

Modelling Actuarial Present Value under Stochastic Discount Function

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Abstract

Most actuarial calculations are based on simplifying assumptions with regard to force of mortality and discount function. Classical, actuarial calculations are determined by the assumption that the discount function is constant over time. To obtain a more realistic assessment, it would be beneficial when the discount function are fluctuated. This paper compares the premium of the term and the endowment insurance which calculated based on constant discount function to stochastic discount function, when the discounting is presented by a stochastic differential equation as in the Hull-White model. In this case, time to maturity in financial valuation models is adjusted with $T(x)$, a continuous random variable representing future lifetime of a life-aged- x . Furthermore, upper bound for the distribution function of present value in convexity order is calculated.

Key words : stochastic discount function, actuarial present value, convexity order.

1 Introduction

The stochastic nature of actuarial processes is not only determined by remaining life times but is also a consequence of the randomness of interest rates. Historically, the theory of life contingencies has ignored explicit consideration of chance fluctuations in interest rate. In contrast, Pollard (1971 and 1976), Boyle (1976), Panjer and Bellhouse (1981) and Dhaene (1989) observed the fluctuations of interest rate by treating interest rate as random variable and employed times series models such as Moving Average (MA), Auto Regressive (AR), Auto Regressive Moving Average (ARMA) and Auto Regressive Integrated Moving Average (ARIMA). In this paper, we observed the fluctuations of interest rate by using the interest rate derivatives model, namely the Hull-White model.

The present value of a series of n payment c_i at times τ_i ($i = 1, 2, \dots, n$) with a discount process $X(\tau)$ is given by

$$V_0 = \sum_{i=1}^n c_i e^{-X(\tau_i)} \quad (1)$$

where

$$X(\tau) = \int_0^\tau r(s) ds \quad (2)$$

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hence

$$V_0 = \sum_{i=1}^n c_i e^{\int_0^{\tau} r(s) ds} \quad (3)$$

The stochastic differential equation for describing the behavior of $r(s)$ is represented by the Hull-White model;

$$dr_t = (\vartheta_t - cr_t)dt + \sigma dW \quad (4)$$

where c and σ are constants. The function ϑ_t can be calculated from the initial term structure:

$$\begin{aligned} \vartheta(t) &= \frac{\partial \ln f^M(0, t)}{\partial T_x} + cf^M(0, t) + \frac{\sigma^2}{2c} (1 - e^{-ct})^2 \\ f^M(0, T_x) &= -\frac{\partial \ln P^M(0, T_x)}{\partial T_x} \end{aligned}$$

$f^M(0, T_x)$ constitutes the market instantaneous forward rate, whereas $P^M(0, T_x)$ represents the market discount factor.

The instantaneous spot interest rate, r_t , is normally distributed with mean and variance given respectively by

$$\begin{aligned} E\{r_t\} &= r_0 e^{-c(t-s)} + \alpha(t) - \alpha(s)e^{-c(t-s)} \\ Var\{r_t\} &= \frac{\sigma^2}{2c} (1 - e^{-2c(t-s)}) \end{aligned}$$

where

$$\alpha_t = f^M(0, t) + \frac{\sigma^2}{2a} [1 - e^{-ct}]^2$$

Concerning with convex upper bound, $X_{T_x} = \int_0^{T_x} r_t dt$ random variable is normally distributed with mean

$$\mu_{T_x} = \frac{r_0}{c} (1 - e^{-cT_x}) - \ln P^M(0, T_x) + \frac{\sigma^2}{2c^2} \left(T_x + \frac{2}{c} e^{-cT_x} - \frac{1}{2c} e^{-2cT_x} - \frac{3}{2} c \right)$$

and variance

$$\sigma_{T_x}^2 = \frac{\sigma^2}{c^2} \left(T_x + \frac{2}{c} e^{-cT_x} - \frac{1}{2c} e^{-2cT_x} - \frac{3}{2} c \right)$$

As solved by Hull-White, bond prices at time t are given by

$$P(t, T_x) = A(t, T_x) e^{-B(t, T_x)r}$$

where

$$B(t, T_x) = \frac{1}{c} (1 - e^{-c(T_x - t)})$$

and

$$\ln A(t, T_x) = \ln \frac{P(0, T_x)}{P(0, t)} - B(t, T_x) \frac{\partial \ln P(0, t)}{\partial t} - \frac{1}{4c} \sigma^2 (1 - e^{-2ct}) B^2(t, T_x)$$

Upper bound for the distribution function of V_0 is needed to determine the fair premium, therefore we only require the information of the distribution function of the $X(\tau_n)$

The remainder of this paper is organized as follows. Section 2 is started with determination of the actuarial quantities based on interest rate derivatives model, the Hull-White model. In section 3, we provide some mathematical background about ordering random variable, comonotonocity and *Fréchet* spaces. In section, 4 we simulate and give empirical results and sensitivity analysis with respect to parameters of insurance contract. Section 5 is closed with a summary.

