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Research Article

Connections between Certain Subclasses of Analytic Univalent Functions Based on Operators

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In this paper, by applying the Hohlov linear operator, connections between the class $\mathrm{SD}(\alpha)$, $\alpha \geq 0$, and two subclasses of the class A of normalized analytic functions are established. Also an integral operator related to hypergeometric function is considered.

1. Introduction

Let A denote the family of functions f that are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let S denote the subclass of functions in A which are also univalent in $\triangle$. A well-known subclass of S is the class ST (see, e.g., [1]) of starlike functions f of the form (1), satisfying $\mathrm{Re}\{zf'(z)/f(z)\} > 0$, $z \in \Delta$. Another class $\mathrm{UCD}(\alpha)$, $\alpha \geq 0$, introduced in [2], consists of functions $f \in A$ satisfying $\mathrm{Re}(f'(z)) \geq \alpha |zf''(z)|$, $z \in \Delta$. Various properties of this class have been obtained in [2–4]. A related class $\mathrm{SD}(\alpha)$ has been recently considered in [5], initially introduced in [6]. A function f of the form (1) is said to be in the class $\mathrm{SD}(\alpha)$ if

$$\mathrm{Re}\left\{\frac{f(z)}{z}\right\} \geq \alpha \left| f'(z) - \frac{f(z)}{z} \right|, \quad \text{for } \alpha \geq 0. \tag{2}$$

Theorem 1 (see [5]). *A function f of the form (1) is in the class $SD(\alpha)$ if*

$$\sum_{n=2}^{\infty} [1 + \alpha(n-1)] |a_n| \leq 1. \tag{3}$$

Ponnusamy and Ronning [7] introduced and studied the class $R_\eta(\beta) \subset S$, $(0 \leq \beta < 1)$ of functions $f \in A$ for which there exists a number $\eta \in (-\pi/2, \pi/2)$ such that $\mathrm{Re}\{e^{i\eta}[f'(z) - \beta]\} > 0$, $z \in \triangle$. If the function f of the form (1) belongs to the class $R_\eta(\beta)$, then

$$|a_n| \leq \frac{2(1-\beta)\cos\eta}{n}, \quad (n \in N \setminus \{1\}). \tag{4}$$

For complex numbers a, b, and c $(c \neq 0, -1, -2, \ldots)$, the Gaussian hypergeometric function ${}_2F_1(z)$ is defined by

$${}_2F_1(z) = {}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{5}$$

where $(\lambda)_n$ is the Pochhammer symbol given by

$$(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & n \in N \end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}. \tag{6}$$

It is known that

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0,\ c \neq 0, -1, -2, \ldots. \tag{7}$$

The Hadamard product (or convolution) of two functions f defined by (1) and g given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{8}$$

Hohlov [8] introduced a linear operator $I_{a,b,c} : A \to A$, corresponding to the Gaussian hypergeometric function ${}_2F_1$ which is defined by the convolution

$$[I_{a,b,c}(f)](z) = z\,{}_2F_1(a,b,c,z) * f(z), \quad f \in A. \tag{9}$$

For a function f of the form (1), we have

$$(I_{a,b,c}(f))(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \quad z \in \triangle. \tag{10}$$

The operator $I_{a,b,c}$ is a natural extension of several operators such as Alexander, Libera, Bernardi, and Carlson-Shaffer operators denoted, respectively, by $\mathscr{A}$, $\mathfrak{L}$, $\mathscr{B}$, and $\mathfrak{L}(a,c)$.

Motivated by the work of Thulasiram et al. [9], in this paper, by applying the linear operator $I_{a,b,c}$, we establish some interesting connections between the class SD(α) and the classes ST and $R_\eta(\beta)$, $(\beta < 1)$ consisting of functions f given by (1). Also we consider an integral operator related to the hypergeometric functions.

2. Main Results

In the sequel the function $f \in A$ is given by (1).

Theorem 2. *Let $a, b \in C\setminus\{0\}$. Also let c be a real number such that $c > |a| + |b| + 2$. If $f \in ST$ and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{|ab|}{(c-|a|-|b|-1)}\left(\frac{\alpha(|a|+1)(|b|+1)}{(c-|a|-|b|-2)} + 2\alpha + 1\right) + 1\right] \le 2 \tag{11}$$

is satisfied, then $I_{a,b,c}(f) \in SD(\alpha)$.

