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Applications of First Order Differential Superordinations to Certain Linear Operators*

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Abstract. In the present paper, we give some applications of first order differential superordinations to obtain sufficient conditions for normalized analytic functions defined by certain linear operators to be superordinated to a given univalent function.

Keywords: Differential subordinations; Differential superordinations; Subordinant.

1. Introduction

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Let $\mathcal{H}$ be the class of functions analytic in $\Delta := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $\mathcal{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \Delta) \tag{1.1}$$

and $\mathcal{A} := \mathcal{A}_1$. If f is subordinate to F, then F is superordinate to f. Recently Miller and Mocanu [14] considered certain first and second order differential superordinations. Using the results of Miller and Mocanu [14], Bulboaca have considered certain classes of first order differential superordinations [4] as well as superordination-preserving integral operators [3]. The authors [1] have used the results of Bulboaca [4] to obtain some sufficient conditions for functions to satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z)$$

where q_1, q_2 are given univalent functions in Δ.

Recently, several authors [10, 15, 16, 19, 20, 21, 22, 28] have obtained sufficient conditions associated with starlikeness in terms of the expression

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)}.$$

In this paper, we give some applications of first order differential superordinations to obtain sufficient conditions for functions defined through Dziok-Srivastava linear operator and the multiplier transformation on the space of multivalent functions $\mathcal{A}_p$.

2. Preliminaries

For two analytic functions $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

their Hadamard product (or convolution) is the function $(f * g)(z)$ defined by

$$(f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k.$$

For $\alpha_j \in \mathbb{C} \quad (j = 1, 2, \ldots, l)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\} (j = 1, 2, \ldots m)$, the *generalized hypergeometric function* ${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$${}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!}$$

$$(l \le m+1; l, m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N} := \{1, 2, 3\ldots\}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^p \ {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

the Dziok-Srivastava operator [8] (see also [26]) $H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$\begin{aligned} H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z) &:= h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \ldots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \ldots (\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!}. \end{aligned} \tag{2.1}$$

To make the notation simple, we write

$$H_p^{l,m}[\alpha_1]f(z) := H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)f(z).$$

It is well known [8] that

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z(H_p^{l,m}[\alpha_1]f(z))' + (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z). \tag{2.2}$$

Special cases of the Dziok-Srivastava linear operator includes the Hohlov linear operator [9], the Carlson-Shaffer linear operator [5], the Ruscheweyh derivative operator [23], the generalized Bernardi-Libera-Livingston linear integral operator (*cf.* [2], [11], [12]) and the Srivastava-Owa fractional derivative operators (*cf.* [17], [18]).

Motivated by the multiplier transformation on $\mathcal{A}$, we define the operator $I_p(n, \lambda)$ on $\mathcal{A}_p$ by the following infinite series

$$I_p(n, \lambda)f(z) := z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k \quad (\lambda \ge 0). \tag{2.3}$$

A straightforward calculation shows that

$$(p+\lambda)I_p(n+1, \lambda)f(z) = z[I_p(n, \lambda)f(z)]' + \lambda I_p(n, \lambda)f(z). \tag{2.4}$$

The operator $I_p(n, \lambda)$ is closely related to the Sălăgean derivative operators [24]. The operator $I_\lambda^n := I_1(n, \lambda)$ was studied recently by Cho-Srivastava [6] and Cho-Kim [7]. The operator $I_n := I_1(n, 1)$ was studied by Uralegaddi-Somanatha [27].

In our present investigation, we need the following:

Definition 2.1. [14, Definition 2, p. 817] *Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\Delta} - E(f)$, where*

$$E(f) = \{\zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 2.2. [1] *Let $q(z)$ be convex univalent in Δ and $\alpha, \beta, \gamma \in \mathbb{C}$. Further assume that*

$$\Re\left[\frac{\alpha}{\gamma} + \frac{2\beta}{\gamma} q(z)\right] > 0.$$

If $\psi(z) \in \mathcal{H}[q(0), 1] \cap Q$, $\alpha\psi(z) + \beta\psi^2(z) + \gamma z\psi'(z)$ is univalent in Δ, then

$$\alpha q(z) + \beta q^2(z) + \gamma z q'(z) \prec \alpha\psi(z) + \beta\psi^2(z) + \gamma z\psi'(z)$$

implies $q(z) \prec \psi(z)$ and $q(z)$ is the best subordinant.