2 Actuarial Quantities

The benefit and insurance payment schedule depends on the interval between the time of policy determination and time of death of the insured. The symbol (x) will be used to denote a life aged x . The time and amount of this payment will be functions of the random variable $T(x)$ as a continuous random variable representing the future lifetime of (x) or $K(x)$ as a discrete random variable representing the number of completed future years lived by (x) . We have a benefit function b_t and a discount function v_t in continuous time. In this model v_t is the discount function of interest rate at time t . The present value function of the benefit is

$$z_t = b_t v_t$$

The maturity period is random and depends on the future life time. The present value of the policy from the benefit payment is z_T . This random variable will be denoted by

$$Z = b_T v_T$$

where

$$v_T = (1 + i)^{-T}$$

The discount factor displays two characteristics: interest rate risk and mortality risk. The expected present value of the random value Z is called the actuarial present value.

2.1 Varying Interest Rates in Simple Contract

In this section we look at the effect of changing interest rates on the present value of a single payment and an annuity-due. We distinguish payments on a yearly base and continuous payments.

In the discrete case the effective interest rates are denoted by i_1, i_2, i_3, \dots . For notational convenience we take $i_0 = 0$. The corresponding sequence of forces of interest is $\delta_1, \delta_2, \dots$ and the respective sequence of discount factors is v_1, v_2, \dots .

In continuous time $\delta(t)$ denotes the force of interest at time t . The compound interest over the time interval $[0, t]$ is given by

$$1 + i(t) = \exp\left(\int_0^t \delta(s) ds\right)$$

and the discount rate is

$$v(t) = \frac{1}{1 + i(t)} = \exp\left(-\int_0^t \delta(s) ds\right)$$

2.2 The Actuarial Present Value

The present value of a **discrete benefit payment** is

$$\begin{aligned} A_{\overline{n}|}^* &= \prod_{t=1}^n \frac{1}{1 + r_t} \\ &= \prod_{t=1}^n v_t \\ &= \exp\left(-\sum_{t=1}^n \delta_t\right) \\ &= v_n^* \end{aligned} \tag{5}$$

As a recursive relation we have

$$A_{\overline{n+1}|}^* = v_{n+1} A_{\overline{n}|}^*$$

The present value of a **continuous benefit payment** is

$$\bar{A}_{\overline{T}|}^* = \exp \left(- \int_0^T \delta(s) ds \right)$$

The present value of **an annuity-due** is

$$\begin{aligned} \ddot{a}_{\overline{n}|}^* &= 1 + v_1^* + \dots + v_{n-1}^* \\ &= \sum_{k=0}^{n-1} \prod_{t=0}^k v_t \end{aligned} \quad (6)$$

and

$$A_{\overline{n}|}^* = \sum_{k=0}^{n-1} \exp \left(- \sum_{t=0}^k \delta_t \right) \quad (7)$$

The present value of a **continuous benefit payment** is

$$\bar{A}_{\overline{T}|}^* = \int_0^T e^{(-\int_0^t \delta(s) ds)} dt \quad (8)$$

2.3 The Premium

The basic concept involved in the determination of annual benefit premiums is the equivalence principle. The insurer's loss, L , as random variable of the present value of benefit to be paid by insurer less the annuity of premiums to be paid by the insured. For the case of the continuous time, L is represented by

$$L = \exp \left(- \int_0^T \delta(s) ds \right) - P \cdot \int_0^T \exp \left(- \int_0^t \delta(s) ds \right) dt$$

The equivalence principle has the requirement that $E(L) = 0$. Equivalently, benefit premiums will be such that

$$E(\text{present value of benefits}) = E(\text{present value of benefit premiums})$$

and the premium is as follows

$$P = \frac{E \left(\exp \left(- \int_0^T \delta(s) ds \right) \right)}{E \left(\int_0^T \exp \left(- \int_0^t \delta(s) ds \right) dt \right)} \quad (9)$$

3 Actuarial Quantities Based on Interest Rate Derivative; The Hull-White Model

Time to maturity in financial valuation model is adjusted with $T(x)$. The actuarial present value for whole life insurance with a unit payable at the moment of death of x , $E(Z)$, and a

continuous whole life annuity are denoted by \bar{A}_{xp} and \bar{a}_{xp} ;

$$\begin{aligned}\bar{A}_{xp} &= E(E(v(t))) \\ &= \int_0^\infty E(v(t)) f(t) dt \\ &= \int_0^\infty E\left(\exp\left(-\int_0^{T_x} r_u du\right)\right) f(t) dt \\ &= \int_0^\infty P(0, T_x) {}_t p_x \mu_{x+t} dt\end{aligned}$$

and

$$\bar{a}_{xp} = \int_0^\infty P(0, T_x) {}_t p_x dt$$

where $P(0, T_x)$ is the price of a zero coupon bond.

