\right\|^2 + \|w_x(t)\|^2 + \varepsilon\left\|w_x'(t)\right\|^2 + hw^2(0,t).$$

Using the following inequality

$$(2.45)\quad \left|\,|x|^{p-2}x - |y|^{p-2}y\right| \le (p-1)M^{p-2}|x-y|,\ \forall x,y \in [-M,M],\ \forall M > 0,\ \forall p \ge 2,$$

with $M = \|u\|_{L^\infty(0,T;V)} + \|v\|_{L^\infty(0,T;V)}$, and note that

$$\int_0^t \left\langle \left|u'(s)\right|^{q-2}u'(s) - \left|v'(s)\right|^{q-2}v'(s), u'(s) - v'(s)\right\rangle ds \ge 0,$$

$$(2.46)\qquad \sigma(t) = \left\|w'(t)\right\|^2 + \|w_x(t)\|^2 + \varepsilon\left\|w_x'(t)\right\|^2 \ge 2\left\|w'(t)\right\|\|w_x(t)\|,$$

we deduce from (2.43), (2.46) that

$$\begin{aligned} \sigma(t) &\le -2K\int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds \\ (2.47)\qquad &\le 2K(p-1)M^{p-2}\int_0^t \|w(s)\|\left\|w'(s)\right\| ds \le K(p-1)M^{p-2}\int_0^t \sigma(s)ds. \end{aligned}$$

By Gronwall's lemma, it follows from (2.47) that $\sigma \equiv 0$, i.e., $u \equiv v$. Theorem 2.1 is proved completely. ∎

3. The regularity of solutions

In this section, we study the regularity of solutions of problem (1.1) – (1.4) corresponding to $(\tilde{u}_0, \tilde{u}_1) \in \left(V \cap H^2\right) \times \left(V \cap H^2\right)$.

Henceforth, we strengthen the hypotheses and assume that:

(H_1') $p \ge 2$, $q \ge 2$, $\lambda > 0$, $K > 0$, $\varepsilon > 0$, $h \ge 0$;

(H_2') $\tilde{u}_0,\ \tilde{u}_1 \in V \cap H^2$;

(H_3') $F,\ F' \in L^1(0,T;L^2)$;

(H_4') $g \in W^{2,1}(0,T)$.

First, we have the following theorem.

Theorem 3.1. *Let $T>0$. Suppose that $(H_1')-(H_4')$ hold. Then problem (1.1)-(1.4) has a unique weak solution*

$$u\in L^\infty\left(0,T;V\cap H^2\right),\ \text{such that } u_t,\ u_{tt}\in L^\infty\left(0,T;V\cap H^2\right). \tag{3.1}$$

Remark 3.1. The regularity obtained by (3.1) shows that problem (1.1)-(1.4) has a unique strong solution

$$u\in C^1\left(0,T;V\cap H^2\right),\ u_{tt}\in L^\infty\left(0,T;V\cap H^2\right). \tag{3.2}$$

Proof. The proof consists of four Steps as follows.

Step 1. The Faedo-Galerkin approximation. By the same argument as in Theorem 2.1, we obtain the approximate solution $u_m(t)$ of problem (1.1) – (1.4) in the form (2.7), where the coefficient functions c_{mj} satisfy the system (2.8), with

$$\tilde{u}_{0m}=\sum_{j=1}^{m}\alpha_{mj}w_j\to\tilde{u}_0 \text{ strongly in } V\cap H^2, \tag{3.3}$$

$$\tilde{u}_{1m}=\sum_{j=1}^{m}\beta_{mj}w_j\to\tilde{u}_1 \text{ strongly in } V\cap H^2. \tag{3.4}$$

Step 2. *A priori estimates* I. Using assumptions $(H_1')-(H_4')$, similarly, we get

$$\begin{aligned} S_m(t)=&\left\|u_m'(t)\right\|^2+\|u_{mx}(t)\|^2+\varepsilon\left\|u_{mx}'(t)\right\|^2+hu_m^2(0,t)\\ &+\frac{2K}{p}\|u_m(t)\|_{L^p}^p+2\lambda\int_0^t\left\|u_m'(s)\right\|_{L^q}^q ds\le C_T, \end{aligned}\tag{3.5}$$

for all $t\in[0,T]$ and for all m, and C_T always indicates a bound depending on T.

A priori estimates II. Now differentiating $(2.8)_1$ with respect to t, we have

$$\begin{aligned} &\left\langle u_m'''(t),w_j\right\rangle+\left\langle u_{mx}'(t)+\varepsilon u_{mx}'''(t),w_{jx}\right\rangle+K(p-1)\left\langle|u_m(t)|^{p-2}u_m'(t),w_j\right\rangle\\ &+\lambda(q-1)\left\langle\left|u_m'(t)\right|^{q-2}u_m''(t),w_j\right\rangle+\left(hu_m'(0,t)+g'(t)\right)w_j(0)=\left\langle F'(t),w_j\right\rangle, \end{aligned}\tag{3.6}$$

for all $1\le j\le m$.

Multiplying the j-th equation of (3.6) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, we obtain

$$\begin{aligned} X_m(t)=&X_m(0)+2g'(0)\tilde{u}_{1m}(0)+2\int_0^t\left\langle F'(s),u_m''(s)\right\rangle ds\\ &-2K(p-1)\int_0^t\left\langle|u_m(s)|^{p-2}u_m'(s),u_m''(s)\right\rangle ds\\ &-2g'(t)u_m'(0,t)+2\int_0^t g''(s)u_m'(0,s)ds\\ \equiv&X_m(0)+2g'(0)\tilde{u}_{1m}(0)+\sum_{j=1}^{4}J_j, \end{aligned}\tag{3.7}$$

where

$$\begin{aligned} X_m(t)=&\left\|u_m''(t)\right\|^2+\left\|u_{mx}'(t)\right\|^2+\varepsilon\left\|u_{mx}''(t)\right\|^2+h\left|u_m'(0,t)\right|^2\\ &+2\lambda(q-1)\int_0^t ds\int_0^1\left|u_m'(x,s)\right|^{q-2}\left|u_m''(x,s)\right|^2dx. \end{aligned}\tag{3.8}$$

First, we estimate $\eta_m = \|u''_m(0)\|^2 + \varepsilon \|u''_{mx}(0)\|^2$.