Lemma 2.3. [1] *Let $q(z) \neq 0$ be convex univalent in Δ and $\alpha, \beta \in \mathbb{C}$. Further assume that $\Re\left[\alpha\overline{\beta} q(z)\right] > 0$ and $zq'(z)/q(z)$ is starlike univalent in Δ. If $\psi(z) \in \mathcal{H}[q(0), 1] \cap Q$, $\psi(z) \neq 0$, $\alpha\psi(z) + \beta\frac{z\psi'(z)}{\psi(z)}$ is univalent in Δ, then*

$$\alpha q(z) + \beta\frac{zq'(z)}{q(z)} \prec \alpha\psi(z) + \beta\frac{z\psi'(z)}{\psi(z)},$$

implies $q(z) \prec \psi(z)$ and $q(z)$ is the best subordinant.

We also need the following result:

Lemma 2.4. *Let $q(z) \neq 0$ be univalent in Δ and $\alpha, \beta \in \mathbb{C}$. Further assume that $\Re\left[\overline{\alpha}\beta q(z)\right] > 0$ and $zq'(z)/q(z)$ is starlike univalent in Δ. If $\psi(z) \in \mathcal{H}[q(0), 1] \cap Q$, $\psi(z) \neq 0$, $\frac{\alpha}{\psi(z)} - \beta\frac{z\psi'(z)}{\psi(z)}$ is univalent in Δ, then*

$$\frac{\alpha}{q(z)} - \beta\frac{zq'(z)}{q(z)} \prec \frac{\alpha}{\psi(z)} - \beta\frac{z\psi'(z)}{\psi(z)},$$

implies $q(z) \prec \psi(z)$ and $q(z)$ is the best subordinant.

The proof of Lemma 2.4 is similar to the proof of Lemma 2.3 and therefore we omit the proof.

3. Sufficient Conditions Involving Dziok-Srivastava Linear Operator

By making use of Lemma 2.2, we first prove the following:

Theorem 3.1. *Let $q(z)$ be convex univalent, $\alpha \neq 0$. Further assume that*

$$\Re\left\{\frac{1+\alpha_1(1-\alpha)}{\alpha}+2\alpha_1 q(z)\right\}>0.$$

If $f(z)\in\mathcal{A}_p$, $H_p^{l,m}[\alpha_1+1]f(z)/H_p^{l,m}[\alpha_1]f(z)\in\mathcal{H}[1,1]\cap Q$,

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\left\{1-\alpha+\alpha\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}\right\}$$

is univalent in Δ, then

$$\begin{aligned}&\frac{1+\alpha_1(1-\alpha)}{1+\alpha_1}q(z)+\frac{\alpha\alpha_1}{1+\alpha_1}q^2(z)+\frac{\alpha}{1+\alpha_1}zq'(z)\\ &\quad\prec\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\left\{1-\alpha+\alpha\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}\right\}\end{aligned}\tag{3.1}$$

implies

$$q(z)\prec\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\tag{3.2}$$

and q(z) is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z):=\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}.\tag{3.3}$$

By a simple computation from (3.3), we get

$$\frac{z\psi'(z)}{\psi(z)}=\frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)}-\frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)}.\tag{3.4}$$

By making use of (2.2) in the equation (3.4), we obtain

$$\frac{z\psi'(z)}{\psi(z)}=(\alpha_1+1)\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}-\alpha_1\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}-1.\tag{3.5}$$

Using (3.3) in (3.5), we get

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}=\frac{1}{1+\alpha_1}\left[\frac{z\psi'(z)}{\psi(z)}+\alpha_1\psi(z)+1\right].\tag{3.6}$$

Therefore we have, from (3.3) and (3.6),

$$\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}\left\{1-\alpha+\alpha\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}\right\}$$
$$=\frac{1+\alpha_1(1-\alpha)}{1+\alpha_1}\psi(z)+\frac{\alpha\alpha_1}{1+\alpha_1}\psi^2(z)+\frac{\alpha}{1+\alpha_1}z\psi'(z). \tag{3.7}$$

In view of the equation (3.7), the subordination (3.1) becomes

$$[1+\alpha_1(1-\alpha)]q(z)+\alpha\alpha_1 q^2(z)+\alpha z q'(z)$$
$$\prec [1+\alpha_1(1-\alpha)]\psi(z)+\alpha\alpha_1\psi^2(z)+\alpha z\psi'(z)$$

and the result now follows by an application of Lemma 2.2. ■

By making use of Lemma 2.3, we now prove the following:

