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CERTAIN SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS OF COMPLEX ORDER

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Abstract

In the present investigation, we consider certain subclasses of starlike and convex functions of complex order, giving necessary and sufficient conditions for functions to belong to these classes.

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1. Introduction

Let $\mathcal{A}$ be the class of all analytic functions

$$(1) \qquad f(z) = z + a_2z^2 + a_3z^3 + \cdots$$

in the open unit disk $\Delta = \{z \in \mathbb{C}; |z| < 1\}$. A function $f \in \mathcal{A}$ is *subordinate to an univalent function* $g \in \mathcal{A}$, written $f(z) \prec g(z)$, if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let Ω be the family of analytic functions $\omega(z)$ in the unit disc Δ satisfying the conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \Delta$. Note that $f(z) \prec g(z)$ if there is a function $w(z) \in \Omega$ such that $f(z) = g(\omega(z))$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. The class $S^*(\phi)$, introduced and studied by Ma and Minda [5], consists of functions in $f \in \mathcal{S}$ for which

$$\frac{zf'(z)}{f(z)} \prec \phi(z), \ (z \in \Delta).$$

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The functions $h_{\phi n}$ $(n = 2, 3, \ldots)$ are defined by

$$\frac{zh'_{\phi n}(z)}{h_{\phi n}(z)} = \phi(z^{n-1}),\ h_{\phi n}(0) = 0 = h'_{\phi n}(0) - 1.$$

The functions $h_{\phi n}$ are all functions in $S^*(\phi)$. We write $h_{\phi 2}$ simply as h_ϕ. Clearly,

$$h_\phi(z) = z \exp\left(\int_0^z \frac{\phi(x) - 1}{x} dx\right). \tag{2}$$

Following Ma and Minda [5], we define a more general class related to the class of starlike functions of complex order as follows.

1.1. Definition. Let $b \neq 0$ be a complex number. Let $\phi(z)$ be an analytic function with positive real part on Δ, which satisfies $\phi(0) = 1$, $\phi'(0) > 0$, and which maps the unit disk Δ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Then the class $S_b^*(\phi)$ consists of all analytic functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z).$$

The class $C_b(\phi)$ consists of the functions $f \in \mathcal{A}$ satisfying

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \phi(z).$$

Moreover, we let $S^*(A, B, b)$ and $C(A, B, b)$ ($b \neq 0$, complex) denote the classes $S_b^*(\phi)$ and $C_b(\phi)$ respectively, where

$$\phi(z) = \frac{1 + Az}{1 + Bz},\ (-1 \leq B < A \leq 1).$$

The class $S^*(A, B, b)$, and therefore the class $S_b^*(\phi)$, specialize to several well-known classes of univalent functions for suitable choices of A, B and b.

The class $S^*(A, B, 1)$ is denoted by $S^*(A, B)$. Some of these classes are listed below:

(1) $S^*(1, -1, 1)$ is the class S^* of starlike functions [1, 2, 7].

(2) $S^*(1, -1, b)$ is the class of starlike functions of complex order introduced by Wiatrowski [12].

(3) $S^*(1, -1, 1 - \beta)$, $0 \leq \beta < 1$, is the class $S^*(\beta)$ of starlike functions of order β. This class was introduced by Robertson [8].

(4) $S^*(1, -1, e^{-i\lambda}\cos\lambda)$, $|\lambda| < \frac{\pi}{2}$ is the class of λ-spirallike functions introduced by Spacek [11].

(5) $S^*(1, -1, (1 - \beta)e^{-i\lambda}\cos\lambda)$, $0 \leq \beta < 1$, $|\lambda| < \frac{\pi}{2}$, is the class of λ-spirallike functions of order β. This class was introduced by Libera [4].

Let $ST(b)$ denote $1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)$. Then we have the following:

(6) $S^*(1, 0, b)$ is the set defined by $|ST(b) - 1| < 1$.

(7) $S^*(\beta, 0, b)$ is the set defined by $|ST(b) - 1| < \beta$, $0 \leq \beta < 1$.

(8) $S^*(\beta, -\beta, b)$ is the set defined by $\left|\frac{ST(b)-1}{(ST(b)+1)}\right| < \beta$, $0 \leq \beta < 1$.

(9) $S^*(1, (-1 + \frac{1}{M}), b)$ is the set defined by $|ST(b) - M| < M$.

(10) $S^*(1 - 2\beta, -1, b)$ is the set defined by $\mathrm{Re}ST(b) > \beta$, $0 \leq \beta < 1$.

To prove our main result, we need the following Lemma due to Miller and Mocanu:

1.2. Lemma. [6, Corollary 3.4h.1, p.135] *Let $q(z)$ be univalent in Δ and let $\varphi(z)$ be analytic in a domain containing $q(\Delta)$. If $zq'(z)/\varphi(q(z))$ is starlike, then*

$$zp'(z)\varphi(p(z)) \prec zq'(z)\varphi(q(z))$$

implies that $p(z) \prec q(z)$, and $q(z)$ is the best dominant.