Proof. Let $f \in$ ST. Applying the well-known estimate due to Nevanlinna [10] for the coefficients of the functions $f \in$ ST, in view of Theorem 1, we need to prove that

$$\sum_{n=2}^{\infty} n(1+\alpha(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| \le 1. \tag{12}$$

By virtue of the relation $|(d)_n| \le (|d|)_n$, and on writing $n+2 = (n+1)+1$ and $(n+2)^2 = (n+1)^2 + 2(n+1) + 1$ and using the fact that $(a)_{n+k} = (a)_k(a+k)_n$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} n(1+\alpha(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| &\le \sum_{n=2}^{\infty} n(1+\alpha(n-1))\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= \sum_{n=1}^{\infty}\left\{(n+1) + \alpha\left[(n+1)^2 - (n+1)\right]\right\}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} \\
&= \alpha\sum_{n=0}^{\infty}(n+2)^2\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + (1-\alpha)\sum_{n=0}^{\infty}(n+2)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= \alpha\sum_{n=0}^{\infty}\frac{(n+1)^2(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + 2\alpha\sum_{n=0}^{\infty}\frac{(n+1)(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&\quad + (1-\alpha)\sum_{n=0}^{\infty}(n+1)\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} + \sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_{n+1}} \\
&= \alpha\sum_{n=0}^{\infty}\frac{(|a|)_{n+2}(|b|)_{n+2}}{(c)_{n+2}(1)_n} + [2\alpha+1]\sum_{n=0}^{\infty}\frac{(|a|)_{n+1}(|b|)_{n+1}}{(c)_{n+1}(1)_n} + \sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} - 1 \\
&= \frac{\alpha(|a|)_2(|b|)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(|a|+2)_n(|b|+2)_n}{(c+2)_n(1)_n} + (2\alpha+1)\frac{|ab|}{c}\sum_{n=0}^{\infty}\frac{(|a|+1)_n(|b|+1)_n}{(c+1)_n(1)_n} \\
&\quad + \sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n} - 1 \\
&= \frac{\alpha(|a|)_2(|b|)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-|a|-|b|-2)}{\Gamma(c-|a|)\Gamma(c-|b|)}
\end{aligned}$$

$$+(2\alpha+1)\frac{|ab|}{c}\frac{\Gamma(c+1)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}+\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1$$
$$=\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left\{\frac{|ab|}{(c-|a|-|b|-1)}\left[\frac{\alpha(|a|+1)(|b|+1)}{(c-|a|-|b|-2)}+2\alpha+1\right]+1\right\}-1$$
$$\leq 1, \tag{13}$$

which is satisfied by the hypothesis. □

On setting $b=\overline{a}$, an improvement of the assertion of Theorem 2 is obtained as given in Theorem 3.

Theorem 3. *Let $a\in C\setminus\{0\}$. Also let c be a real number such that $c>\max\{0,2\operatorname{Re}(a)+2\}$. If $f\in ST$, and if the inequality*

$$\frac{\Gamma(c)\Gamma(c-2\operatorname{Re}(a))}{\Gamma(c-a)\Gamma(c-\overline{a})}\left\{\frac{|a|^2}{(c-2\operatorname{Re}(a)-1)}\left[\frac{\alpha(a+1)(\overline{a}+1)}{c-2\operatorname{Re}(a)-2}+2\alpha+1\right]+1\right\}\leq 2 \tag{14}$$

is satisfied, then $I_{a,\overline{a},c}(f)\in SD(\alpha)$.

Theorem 4. *Let $a,b\in C\setminus\{0\}$. Also, let c be a real number such that $c>|a|+|b|$, $|a|\neq 1$, $|b|\neq 1$. If $f\in R_\eta(\beta)$, $(\beta<1)$ and the inequality*

$$\frac{2(1-\beta)\cos\eta}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}[c-|a|-|b|+\alpha(|ab|+1-c)]-[c+|ab|-|a|-|b|-\alpha(c-1)]\right]\leq 1 \tag{15}$$

is satisfied, then $I_{a,b,c}(f)\in SD(\alpha)$.

Proof. Let f be of the form (1) and let $f\in R_\eta(\beta)$, $(\beta<1)$. By virtue of Theorem 1 and in view of (10), it remains to show that

$$\sum_{n=2}^{\infty}(1+\alpha(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 1. \tag{16}$$

Using the inequality (4) and the relations $(d)_n=d(d+1)_{n-1}$ and $|(d)_n|\leq(|d|)_n$, we obtain that

$$\sum_{n=2}^{\infty}(1+\alpha(n-1))\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}a_n\right|\leq 2(1-\beta)\cdot\cos\eta\sum_{n=2}^{\infty}\frac{(1+\alpha(n-1))}{n}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}=2\alpha(1-\beta)$$
$$\cdot\cos\eta\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}+2(1-\alpha)(1-\beta)\cdot\cos\eta\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n}}=2\alpha(1-\beta)\cos\eta\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}$$
$$+2(1-\alpha)(1-\beta)\cdot\cos\eta\sum_{n=2}^{\infty}\frac{((|a|-1)_n/(|a|-1))((|b|-1)_n/(|b|-1))}{((c-1)_n/(c-1))(1)_n}=2\alpha(1-\beta)\cos\eta\left[\sum_{n=0}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}-1\right]$$
$$+\frac{2(1-\alpha)(1-\beta)(c-1)\cos\eta}{(|a|-1)(|b|-1)}\left[\sum_{n=0}^{\infty}\frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n}-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right]=2\alpha(1-\beta)\cos\eta\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]$$
$$+\frac{2(1-\alpha)(1-\beta)(c-1)\cos\eta}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c-1)\Gamma(c-|a|-|b|+1)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1-\frac{(|a|-1)(|b|-1)}{(c-1)}\right]=2\alpha(1-\beta)\cos\eta\cdot\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-1\right]$$
$$+\frac{2(1-\alpha)(1-\beta)\cos\eta}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)\cdot(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}-(c-1)-(|a|-1)(|b|-1)\right]=2(1-\beta)\cos\eta$$
$$\cdot\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\alpha+\frac{(1-\alpha)(c-|a|-|b|)}{(|a|-1)(|b|-1)}\right]+2(1-\beta)\cos\eta\left[-\alpha-\frac{(1-\alpha)(c-1)}{(|a|-1)(|b|-1)}-(1-\alpha)\right]$$
$$=\frac{2(1-\beta)\cos\eta}{(|a|-1)(|b|-1)}\left[\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}[c-|a|-|b|+\alpha(|ab|+1-c)]-[c+|ab|-|a|-|b|-\alpha(c-1)]\right]\leq 1. \tag{17}$$