Theorem 3.2. *Let $q(z)\neq 0$ be convex univalent in Δ, $q(0)=1$. Let $zq'(z)/q(z)$ be starlike univalent in Δ. If $f(z)\in\mathcal{A}_p$, $0\neq \frac{z^{p(\alpha-1)}H_p^{l,m}[\alpha_1+1]f(z)}{(H_p^{l,m}[\alpha_1]f(z))^\alpha}\in\mathcal{H}[1,1]\cap Q$ and*

$$(\alpha_1+1)\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}-\alpha\alpha_1\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}$$

is univalent in Δ, then

$$\frac{zq'(z)}{q(z)}+(1-\alpha)\alpha_1+1$$
$$\prec(\alpha_1+1)\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}-\alpha\alpha_1\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)}, \tag{3.8}$$

implies

$$q(z)\prec\frac{z^{p(\alpha-1)}H_p^{l,m}[\alpha_1+1]f(z)}{(H_p^{l,m}[\alpha_1]f(z))^\alpha} \tag{3.9}$$

and q(z) is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z):=\frac{z^{p(\alpha-1)}H_p^{l,m}[\alpha_1+1]f(z)}{(H_p^{l,m}[\alpha_1]f(z))^\alpha}. \tag{3.10}$$

By a simple computation from (3.10), we get

$$\frac{z\psi'(z)}{\psi(z)}=p(\alpha-1)+\frac{z[H_p^{l,m}[\alpha_1+1]f(z)]'}{H_p^{l,m}[\alpha_1+1]f(z)}-\alpha\frac{z[H_p^{l,m}[\alpha_1]f(z)]'}{H_p^{l,m}[\alpha_1]f(z)}. \tag{3.11}$$

By making use of (2.2) in the equation (3.11), we obtain

$$(\alpha_1+1)\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} - \alpha\alpha_1\frac{H_p^{l,m}[\alpha_1+1]f(z)}{H_p^{l,m}[\alpha_1]f(z)} = \frac{z\psi'(z)}{\psi(z)} + (1-\alpha)\alpha_1 + 1. \tag{3.12}$$

In view of the equation (3.12), the subordination (3.8) becomes

$$\frac{zq'(z)}{q(z)} \prec \frac{z\psi'(z)}{\psi(z)}$$

and the result now follows by an application of Lemma 2.3. ■

By applying Lemma 2.4, we now prove the following:

Theorem 3.3. *Let $q(z)$ be univalent, $\Re(\overline{\alpha}_1 q(z)) > 0$ and $zq'(z)/q(z)$ be starlike univalent in Δ. If $f(z) \in \mathcal{A}_p$, $0 \neq H_p^{l,m}[\alpha_1]f(z)/H_p^{l,m}[\alpha_1+1]f(z) \in \mathcal{H}[1,1] \cap Q$, and*

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}$$

is univalent in Δ, then

$$\frac{1}{1+\alpha_1}\left[1 + \frac{\alpha_1}{q(z)} - \frac{zq'(z)}{q(z)}\right] \prec \frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} \tag{3.13}$$

implies

$$q(z) \prec \frac{H_p^{l,m}[\alpha_1]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} \tag{3.14}$$

and $q(z)$ is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := \frac{H_p^{l,m}[\alpha_1]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)}. \tag{3.15}$$

Then a computation shows that

$$\frac{H_p^{l,m}[\alpha_1+2]f(z)}{H_p^{l,m}[\alpha_1+1]f(z)} = \frac{1}{1+\alpha_1}\left[1 + \frac{\alpha_1}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)}\right]$$

and the superordination (3.13) becomes

$$\frac{\alpha_1}{q(z)} - \frac{zq'(z)}{q(z)} \prec \frac{\alpha_1}{\psi(z)} - \frac{z\psi'(z)}{\psi(z)}.$$

The result now follows from Lemma 2.4. ■

4. Sufficient Conditions Involving Multiplier Transformation

By making use of Lemma 2.2, we prove the following:

Theorem 4.1. *Let $q(z)$ be convex univalent, $\alpha \neq 0$. Further assume that*

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z)\right\} > 0.$$

If $f(z) \in \mathcal{A}_p$, $\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \in \mathcal{H}[1,1] \cap Q$,

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\left\{1-\alpha+\alpha\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}\right\}$$

is univalent in Δ, then

$$\begin{aligned}(1-\alpha)q(z) + \alpha q^2(z) + \frac{\alpha}{p+\lambda}zq'(z)\\ \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\left\{1-\alpha+\alpha\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}\right\},\end{aligned} \tag{4.1}$$

implies

$$q(z) \prec \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \tag{4.2}$$

and q(z) is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}. \tag{4.3}$$

By a simple computation from (4.3), we get

$$\frac{z\psi'(z)}{\psi(z)} = \frac{z[I_p(n+1,\lambda)f(z)]'}{I_p(n+1,\lambda)f(z)} - \frac{z[I_p(n,\lambda)f(z)]'}{I_p(n,\lambda)f(z)}. \tag{4.4}$$