Letting $t \to 0_+$ in equation $(2.8)_1$, multiplying the result by $c''_{mj}(0)$, then

$$\begin{aligned}
&\|u''_m(0)\|^2 + \varepsilon \|u''_{mx}(0)\|^2 + \langle \tilde{u}_{0mx}, u''_{mx}(0)\rangle + \lambda \left\langle |\tilde{u}_{1m}|^{q-2}\tilde{u}_{1m}, u''_m(0)\right\rangle \\
&\quad + (h\tilde{u}_{0m}(0) + g(0))\, u''_m(0,0) \\
&\quad + K\left\langle |\tilde{u}_{0m}|^{p-2}\tilde{u}_{0m}, u''_m(0)\right\rangle = \langle F(0), u''_m(0)\rangle . \qquad (3.9)
\end{aligned}$$

Note that

$$\left|u''_m(0,0)\right| \le \|u''_m(0)\|_{C^0([0,1])} \le \|u''_{mx}(0)\| \le \frac{1}{\sqrt{\varepsilon}}\sqrt{\eta_m}. \qquad (3.10)$$

This implies that

$$\begin{aligned}
\eta_m &= \|u''_m(0)\|^2 + \varepsilon \|u''_{mx}(0)\|^2 \le \|\tilde{u}_{0mx}\| \, \|u''_{mx}(0)\| + |h\tilde{u}_{0m}(0) + g(0)| \, |u''_m(0,0)| \\
&\quad + \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1}\right\| + K\left\| |\tilde{u}_{0m}|^{p-1}\right\| + \|F(0)\| \right] \|u''_m(0)\| \\
&\le \frac{1}{2\gamma}\|\tilde{u}_{0mx}\|^2 + \frac{\gamma}{2}\|u''_{mx}(0)\|^2 + \frac{1}{2\gamma}(|h\tilde{u}_{0m}(0) + g(0)|)^2 + \frac{1}{2\varepsilon}\gamma\eta_m \\
&\quad + \frac{1}{2\gamma}\left[\lambda \left\| |\tilde{u}_{1m}|^{q-1}\right\| + K\left\| |\tilde{u}_{0m}|^{p-1}\right\| + \|F(0)\| \right]^2 + \frac{\gamma}{2}\|u''_m(0)\|^2 \\
&\le \frac{1}{2\gamma}\|\tilde{u}_{0mx}\|^2 + \frac{\gamma}{2\varepsilon}\eta_m + \frac{1}{2\gamma}(|h\tilde{u}_{0m}(0) + g(0)|)^2 + \frac{1}{2\varepsilon}\gamma\eta_m \\
&\quad + \frac{1}{2\gamma}\left[\lambda \left\| |\tilde{u}_{1m}|^{q-1}\right\| + K\left\| |\tilde{u}_{0m}|^{p-1}\right\| + \|F(0)\| \right]^2 + \frac{\gamma}{2}\eta_m \\
&\le \frac{1}{2\gamma}\|\tilde{u}_{0mx}\|^2 + \frac{1}{2\gamma}(|h\tilde{u}_{0m}(0) + g(0)|)^2 \\
&\quad + \frac{1}{2\gamma}\left[\lambda \left\| |\tilde{u}_{1m}|^{q-1}\right\| + K\left\| |\tilde{u}_{0m}|^{p-1}\right\| + \|F(0)\| \right]^2 \\
&\quad + \frac{\gamma}{2}\left[1 + \frac{2}{\varepsilon}\right]\eta_m, \ \mathit{for\ all}\ \gamma > 0. \qquad (3.11)
\end{aligned}$$

Choose $\gamma > 0$, such that $\frac{\gamma}{2}\left[1 + \frac{2}{\varepsilon}\right] \le \frac{1}{2}$, we have

$$\begin{aligned}
\eta_m &= \|u''_m(0)\|^2 + \varepsilon \|u''_{mx}(0)\|^2 \le \frac{1}{\gamma}\|\tilde{u}_{0mx}\|^2 + \frac{1}{\gamma}(|h\tilde{u}_{0m}(0) + g(0)|)^2 \\
&\quad + \frac{1}{\gamma}\left[\lambda \left\| |\tilde{u}_{1m}|^{q-1}\right\| + K\left\| |\tilde{u}_{0m}|^{p-1}\right\| + \|F(0)\| \right]^2 \le \overline{X}_0 \ \mathit{for\ all}\ m, \qquad (3.12)
\end{aligned}$$

where $\overline{X}_0$ is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h, $g(0)$ and ε.

By (3.4), (3.8) and (3.12), we get

$$\begin{aligned}
&X_m(0) + 2g'(0)\tilde{u}_{1m}(0) = \eta_m + \|\tilde{u}_{1mx}\|^2 + h\tilde{u}^2_{1mx}(0) + 2g'(0)\tilde{u}_{1m}(0) \\
&\le \overline{X}_0 + \|\tilde{u}_{1mx}\|^2 + h\tilde{u}^2_{1mx}(0) + 2g'(0)\tilde{u}_{1m}(0) \le \frac{1}{2}X_0, \ \mathit{for\ all}\ m, \qquad (3.13)
\end{aligned}$$

where X_0 is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h, $g(0)$ and ε.