Let C be the class of convex analytic functions in Δ. We will also need the following result:

1.3. Lemma. [10, Theorem 2.36, p. 86] *For $f, h \in C$ and $g \prec h$, we have $f * g \prec f * h$.*

2. A necessary and Sufficient Condition

We begin with the following:

2.1. Lemma. *Let ϕ be a convex function defined on Δ and satisfying $\phi(0) = 1$. As in Equation* (1) *let $h_\phi(z) = z\exp\left(\int_0^z \frac{\phi(x)-1}{x}dx\right)$, and let $q(z) = 1 + c_1 z + \cdots$ be analytic in Δ. Then*

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z) \tag{3}$$

if and only if for all $|s| \le 1$ and $|t| \le 1$, we have

$$\frac{q(tz)}{q(sz)} \prec \frac{sh_\phi(tz)}{th_\phi(sz)}. \tag{4}$$

Proof. Our result and its proof are motivated by a similar result of Ruscheweyh [rus] for functions in the class $S^*(\phi)$. Also see Ruscheweyh [10, Theorem 2.37, pages 86-88].

Let $q(z)$ satisfy (3). Since the function

$$p(z) = \int_0^z \left(\frac{s}{1-sx} - \frac{t}{1-tx}\right) dx$$

is convex and univalent in Δ for $s, t \in \overline{\Delta} := \Delta \cup \{z \in \mathbb{C} : |z| = 1\}$, $s \neq t$, by Lemma 1.2 we have:

$$\left(\frac{zq'(z)}{q(z)}\right) * p(z) \prec (\phi(z) - 1) * p(z). \tag{5}$$

For an analytic function $h(z)$ with $h(0) = 0$, we have

$$(h * p)(z) = \int_{sz}^{tz} h(x)\frac{dx}{x}, \tag{6}$$

and using (6), we see that (5) is equivalent to

$$\int_{sz}^{tz} \left(\frac{q'(x)}{q(x)}\right) dx \prec \int_{sz}^{tz} \left(\frac{\phi(x)-1}{x}\right) dx,$$

which gives the desired assertion (4) upon exponentiation.

To prove the converse, let us assume that (4) holds. By taking $t = 1$ in (4), we have

$$\frac{q(z)}{q(sz)} \prec \frac{sh_\phi(z)}{h_\phi(sz)}, \tag{7}$$

and therefore we have

$$\frac{q(z)}{q(sz)} = \frac{sh_\phi(\phi_s(z))}{h_\phi(s\phi_s(z))}, \tag{8}$$

where $\phi_s(z)$ are analytic in Δ and satisfy $|\phi_s(z)| \leq |z|$. Thus we can find a sequence $s_k \to 1$ such that $\phi_{s_k} \to \phi^*$ locally uniformly in Δ, where $|\phi^*(z)| \leq |z|$ $(z \in \Delta)$. Therefore, by making use of (8), we have for any fixed $z \in \Delta$,

$$\begin{aligned} 1 + \frac{zq'(z)}{q(z)} &= \lim_{k\to\infty} \left[\frac{s_k q(s_k z) - q(z)}{(s_k - 1)q(z)} \right] \\ &= \lim_{k\to\infty} \frac{\phi_{s_k}(z)}{h_\phi(\phi_{s_k}(z))} \left[\frac{h_\phi(s_k\phi_{s_k}(z)) - h_\phi(\phi_{s_k}(z))}{s_k\phi_{s_k}(z) - \phi_{s_k}(z)} \right] \\ &= \frac{\phi^*(z) h'_\phi(\phi^*(z))}{h_\phi(\phi^*(z))}. \end{aligned}$$

This shows that

$$1 + \frac{zq'(z)}{q(z)} \in \left(\frac{zh'_\phi}{h_\phi} \right)(\Delta) = \phi(\Delta), \ (z \in \Delta),$$

which completes the proof of our Lemma 2.1. □

By making use of Lemma 2.1, we now have the following:

2.2. Theorem. *Let ϕ be a convex function defined on Δ which satisfies $\phi(0) = 1$, and $h_\phi(z) = z \exp\left(\int_0^z \frac{\phi(x)-1}{x} dx \right)$ be as in Equation'(1). The the function f belongs to $S_b^*(\phi)$ if and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have*

$$\left(\frac{sf(tz)}{tf(sz)} \right)^{\frac{1}{b}} \prec \frac{sh_\phi(tz)}{th_\phi(sz)}. \tag{9}$$

Proof. Define the function $q(z)$ by

$$q(z) := \left(\frac{f(z)}{z} \right)^{1/b}. \tag{10}$$

Then a computation show that

$$1 + \frac{zq'(z)}{q(z)} = 1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right).$$

The result now follows from Lemma 2.1. □

As an immediate consequence of Theorem 2.2, we have:

2.3. Corollary. *Let $\phi(z)$ and $h_\phi(z)$ be as in Theorem 2.2. If $f \in S_b^*(\phi)$, then we have*

$$\left(\frac{f(z)}{z} \right)^{\frac{1}{b}} \prec \frac{h_\phi(z)}{z}. \tag{11}$$

3. Another Subordination Result

In this section, we prove the following without the assumption that the function ϕ is convex. We only require that the function ϕ be starlike with respect to the origin.