By making use of (2.4) in the equation (4.4), we obtain

$$\frac{z\psi'(z)}{\psi(z)} = (p+\lambda)\left[\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} - \frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] \tag{4.5}$$

Using (4.3) in (4.5), we get

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \frac{1}{p+\lambda}\left[\frac{z\psi'(z)}{\psi(z)} + (p+\lambda)\psi(z)\right]. \tag{4.6}$$

Therefore we have from (4.6),

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\left\{1-\alpha+\alpha\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}\right\}$$
$$=(1-\alpha)\psi(z)+\alpha\psi^2(z)+\frac{\alpha}{p+\lambda}z\psi'(z). \tag{4.7}$$

In view of the equation (4.7), the subordination (4.1) becomes

$$(1-\alpha)q(z)+\alpha q^2(z)+\frac{\alpha}{p+\lambda}zq'(z)$$
$$\prec (1-\alpha)\psi(z)+\alpha\psi^2(z)+\frac{\alpha}{p+\lambda}z\psi'(z)$$

and the result now follows by an application of Lemma 2.2. ∎

By using Lemma 2.3, we now prove the following theorem.

Theorem 4.2. *Let $q(z)\neq 0$ be convex univalent in Δ, $q(0)=1$. Let $zq'(z)/q(z)$ be starlike univalent in Δ. If $f(z)\in\mathcal{A}_p$, $0\neq\frac{z^{p(\alpha-1)}I_p(n+1,\lambda)f(z)}{(I_p(n,\lambda)f(z))^\alpha}\in\mathcal{H}[1,1]\cap Q$,*

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}$$

is univalent in Δ, then

$$\frac{1}{p+\lambda}\frac{zq'(z)}{q(z)}+1-\alpha\prec\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \tag{4.8}$$

implies

$$q(z)\prec\frac{z^{p(\alpha-1)}I_p(n+1,\lambda)f(z)}{(I_p(n,\lambda)f(z))^\alpha} \tag{4.9}$$

and q(z) is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z):=\frac{z^{p(\alpha-1)}I_p(n+1,\lambda)f(z)}{(I_p(n,\lambda)f(z))^\alpha}. \tag{4.10}$$

By a simple computation from (4.10), we get

$$\frac{z\psi'(z)}{\psi(z)}=p(\alpha-1)+\frac{z[I_p(n+1,\lambda)f(z)]'}{I_p(n+1,\lambda)f(z)}-\alpha\frac{z[I_p(n,\lambda)f(z)]'}{I_p(n,\lambda)f(z)}. \tag{4.11}$$

By making use of (2.4) in the equation (4.11), we obtain

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}-\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}=\frac{1}{p+\lambda}\frac{z\psi'(z)}{\psi(z)}+1-\alpha. \tag{4.12}$$

In view of the equation (4.12), the subordination (4.8) becomes

$$\frac{zq'(z)}{q(z)} \prec \frac{z\psi'(z)}{\psi(z)}$$

and the result now follows by an application of Lemma 2.3. ■

We now prove the following theorem by using Lemma 2.4.

Theorem 4.3. *Let $q(z)$ be univalent, $\Re(q(z)) > 0$ and $zq'(z)/q(z)$ be starlike univalent in Δ. If $f(z) \in \mathcal{A}_p$, $0 \neq I_p(n,\lambda)f(z)/I_p(n+1,\lambda)f(z) \in \mathcal{H}[1,1] \cap Q$, and*

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}$$

is univalent in Δ, then

$$\frac{1}{q(z)} - \frac{1}{p+\lambda}\frac{zq'(z)}{q(z)} \prec \frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} \tag{4.13}$$

implies

$$q(z) \prec \frac{I_p(n,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} \tag{4.14}$$

and q(z) is the best subordinant.

Proof. Define the function $\psi(z)$ by

$$\psi(z) := \frac{I_p(n,\lambda)f(z)}{I_p(n+1,\lambda)f(z)}. \tag{4.15}$$

Then a computation shows that

$$\frac{I_p(n+2,\lambda)f(z)}{I_p(n+1,\lambda)f(z)} = \frac{1}{\psi(z)} - \frac{1}{p+\lambda}\frac{z\psi'(z)}{\psi(z)}$$

and the superordination (4.13) becomes

$$\frac{1}{q(z)} - \frac{1}{p+\lambda}\frac{zq'(z)}{q(z)} \prec \frac{1}{\psi(z)} - \frac{1}{p+\lambda}\frac{z\psi'(z)}{\psi(z)}.$$

The result now follows from Lemma 2.4. ■

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