A combination of (2.3), (2.14), (3.8) and the following inequalities

$$X_m(t) \geq \left\|u_m''(t)\right\|^2 + \left\|u_{mx}'(t)\right\|^2 + \varepsilon \left\|u_{mx}''(t)\right\|^2, \tag{3.14}$$

$$\left|u_m'(0,t)\right| \leq \left\|u_m'(t)\right\|_{C^0(\overline{\Omega})} \leq \left\|u_{mx}'(t)\right\| \leq \sqrt{X_m(t)}, \tag{3.15}$$

all terms on the right-hand side of (3.7) are estimated as follows

$$\begin{aligned} J_1 &= 2\int_0^t \left\langle F'(s), u_m''(s)\right\rangle ds \leq \left\|F'\right\|_{L^1\left(0,T;L^2\right)} + \int_0^t \left\|F'(s)\right\| X_m(s)\, ds \\ &\leq C_T + \int_0^t \left\|F'(s)\right\| X_m(s)\, ds; \end{aligned} \tag{3.16}$$

$$\begin{aligned} J_2 &= -2K(p-1)\int_0^t \left\langle |u_m(s)|^{p-2} u_m'(s), u_m''(s)\right\rangle ds \\ &\leq 2K(p-1)\int_0^t \|u_{mx}(s)\|^{p-2} \left\|u_m'(s)\right\| \left\|u_m''(s)\right\| ds \\ &\leq 2K(p-1)\int_0^t \left(\sqrt{S_m(s)}\right)^{p-2} \sqrt{S_m(s)}\sqrt{X_m(s)} ds \\ &\leq 2(p-1)\sqrt{C_T^{p-1}} \int_0^t \sqrt{X_m(s)} ds \leq C_T + \int_0^t X_m(s) ds; \end{aligned} \tag{3.17}$$

$$\begin{aligned} J_3 &= -2g'(t)u_m'(0,t) \leq 2\left|g'(t)\right| \left|u_m'(0,t)\right| \leq 2\left|g'(t)\right| \sqrt{X_m(t)} \\ &\leq \frac{1}{\beta}\left\|g'\right\|_{L^\infty(0,T)}^2 + \beta X_m(t) \leq \frac{1}{\beta} C_T + \beta X_m(t); \end{aligned} \tag{3.18}$$

$$\begin{aligned} J_4 &= 2\int_0^t g''(s)u_m'(0,s)ds \leq 2\int_0^t \left|g''(s)\right| \sqrt{X_m(s)} ds \\ &\leq \int_0^t \left|g''(s)\right| \left[1 + X_m(s)\right] ds \leq C_T + \int_0^t \left|g''(s)\right| X_m(s)\, ds, \end{aligned} \tag{3.19}$$

where C_T also indicates a bound depending on T and $C_T \geq \int_0^T |g''(s)|\, ds$.

Combining (3.7), (3.13), (3.16) – (3.19) and choose $\beta = \frac{1}{2}$, the result is

$$X_m(t) \leq C_T + 2\int_0^t \left(1 + \left|g''(s)\right| + \left\|F'(s)\right\|\right) X_m(s)\, ds, \quad 0 \leq t \leq T, \tag{3.20}$$

where C_T indicates a bound depending on T as above.

By Gronwall's lemma, we deduce from (3.20) that

$$X_m(t) \leq C_T \exp\left[2\int_0^T \left(1 + \left|g''(s)\right| + \left\|F'(s)\right\|\right) ds\right] \leq C_T, \ \textit{for all } t \in [0,T], \tag{3.21}$$

where C_T always indicates a bound depending on T.

Step 3. *Limiting process.* From (3.5), (3.8), (3.21), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}, \\ u_m' \to u' & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}, \\ u_m'' \to u'' & \text{in} \quad L^\infty(0,T;V) \quad \text{weakly*}. \end{cases} \tag{3.22}$$

By the compactness lemma of Lions ([6], p. 57), from (3.22), there exists a subsequence of $\{u_m\}$, denoted by the same symbol, such that

$$\begin{cases} u_m \to u & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \\ u'_m \to u' & \text{strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \end{cases} \tag{3.23}$$

Using again the inequality (2.45), with $M = C_T$, we deduce from (3.23) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \text{ strongly in } L^2(Q_T), \tag{3.24}$$

$$|u'_m|^{q-2}u'_m \to |u'|^{q-2}u' \text{ strongly in } L^2(Q_T). \tag{3.25}$$

Passing to the limit in (2.8), by (3.4), (3.22) – (3.25), we have u satisfying the problem

$$\begin{cases} \langle u''(t),v\rangle + \langle u_x(t)+\varepsilon u''_x(t),v_x\rangle + \lambda\left\langle |u'(t)|^{q-2}u'(t),v\right\rangle + K\left\langle |u(t)|^{p-2}u(t),v\right\rangle \\ \qquad + (hu(0,t)+g(t))\,v(0) = \langle F(t),v\rangle, \text{ for all } v\in V, \\ u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1. \end{cases} \tag{3.26}$$

On the other hand, (3.22) and $(3.26)_1$ yield

$$\frac{\partial^2}{\partial x^2}(u+\varepsilon u_{tt}) = u_{tt} + \lambda|u_t|^{q-2}u_t + K|u|^{p-2}u - F(t) \in L^\infty(0,T;L^2). \tag{3.27}$$