3.1. Corollary. *If $f \in S_b^*(\phi)$, then we have*

$$\left(\frac{f(z)}{z} \right)^{\frac{1}{b}} \prec \frac{h_\phi(z)}{z}, \tag{12}$$

where $h_\phi(z)$ is given by (2).

Proof. Define the functions $p(z)$ and $q(z)$ by

$$p(z) := \left(\frac{f(z)}{z}\right)^{1/b}, \quad q(z) := \frac{h_\phi(z)}{z}.$$

Then a computation yields

$$1 + \frac{zp'(z)}{p(z)} = 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)$$

and

$$\frac{zq'(z)}{q(z)} = \frac{zh'_\phi(z)}{h_\phi(z)} - 1 = \phi(z) - 1.$$

Since $f \in S_b^*(\phi)$, we have

$$\frac{zp'(z)}{p(z)} = \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \phi(z) - 1 = \frac{zq'(z)}{q(z)}.$$

The result now follows by an application of Lemma 1.1. □

4. The Fekete-Szegö inequality

In this section, we obtain the Fekete-Szegö inequality for functions in the class $S_b^*(\phi)$.

4.1. Theorem. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If $f(z)$ given by Equation (1) belongs to $S_b^*(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq 2 \max\left\{1; \left|\frac{B_2}{B_1} + (1-2\mu)bB_1\right|\right\}.$$

The result is sharp.

Proof. If $f(z) \in S_b^*(\phi)$, then there is a Schwarz function $w(z)$, analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$ in Δ and such that

$$1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) = \phi(w(z)). \tag{13}$$

Define the function $p_1(z)$ by

$$p_1(z) := \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \tag{14}$$

Since $w(z)$ is a Schwarz function, we see that $\Re p_1(z) > 0$ and $p_1(0) = 1$. Define the function $p(z)$ by

$$p(z) := 1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right) = 1 + b_1 z + b_2 z^2 + \cdots. \tag{15}$$

In view of the equations (13), (14) and (15), we have

$$p(z) = \phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right). \tag{16}$$

Since

$$\frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}\left[c_1 z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 + \frac{c_1^3}{4} - c_1 c_2)z^3 + \cdots\right]$$

and therefore

$$\phi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left[\frac{1}{2}B_1(c_2 - \frac{1}{2}c_1^2) + \frac{1}{4}B_2 c_1^2\right] z^2 + \cdots,$$

from this equation and (16), we obtain

$$b_1 = \frac{1}{2} B_1 c_1$$

and

$$b_2 = \frac{1}{2} B_1 (c_2 - \frac{1}{2} c_1^2) + \frac{1}{4} B_2 c_1^2.$$

Since

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2) z^2 + (3a_4 + a_2^3 - 3a_3 a_2) z^3 + \cdots ,$$

from Equation (15), we see that

$$bb_1 = a_2, \tag{17}$$

$$bb_2 = 2a_3 - a_2^2, \tag{18}$$

or equivalently we have

$$\begin{aligned} a_2 &= bb_1 = \frac{bB_1 c_1}{2}, \\ a_3 &= \frac{1}{2} \left\{ bb_2 + b^2 b_1^2 \right\} \\ &= \frac{b}{4} B_1 c_1 + \frac{c_1^2}{8} \left\{ b^2 B_1^2 - b(B_1 - B_2) \right\}. \end{aligned}$$

Therefore we have

$$a_3 - \mu a_2^2 = \frac{bB_1}{4} \left\{ c_2 - v c_1^2 \right\}, \tag{19}$$

where

$$v := \frac{1}{2} \left[1 - \frac{B_2}{B_1} + (2\mu - 1) bB_1 \right].$$

We recall from [5] that if $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part, then

$$|c_2 - \mu c_1^2| \leq 2 \max\{1, |2\mu - 1|\},$$

the result being sharp for the functions given by

$$p(z) = \frac{1 + z^2}{1 - z^2}, \quad p(z) = \frac{1 + z}{1 - z}.$$

Our result now follows from an application of the above inequality, and we see that he result is sharp for the functions defined by

$$1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) = \phi(z^2)$$

and

$$1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) = \phi(z).$$

This completes the proof of the theorem. □

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