□

On taking $b=\overline{a}$, an improvement of the assertion of Theorem 4 is obtained as given in Theorem 5.

Theorem 5. *Let $a\in C\setminus\{0\}$. Further let c be a real number such that $c>\max\{0,2\operatorname{Re}(a)\}$, $a\neq 1$. If $f\in R_\eta(\beta)$, $(\beta<1)$ and the inequality*

$$\frac{2(1-\beta)\cos\eta}{(a-1)(\overline{a}-1)}\left[\frac{\Gamma(c)\Gamma(c-2\operatorname{Re}(a))}{\Gamma(c-a)\Gamma(c-\overline{a})}[c-2\operatorname{Re}(a)\right.$$

$$+\alpha\left(|a|^2+1-c\right)\Big]-\Big[c-|a|^2-2\operatorname{Re}(a)$$
$$-\alpha(c-1)\Big]\Big]\le 1 \tag{18}$$

is satisfied, then $I_{a,\bar{a},c}(f)\in SD(\alpha)$.

3. An Integral Operator

We now obtain results in connection with a particular integral operator [11] $G(a,b;c;z)$ defined by

$$G(a,b;c;z)=\int_0^z F(a,b;c;t)\,dt, \tag{19}$$

where $F(a,b;c;z)={}_2F_1(z)$ is given by (5).

Theorem 6. *Let $a,b\in C-\{0\}$. Also let c be a real number such that $c>|a|+|b|$, $|a|\neq 1$, $|b|\neq 1$. Let $G(a,b;c;z)$ be given by (19). If the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{c(1-\alpha)+\alpha(|ab|+1)-(|a|+|b|)}{(|a|-1)(|b|-1)}\right]-\frac{(1-\alpha)(c-1)}{(|a|-1)(|b|-1)}\le 2 \tag{20}$$

is satisfied, then $G(a,b,c,z)\in SD(\alpha)$.

Proof. The function $G(a,b;c;z)$ has the series representation given by

$$z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}z^n. \tag{21}$$

In view of Theorem 1, it is enough to prove that

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\right|\le 1. \tag{22}$$

Now

$$\sum_{n=2}^{\infty}[1+\alpha(n-1)]\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_n}\right|\le(1-\alpha)\cdot\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n}+\alpha\sum_{n=2}^{\infty}\frac{n(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_n}=(1-\alpha)\sum_{n=2}^{\infty}\frac{(|a|-1)_n(|b|-1)_n(c-1)}{(|a|-1)(|b|-1)(c-1)_n(1)_n}+\alpha\sum_{n=2}^{\infty}\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$
$$=\frac{(1-\alpha)(c-1)}{(|a|-1)(|b|-1)}\left[\sum_{n=0}^{\infty}\frac{(|a|-1)_n(|b|-1)_n}{(c-1)_n(1)_n}-1-\frac{(|a|-1)(|b|-1)}{c-1}\right]+\alpha\left[\sum_{n=1}^{\infty}\frac{(|a|)_n(|b|)_n}{(c)_n(1)_n}\right]$$
$$=\frac{\Gamma(c)\Gamma(c-|a|-|b|)}{\Gamma(c-|a|)\Gamma(c-|b|)}\left[\frac{(1-\alpha)(c-|a|-|b|)}{(|a|-1)(|b|-1)}+\alpha\right]-\frac{(1-\alpha)(c-1)}{(|a|-1)(|b|-1)}-1\le 1, \tag{23}$$

by hypothesis. □

A result analogous to Theorem 6 can be stated for the class UCD(α) in Theorem 7.

Theorem 7. *Let $a,b\in C-\{0\}$. Also let c be a real number such that $c>|a|+|b|+1$. Let $f\in A$ and be of the form (1). If the hypergeometric inequality*

$$\frac{\Gamma(c)\Gamma(c-|a|-|b|-1)}{\Gamma(c-|a|)\Gamma(c-|b|)}[c-|a|-|b|+\alpha|ab|-1]\le 2 \tag{24}$$

is satisfied, then $G(a,b,c,z)\in UCD(\alpha)$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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