Hence

$$u+\varepsilon u_{tt} \equiv \Psi \in L^\infty(0,T;V\cap H^2). \tag{3.28}$$

Furthermore, by $u_{tt}+\frac{1}{\varepsilon}u \equiv \frac{1}{\varepsilon}\Psi$, it follows that

$$\begin{aligned} u(t) &= \cos\left(\sqrt{\frac{1}{\varepsilon}}t\right)\tilde{u}_0 + \sqrt{\varepsilon}\sin\left(\sqrt{\frac{1}{\varepsilon}}t\right)\tilde{u}_1 \\ &\quad + \sqrt{\varepsilon}\int_0^t \sin\left(\sqrt{\frac{1}{\varepsilon}}(t-s)\right)\frac{1}{\varepsilon}\Psi(s)ds \in L^\infty(0,T;V\cap H^2). \end{aligned} \tag{3.29}$$

Then

$$u_{tt} = \frac{1}{\varepsilon}(\Psi-u) \in L^\infty(0,T;V\cap H^2), \text{ and } u_t = \tilde{u}_1 + \int_0^t u_{tt}(s)ds \in L^\infty(0,T;V\cap H^2). \tag{3.30}$$

Thus u, u_t, $u_{tt} \in L^\infty(0,T;V\cap H^2)$ and the existence of the solution is proved completely.

Step 4. *Uniqueness of the solution*. Let u, v be two weak solutions of problem (1.1)-(1.4), such that

$$u,v\in C^1\left(0,T;V\cap H^2\right), \text{ with } u',\ v',\ u'',\ v''\in L^\infty\left(0,T;V\cap H^2\right). \tag{3.31}$$

Then $w = u - v$ verifies

$$\begin{cases} \langle w''(t),z\rangle + \langle w_x(t)+\varepsilon w''_x(t),z_x\rangle + \lambda\left\langle |u'(t)|^{q-2}u'(t) - |v'(t)|^{q-2}v(t),z\right\rangle \\ \qquad + hw(0,t)z(0) = -K\left\langle |u(t)|^{p-2}u(t) - |v(t)|^{p-2}v(t),z\right\rangle, \text{ for all } z\in V, \\ w(0)=w'(0)=0. \end{cases} \tag{3.32}$$

We take $z = w = u - v$ in (3.32) and integrating with respect to t, we obtain

$$\sigma(t) = -2K\int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds, \tag{3.33}$$

where

$$\sigma(t) = \left\|w'(t)\right\|^2 + \varepsilon\left\|w_x'(t)\right\|^2 + \|w_x(t)\|^2 + hw^2(0,t)$$

$$+2\lambda\int_0^t \left\langle \left|u'(s)\right|^{q-2}u'(s) - \left|v'(s)\right|^{q-2}v'(s), u'(s) - v'(s)\right\rangle ds. \tag{3.34}$$

Using again the inequality (2.45), with $M = \max\{\|u\|_{L^\infty(0,T;V)}, \|v\|_{L^\infty(0,T;V)}\}$, we get

(3.35)
$$\left|\,|u(x,s)|^{p-2}u(x,s) - |v(x,s)|^{p-2}v(x,s)\right| \le (p-1)M^{p-2}|w(x,s)|, \text{ for all } (x,s) \in Q_T,$$

and the following inequalities

$$\int_0^t \left\langle \left|u'(s)\right|^{q-2}u'(s) - \left|v'(s)\right|^{q-2}v'(s), u'(s) - v'(s)\right\rangle ds \ge 0,$$

$$\sigma(t) \ge \left\|w'(t)\right\|^2 + \varepsilon\left\|w_x'(t)\right\|^2 + \|w_x(t)\|^2 \ge 2\left\|w'(t)\right\|\|w_x(t)\|, \tag{3.36}$$

so

$$\sigma(t) \le -2K\int_0^t \left\langle |u(s)|^{p-2}u(s) - |v(s)|^{p-2}v(s), w'(s)\right\rangle ds$$

$$\le 2K(p-1)M^{p-2}\int_0^t \|w(s)\|\left\|w'(s)\right\| ds \le K(p-1)M^{p-2}\int_0^t \sigma(s)ds. \tag{3.37}$$

By Gronwall's lemma, it follows from (3.37) that $\sigma \equiv 0$, i.e., $u \equiv v$.
Theorem 3.1 is proved completely. ∎

Next, we continue to consider the regularity of solution of problem (1.1)-(1.4), corresponding to $p = q = 2$.

$$\begin{cases} Lu \equiv u'' - u_{xx} - \varepsilon u_{xx}'' + \lambda u' + Ku = F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ L_0 u \equiv \varepsilon u_x''(0,t) + u_x(0,t) - hu(0,t) = g(t), \\ u(1,t) = 0, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{3.38}$$

For this purpose, we also assume that $\varepsilon > 0$, $K > 0$, $\lambda > 0$, $h \ge 0$. Furthermore, we will impose stronger assumptions. With $r \in N$, we assume that

$(H_2^{[r]})$ $\tilde{u}_0, \tilde{u}_1 \in V \cap H^{r+2}$.

$(H_3^{[r]})$ The function F satisfies

$$\begin{cases} \frac{\partial^j F}{\partial t^j} \in L^\infty(0,T;V \cap H^r),\ \ 0 \le j \le r, \\ \frac{\partial^{r+1} F}{\partial t^{r+1}} \in L^1(0,T;V \cap H^r). \end{cases}$$

$(H_4^{[r]})$ $g \in W^{r+1,1}(0,T)$, $r \ge 1$.