The premium is as follows;

$$P = \frac{\int_0^\infty P(0, T_x) {}_t p_x \mu_{x+t} dt}{\int_0^\infty P(0, T_x) {}_t p_x dt}$$

4 Convex Upper Bound

4.1 Ordering Random Variable

◇ Stop-loss Premium

Reinsurance treaties usually cover only part of the risk. Stop-loss reinsurance covers the top part. It is defined as follows: if the loss is X_1, X_2, \dots, X_n and it is assumed $X \geq 0$, the payment equals

$$(X - d)_+ := \max\{X - d, 0\} = \begin{cases} X - d & , \text{ if } X > d \\ 0 & , \text{ if } X \leq d \end{cases}$$

where d is the retention, stated that the maximum amount to be paid by reinsurance. A stop-loss premium means the net premium $E[(X - d)_+]$ for a stop-loss contract and is defined as follows:

$$\begin{aligned}E[(X - d)_+] &= \int_d^\infty (x - d) f_X(x) dx \\ &= -(x - d)[1 - F_X(x)]|_d^\infty + \int_d^\infty [1 - F_X(x)] dx \\ &= \int_d^\infty [1 - F_X(x)] dx\end{aligned}$$

◇ Stop-loss Order

A random variable X is smaller than a random variable Y in the *stop-loss order*, written $X \leq_{sl} Y$, if

$$E[(X - d)_+] \leq E[(Y - d)_+] \quad \forall d \geq 0$$

◇ Convex Order

A random variable X is less than a random variable Y in *convex order*, written $X \leq_{cx} Y$, if, and only if

$$\begin{aligned}E[X] &= E[Y] \\ E[(X - d)_+] &\leq E[(Y - d)_+] \quad \forall d \geq 0\end{aligned}$$

4.2 Comonotonicity and Fréchet Spaces

For any (n -dimensional) random vector $X = (X_1, X_2, \dots, X_n)$, let the distribution function and the survival function be denoted by $F_{\mathbf{X}}(\mathbf{x})$ and $S_{\mathbf{X}}(\mathbf{x})$ respectively, i.e.

$$\begin{aligned} F_{\mathbf{X}}(\mathbf{x}) &= P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \\ S_{\mathbf{X}}(\mathbf{x}) &= P(X_1 \geq x_1, X_2 \geq x_2, \dots, X_n \geq x_n) \end{aligned}$$

Let $R_n(F_1, F_2, \dots, F_n)$ denote the *Fréchet* class of all random vectors $\mathbf{X} = (X_1, X_2, \dots, X_n)$ with fixed marginal distribution function F_1, F_2, \dots, F_n . The extremal distribution of all random vector in $R_n(F_1, F_2, \dots, F_n)$ is the distribution function $W_n(x)$ is given by

$$W_n(\mathbf{x}) = \min \{F_1(x_1), F_2(x_2), \dots, F_n(x_n)\}$$

in the sense that

$$F_{\mathbf{X}}(\mathbf{x}) \leq W_n(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

The distribution function $W_n(\mathbf{x})$ is known as the *Fréchet* upper bound in $R_n(F_1, F_2, \dots, F_n)$ and we have that

$$W_n(\mathbf{x}) = P(F_1^{-1}(U) \leq x_1, F_2^{-1}(U) \leq x_2, \dots, F_n^{-1}(U) \leq x_n)$$

According to Goovaert et al. (2000) and Dhaene et al. (2002), random vectors $\mathbf{X} = (X_1, X_2, \dots, X_n)$ with $W_n(\mathbf{x})$ as their distribution function are said to be *comonotonic*. *Comonotonic* random variables possess a very strong positive dependency. This means that if $\mathbf{X} = (X_1, X_2, \dots, X_n)$ give the different values, (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_n) and also if

$$x_i \geq y_i$$

then

$$x_j \geq y_j, \quad i \neq j.$$

Let $\mathbf{X} = (X_1, X_2, \dots, X_n) \in R_n(F_1, F_2, \dots, F_n)$ and

$$V = X_1 + X_2 + \dots + X_n \tag{10}$$

The upper bound of V ;

$$W = Y_1 + Y_2 + \dots + Y_n$$

where $(Y_1, Y_2, \dots, Y_n) \in R_n(F_1, F_2, \dots, F_n)$ then

$$V \leq W$$

The upper bound of equation ?? will be obtained through the inverse of distribution function; $F_{X(\tau_i)}$.