First, we define the sequences $\{\tilde{u}_0^{[k]}\}$, $\{\tilde{u}_1^{[k]}\}$, $k = 0,1,...,r+2$ by the following recurrent formulas

$$\begin{cases} \tilde{u}_0^{[0]} = \tilde{u}_0,\ \tilde{u}_1^{[0]} = \tilde{u}_1, \\ \tilde{u}_0^{[k]} = \tilde{u}_1^{[k-1]},\ k \in \{1,2,...,r+1\},\ r \ge 1, \end{cases} \tag{3.39}$$

where $\tilde{u}_0^{[k]}$ is defined by the following problem

$$(3.40)\quad \begin{cases} -\varepsilon\Delta\tilde{u}_0^{[k]} + \tilde{u}_0^{[k]} = \frac{\partial^{k-2}F}{\partial t^{k-2}}(\cdot,0) + \Delta\tilde{u}_0^{[k-2]} - K\tilde{u}_0^{[k-2]} - \lambda\tilde{u}_1^{[k-2]} \equiv \Phi^{[k]},\ 0<x<1, \\ \varepsilon\tilde{u}_{0x}^{[k]}(0) = -\tilde{u}_{0x}^{[k-2]}(0) + h\tilde{u}_0^{[k-2]}(0) + \frac{d^{k-2}g}{dt^{k-2}}(0) \equiv \Phi_0^{[k]},\ \ \tilde{u}_0^{[k]}(1) = 0. \end{cases}$$

Then, we have the following Lemma.

Lemma 3.1. *Suppose that* $(H_2^{[r]}) - (H_4^{[r]})$ *hold. Then problem has a unique weak solution* $\tilde{u}_0^{[k]} \in V$. *Furthermore, we have* $\tilde{u}_0^{[k]} \in V \cap H^{r+2}$, $k = 2,3,...,r+1$.

Proof. The weak solution of problem (3.40) is obtained from the following variational problem.

Find $U \in V$ such that

$$(3.41)\qquad a(U,w) = \langle ,w\rangle,\ forall\ w \in V,$$

where

$$(3.42)\qquad \begin{cases} a(U,w) = \langle \varepsilon U_x, w_x\rangle + \langle U,w\rangle, \\ \langle ,w\rangle = \langle \Phi^{[k]},w\rangle - \Phi_0^{[k]}w(0). \end{cases}$$

Using the Lax-Milgram's theorem, Problem (3.41) has a unique weak solution $\tilde{u}_0^{[k]} \in V$. We shall prove that

$$(3.43)\qquad \tilde{u}_0^{[k]} \in V \cap H^{r+2},\ k \in \{1,2,...,r+1\},\ r \geq 1.$$

(i) $k=1 : \tilde{u}_0^{[1]} = \tilde{u}_1^{[0]} = \tilde{u}_1 \in V \cap H^{r+2}$. (by $(H_2^{[r]})$).

(ii) Suppose by induction that $\tilde{u}_0^{[1]},...,\tilde{u}_0^{[k-1]} \in V \cap H^{r+2}$ hold. We shall prove that $\tilde{u}_0^{[k]} \in V \cap H^{r+2}$ holds.

In fact, by $(H_3^{[r]})$, we have $\frac{\partial^{k-2}F}{\partial t^{k-2}}(\cdot,0) \in V \cap H^r$, $2 \leq k \leq r+2$. Hence, by induction we obtain

$$(3.44)\qquad \Phi^{[k]} = \frac{\partial^{k-2}F}{\partial t^{k-2}}(\cdot,0) + \Delta\tilde{u}_0^{[k-2]} - K\tilde{u}_0^{[k-2]} - \lambda\tilde{u}_0^{[k-1]} \in V \cap H^r.$$

On the other hand, by $\tilde{u}_0^{[k]} \in V$ and (3.44), we obtain

$$(3.45)\qquad \varepsilon\Delta\tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in V.$$

Then $\tilde{u}_0^{[k]} \in V \cap H^3$.

Similarly, we have also $\tilde{u}_0^{[k]} \in V \cap H^{2s+1}$, with $s \in \mathbb{N}$, $2s-1 \leq r < 2s+1$. Then

$$(3.46)\qquad \varepsilon\Delta\tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in V \cap H^r.$$

Thus

$$(3.47)\qquad \tilde{u}_0^{[k]} \in V \cap H^{r+2}.$$

Lemma 3.1 is proved completely. ∎

Now, formally differentiating problem (3.38) with respect to time up to order r and letting $u^{[r]}=\frac{\partial^r u}{\partial t^r}$ we are led to consider the solution $u^{[r]}$ of problem $(Q^{[r]})$:

$$(Q^{[r]})\begin{cases} Lu^{[r]}=\frac{\partial^r F}{\partial t^r}(x,t), \ \ (x,t)\in Q_T, \\ L_0 u^{[r]}=\frac{d^r g}{dt^r}(t), \ \ u^{[r]}(1,t)=0, \\ u^{[r]}(0)=\tilde{u}_0^{[r]}, \ \ u_t^{[r]}(0)=\tilde{u}_1^{[r]}. \end{cases} \tag{3.48}$$

From the assumptions $(H_2^{[r]})-(H_4^{[r]})$ we deduce that $\tilde{u}_0^{[r]}$, $\tilde{u}_1^{[r]}$, $\frac{\partial^r F}{\partial t^r}$ and $\frac{d^r g}{dt^r}$ satisfy the conditions of Theorem 3.1. So, the problem $(Q^{[r]})$ has a unique weak solution $u^{[r]}$ such that

$$u^{[r]}\in C^1\left(0,T;V\cap H^2\right), \ \ u_{tt}^{[r]}\in L^\infty\left(0,T;V\cap H^2\right). \tag{3.49}$$

Moreover, from the uniqueness of a weak solution we have $u^{[r]}=\frac{\partial^r u}{\partial t^r}$. Hence we deduce from (3.49) that the solution u of problem (3.38) satisfy

$$u\in C^{r+1}\left(0,T;V\cap H^2\right), \ \frac{\partial^{r+2}u}{\partial t^{r+2}}\in L^\infty\left(0,T;V\cap H^2\right). \tag{3.50}$$

Next we shall prove by induction on r that

$$u\in C^{r+1}\left(0,T;V\cap H^{r+2}\right), \ \frac{\partial^{r+2}u}{\partial t^{r+2}}\in L^\infty(0,T;V\cap H^{r+2}), \ r\geq 1. \tag{3.51}$$

(i) In the case of $r=1$, the proof of (3.51) is easy, hence we omit the details. We only prove with $r\geq 2$.

(ii) Suppose by induction that (3.51) holds for $r-1$. i.e.,

$$u\in C^{r}\left(0,T;V\cap H^{r+1}\right), \ \frac{\partial^{r+1}u}{\partial t^{r+1}}\in L^\infty(0,T;V\cap H^{r+1}). \tag{3.52}$$

We need prove that (3.51) holds. To achieve this, we only have to prove that

$$\begin{cases} \frac{\partial^r u}{\partial t^r}\in L^\infty(0,T;V\cap H^{r+2}), \\ \frac{\partial^{r+1} u}{\partial t^{r+1}}\in L^\infty(0,T;V\cap H^{r+2}), \\ \frac{\partial^{r+2} u}{\partial t^{r+2}}\in L^\infty(0,T;V\cap H^{r+2}), \ r\geq 1. \end{cases} \tag{3.53}$$

By $(Q^{[r]})_1$, we have

$$\left(u^{[r]}-\varepsilon\Delta u^{[r]}\right)''-\Delta u^{[r]}+Ku^{[r]}+\lambda u_t^{[r]}=\frac{\partial^r F}{\partial t^r}. \tag{3.54}$$

Put

$$\begin{cases} W=u^{[r]}-\varepsilon\Delta u^{[r]}, \\ \tilde{w}_0=\tilde{u}_0^{[r]}-\varepsilon\Delta\tilde{u}_0^{[r]}, \\ \tilde{w}_1=\tilde{u}_1^{[r]}-\varepsilon\Delta\tilde{u}_1^{[r]}=\tilde{u}_0^{[r+1]}-\varepsilon\Delta\tilde{u}_0^{[r+1]}, \end{cases} \tag{3.55}$$

it follows that

$$\begin{cases} W''+\frac{1}{\varepsilon}W=\frac{1}{\varepsilon}u^{[r]}-Ku^{[r]}-\lambda u_t^{[r]}+\frac{\partial^r F}{\partial t^r}\equiv\Psi^{[r]}\in L^\infty(0,T;V\cap H^r), \\ W(0)=\tilde{w}_0\in V\cap H^r, \\ W'(0)=\tilde{w}_1\in V\cap H^r. \end{cases} \tag{3.56}$$

Thus

$$W(t) = \cos\left(\sqrt{\frac{1}{\varepsilon}}t\right)\tilde{w}_0 + \sqrt{\varepsilon}\sin\left(\sqrt{\frac{1}{\varepsilon}}t\right)\tilde{w}_1$$

$$+\sqrt{\varepsilon}\int_0^t \sin\left(\sqrt{\frac{1}{\varepsilon}}(t-s)\right)\Psi^{[r]}(s)ds \in L^\infty(0,T;V\cap H^r). \tag{3.57}$$

By (3.52) and (3.57), it follows that

$$\Delta u^{[r]} = \frac{1}{\varepsilon}u^{[r]} - \frac{1}{\varepsilon}W \in L^\infty(0,T;V\cap H^r). \tag{3.58}$$

Thus

$$u^{[r]} \in L^\infty(0,T;V\cap H^{r+2}). \tag{3.59}$$

On the other hand, by $(3.56)_1$, we obtain

$$W'' = -\frac{1}{\varepsilon}W + \Psi^{[r]} \in L^\infty(0,T;V\cap H^r). \tag{3.60}$$

It follows from (3.49), (3.60) and $r \geq 2$, that

$$\Delta u_{tt}^{[r]} = \frac{1}{\varepsilon}u_{tt}^{[r]} - \frac{1}{\varepsilon}W'' \in L^\infty(0,T;V\cap H^2). \tag{3.61}$$

Consequently

$$u_{tt}^{[r]} \in L^\infty(0,T;V\cap H^4). \tag{3.62}$$

Similarly, we have also $u_{tt}^{[r]} \in L^\infty(0,T;H^{2s})$, with $s\in\mathbb{N}$, $2s-2\leq r<2s$. Then

$$\Delta u_{tt}^{[r]} = \frac{1}{\varepsilon}u_{tt}^{[r]} - \frac{1}{\varepsilon}W'' \in L^\infty(0,T;V\cap H^r). \tag{3.63}$$

So

$$u_{tt}^{[r]} \in L^\infty(0,T;V\cap H^{r+2}). \tag{3.64}$$

On the other hand

$$u_t^{[r]} = \tilde{u}_1^{[r]} + \int_0^t u_{tt}^{[r]}(s)ds \in L^\infty(0,T;V\cap H^{r+2}). \tag{3.65}$$

Combining (3.59), (3.64) and (3.65), by induction arguments on r, we conclude that (3.51) holds and the following theorem is proved.

Theorem 3.2. *Let $(H_2^{[r]}) - (H_4^{[r]})$ hold. Then the unique solution $u(x,t)$ of problem (3.38) satisfies (3.51).*

4. Asymptotic behavior of solutions as $\varepsilon \to 0_+$

In this part, we assume that $p>2$, $q>1$, $\lambda>0$, $K>0$, $h\geq 0$ and $(\tilde{u}_0, \tilde{u}_1, F)$ satisfy the assumptions (H_2), (H_3). Let $\varepsilon>0$. By theorem 2.3, the problem (1.1) – (1.4) has a unique weak solution $u=u_\varepsilon$ depending on ε.