For any U random variable uniformly distributed on $[0, 1]$, we have

$$(X_1, \dots, X_n) =^d (F_1^{-1}(U), \dots, F_n^{-1}(U))$$

and

$$\phi_1(X_1) + \dots + \phi_n(X_n) \leq_{cx} \phi_1(F_1^{-1}(U)) + \dots + \phi_n(F_n^{-1}(U))$$

Let

$$\phi_i(X) := c_i e^{-X(\tau_i)} \tag{11}$$

It is obtained the convex upper bound

$$W = \sum_{i=1}^n \phi_i(F_{X(\tau_i)}^{-1}(U)) = \sum_{i=1}^n c_i e^{-F_{X(\tau_i)}^{-1}(U)} \tag{12}$$

5 Empirical Results and Sensitivity Analysis

In this section, we present empirical results and sensitivity analysis with respect to parameters of insurance contract. Figures ?? display the distribution function compared with the empirical distribution based on 10000 randomly generated, normally distributed vectors.

The mean present value $E(V_{cub}) = 10.381$ and $E(V_{mc}) = 10.390$ for $i = 1, 2, \dots, 30$ and $E(V_{cub}) = 11.0154$ and $E(V_{mc}) = 11.0610$ for $i = 1, 2, \dots, K(x)$. $K(x)$ discrete random variable represent the number of completed future years lived by (x) . This results show that the distribution function is equivalent to the simulated.

Figures ?? describe the Hull-White model premium vs the constant interest rate premium, based on the endowment and the term insurance and also show the changes of parameters. The premiums are negatively related to c , the speed of adjustment in the mean reverting process whereas they are positively related to the standard deviation of the interest rate.

6 Summary

This paper combined Financial and Actuarial approaches to price life insurance contract. We quantified the premium based on the Hull-White financial valuation model. Moreover, sensitivity analysis is further utilized to exam the differences between the constant and the stochastic interest rate. It can be seen that at the certain stochastic interest rate gives a fair premium value when it is compared with a constant interest rate. The premiums are negatively related to the speed of adjustment in the mean reverting process whereas they are positively related to the standard deviation of the interest rate.

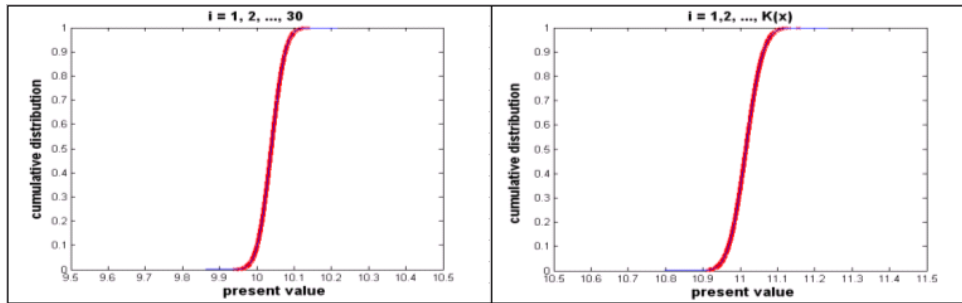


Figure 1: Convex Upper Bound (red) vs. Montecarlo Simulation (blue), $i = 1, 2, \dots, 30$ (left) and Convex Upper Bound (red) vs. Montecarlo Simulation (blue), $i = 1, 2, \dots, K(x)$ (right)

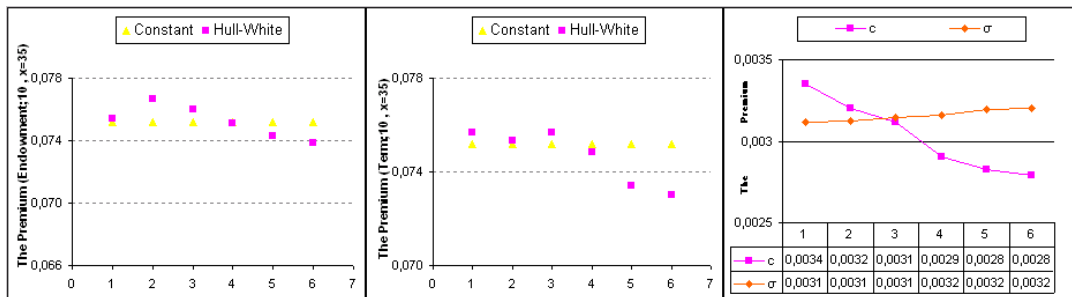


Figure 2: The Hull-White Model Premium vs The Constant Interest Rate Premium

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