We consider the following perturbed problem, where ε is a small parameter:

(4.1)
$$(P_\varepsilon)\begin{cases} u_{tt}-u_{xx}-\varepsilon u_{xxtt}+\lambda\left|u_t\right|^{q-2}u_t+K\left|u\right|^{p-2}u=F(x,t),\ 0<x<1,\ 0<t<T,\\ \varepsilon u_{xtt}(0,t)+u_x(0,t)=hu(0,t)+g(t),\ \ u(1,t)=0,\\ u(0)=\tilde{u}_0,\ u'(0)=\tilde{u}_1. \end{cases}$$

We shall study the asymptotic behavior of the solution u_ε of problem (P_ε) as $\varepsilon\to 0_+$.

Theorem 4.1. *Let $T>0$, $p>2$, $q>1$, $\lambda>0$, $K>0$. Let (H_2), (H_3) hold. Then*

(i) *The problem (P_0) corresponding to $\varepsilon=0$ has a unique weak solution $\bar{u}_0$ satisfying*

$$\bar{u}_0\in L^\infty(0,T;V),\ \bar{u}_0'\in L^\infty\left(0,T;L^2\right). \tag{4.2}$$

(ii) *If $\bar{u}_0''\in L^2(0,T;H^2)$, then solution u_ε converges strongly in W_T to $\bar{u}_0$, as $\varepsilon\to 0_+$, where*

$$W_T=\{v\in L^\infty(0,T;V): v'\in L^\infty(0,T;L^2)\}. \tag{4.3}$$

Furthermore, we have the estimation

$$\left\|u_\varepsilon'-\bar{u}_0'\right\|_{L^\infty(0,T;L^2)}+\left\|u_\varepsilon-\bar{u}_0\right\|_{L^\infty(0,T;V)}\le C_T\sqrt{\varepsilon}, \tag{4.4}$$

where C_T is a posistive constant depending only on T.

Proof. First, we note that if the small parameter $\varepsilon>0$ satisfy $0<\varepsilon<1$ then a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.1 for problem (P_ε) satisfy

$$\left\|u_m'(t)\right\|^2+\left\|u_{mx}(t)\right\|^2+\varepsilon\left\|u_{mx}'(t)\right\|^2+\left\|u_m(t)\right\|_{L^p}^p+\int_0^t\left\|u_m'(s)\right\|_{L^q}^q ds\le C_T, \tag{4.5}$$

for all $t\in[0,T]$ and for all m, and C_T is a constant depending only on T, p, q, λ, K, $\tilde{u}_0$, $\tilde{u}_1$, F (independent of ε). Hence, the limit $u=u_\varepsilon$ of the sequence $\{u_m\}$ as $m\to+\infty$, in suitable function spaces is a unique weak solution of problem (P_ε) satisfying

$$\left\|u_\varepsilon'(t)\right\|^2+\left\|u_{\varepsilon x}(t)\right\|^2+\varepsilon\left\|u_{\varepsilon x}'(t)\right\|^2+\left\|u_\varepsilon(t)\right\|_{L^p}^p+\int_0^t\left\|u_\varepsilon'(s)\right\|_{L^q}^q ds\le C_T, \tag{4.6}$$

for all $t\in[0,T]$ and for all $\varepsilon\in(0,1)$.

Let $\{\varepsilon_m\}$ be a sequence such that $\varepsilon_m>0$, $\varepsilon_m\to 0$ as $m\to+\infty$. We put $u_m=u_{\varepsilon_m}$, we deduce from (4.6) that, there exists a subsequence of the sequence $\{u_m\}$ still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m\to\bar{u}_0 & \text{in} & L^\infty(0,T;V) & \text{weakly*},\\ u_m'\to\bar{u}_0' & \text{in} & L^\infty(0,T;L^2) & \text{weakly*},\\ \sqrt{\varepsilon_m}u_m'\to\zeta & \text{in} & L^\infty(0,T;V) & \text{weakly*},\\ u_m\to\bar{u}_0 & \text{in} & L^\infty(0,T;L^p) & \text{weakly*},\\ u_m'\to\bar{u}_0' & \text{in} & L^q(Q_T) & \text{weakly},\\ \left|u_m\right|^{p-2}u_m\to\chi_0 & \text{in} & L^\infty(0,T;L^{p'}) & \text{weakly*},\\ \left|u_m'\right|^{p-2}u_m'\to\chi_1 & \text{in} & L^{q'}(Q_T) & \text{weakly}. \end{cases} \tag{4.7}$$

By the compactness lemma of Lions [6, p. 57], $(4.7)_{1,2}$ lead to the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m\to\bar{u}_0 \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T. \tag{4.8}$$

It follows from $(4.7)_{2,3}$, that $\zeta = 0$. Hence, we obtain from $(4.7)_3$ that

$$\sqrt{\varepsilon_m} u'_m \to 0 \text{ in } L^\infty(0,T;V) \text{ weakly*}. \tag{4.9}$$

Similarly

$$|u_m|^{p-2} u_m \to |\bar{u}_0|^{p-2} \bar{u}_0 = \chi_0 \text{ strongly in } L^2(Q_T), \tag{4.10}$$

and

$$\chi_1 = |\bar{u}'_0|^{q-2} \bar{u}'_0. \tag{4.11}$$

By passing to the limit, as in the proof of Theorem 2.1, we conclude that $\bar{u}_0$ is a unique weak solution of problem (P_0) corresponding to $\varepsilon = 0$ satisfying

$$\bar{u}_0 \in L^\infty(0,T;V), \ \bar{u}'_0 \in L^\infty\left(0,T;L^2\right). \tag{4.12}$$

(ii) Put $u = u_\varepsilon - \bar{u}_0$, then u is the weak solution of the following problem

$$\begin{cases} u'' - \Delta u - \varepsilon \Delta u'' + \lambda \left(|u'_\varepsilon|^{q-2} u'_\varepsilon - \left|\bar{u}'_0\right|^{q-2} \bar{u}'_0 \right) + K \left(|u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0 \right) \\ \qquad = \varepsilon \Delta \bar{u}''_0, \ 0 < x < 1, \ 0 < t < T, \\ \varepsilon u''_x(0,t) + u_x(0,t) = h u(0,t) - \varepsilon \bar{u}''_0(0,t), \ \ u(1,t) = 0, \\ u(0) = u'(0) = 0. \end{cases} \tag{4.13}$$

Using again Lemma 2.3, in a manner similar to the above part, we obtain

$$\begin{aligned} \sigma(t) &= 2\varepsilon \int_0^t \left\langle \Delta \bar{u}''_0, u'(s) \right\rangle ds + 2\varepsilon \int_0^t \bar{u}''_{0x}(0,s) u'(0,s) ds \\ &\quad - 2K \int_0^t \left\langle |u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0, u'(s) \right\rangle ds, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned} \sigma(t) &= \left\|u'(t)\right\|^2 + \varepsilon \left\|u'_x(t)\right\|^2 + \|u_x(t)\|^2 + h u^2(0,t) \\ &\quad + 2\lambda \int_0^t \left\langle \left|u'_\varepsilon\right|^{q-2} u'_\varepsilon - \left|\bar{u}'_0\right|^{q-2} \bar{u}'_0, u'(s) \right\rangle ds. \end{aligned} \tag{4.15}$$

Note that

$$\begin{cases} \int_0^t \left\langle |u'_\varepsilon|^{q-2} u'_\varepsilon - \left|\bar{u}'_0\right|^{q-2} \bar{u}'_0, u'(s) \right\rangle ds \geq 0, \\ \sigma(t) \geq \varepsilon \left\|u'_x(t)\right\|^2, \\ \sigma(t) \geq \left\|u'(t)\right\|^2 + \|u_x(t)\|^2 \geq 2 \|u_x(t)\| \left\|u'(t)\right\|. \end{cases} \tag{4.16}$$

By (2.45), (4.6), (4.16), we estimate all terms in the right – hand side of (4.14) as follows

$$\begin{aligned} & 2\varepsilon \int_0^t \left\langle \Delta \bar{u}''_0(s), u'(s) \right\rangle ds \leq 2\varepsilon \int_0^t \left\|\Delta \bar{u}''_0(s)\right\| \left\|u'(s)\right\| ds \\ & \leq 2\varepsilon \int_0^t \left\|\bar{u}''_0(s)\right\|_{H^2} \left\|u'(s)\right\| ds \leq \varepsilon^2 \int_0^t \left\|\bar{u}''_0(s)\right\|_{H^2}^2 ds + \int_0^t \left\|u'(s)\right\|^2 ds \\ & \leq \varepsilon^2 \left\|\bar{u}''_0\right\|^2_{L^2(0,T;H^2)} + \int_0^t \sigma(s) ds; \end{aligned} \tag{4.17}$$

$$2\varepsilon \int_0^t \bar{u}''_{0x}(0,s) u'(0,s) ds \leq 2\sqrt{2}\varepsilon \int_0^t \left\|\bar{u}''_{0x}(s)\right\|_{H^1} \left\|u'_x(s)\right\| ds \leq 2\sqrt{2}\varepsilon \int_0^t \left\|\bar{u}''_0(s)\right\|_{H^2} \left\|u'_x(s)\right\| ds$$

(4.18)

$$\leq 2\varepsilon \int_0^t \left\|\bar{u}_0''(s)\right\|_{H^2}^2 ds + \varepsilon \int_0^t \left\|u_x'(s)\right\|^2 ds \leq 2\varepsilon \left\|\bar{u}_0''\right\|_{L^2(0,T;H^2)}^2 + \int_0^t \sigma(s)ds;$$

$$-2K \int_0^t \left\langle |u_\varepsilon|^{p-2} u_\varepsilon - |\bar{u}_0|^{p-2} \bar{u}_0, u'(s) \right\rangle ds \leq 2K(p-1)C_T^{p-2} \int_0^t \|u(s)\| \left\|u'(s)\right\| ds$$

$$\leq K(p-1)C_T^{p-2} \int_0^t \sigma(s)ds. \tag{4.19}$$

Combining (4.14), (4.17)-(4.19), it implies that

$$\sigma(t) \leq 3\varepsilon \left\|\bar{u}_0''\right\|_{L^2(0,T;H^2)}^2 + \left[2 + K(p-1)C_T^{p-2}\right] \int_0^t \sigma(s)ds. \tag{4.20}$$

By Gronwall's lemma, (4.20) leads to

$$\sigma(t) \leq 3\varepsilon \left\|\bar{u}_0''\right\|_{L^2(0,T;H^2)}^2 \exp(T \left[2 + K(p-1)C_T^{p-2}\right]) \equiv \bar{C}_T \varepsilon, \ \forall t \in [0,T]. \tag{4.21}$$

Hence

$$\left\|u_\varepsilon' - \bar{u}_0'\right\|_{L^\infty(0,T;L^2)} + \|u_\varepsilon - \bar{u}_0\|_{L^\infty(0,T;H^1)} \leq C_T \sqrt{\varepsilon}, \tag{4.22}$$

where C_T is a constant depending only on T. Theorem 4.1 is proved completely. ∎

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