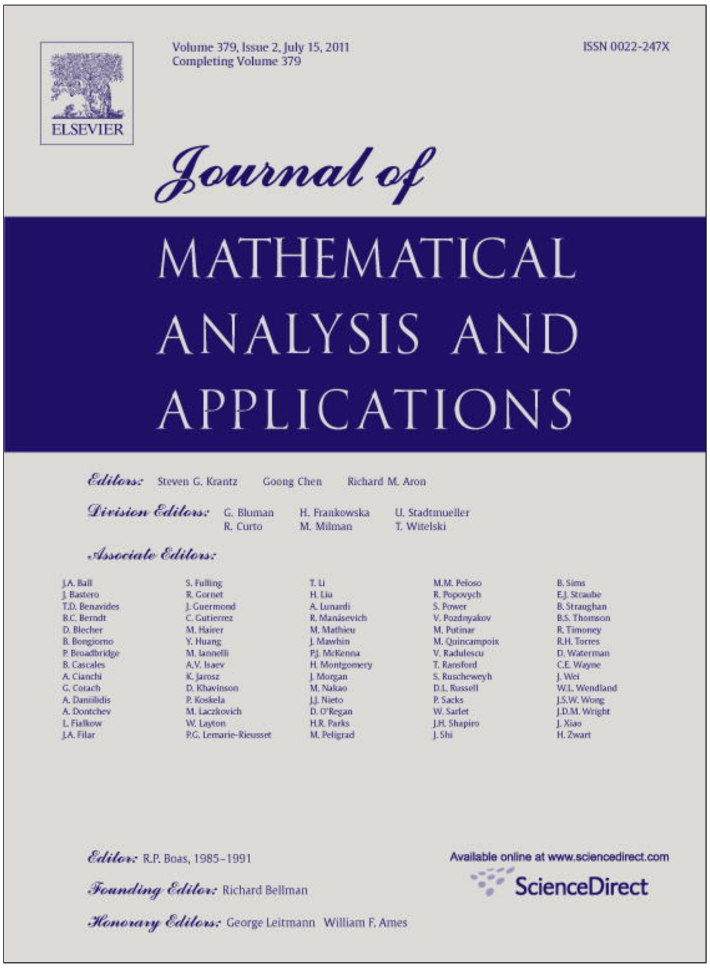
Volume 379, Issue 2, July 15, 2011
Completing Volume 379
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
Division Editors: G. Bluman R. Curto H. Frankowska M. Milman U. Stadtmueller T. Witelski
Associate Editors:
J.A. Ball
J. Bastero
T.D. Benavides
B.C. Berndt
D. Blecher
B. Bongiorno
P. Broadbridge
B. Cascales
A. Cianchi
G. Corach
A. Daniilidis
A. Dontchev
L. Fialkow
J.A. Filar
S. Fulling
R. Gornet
J. Guermond
C. Gutierrez
M. Hairer
Y. Huang
M. Iannelli
A.V. Isaev
K. Jarosz
D. Khavinson
P. Koskela
M. Laczkovich
W. Layton
P.G. Lemarie-Rieusset
T. Li
H. Liu
A. Lunardi
R. Manásevich
M. Mathieu
J. Mawhin
P.J. McKenna
H. Montgomery
J. Morgan
M. Nakao
J.J. Nieto
D. O'Regan
H.R. Parks
M. Peligrad
M.M. Peloso
R. Popovych
S. Power
V. Pozdnyakov
M. Putinar
M. Quincampoix
V. Radulescu
T. Ransford
S. Ruscheweyh
D.L. Russell
P. Sacks
W. Sarlet
J.H. Shapiro
J. Shi
B. Sims
E.J. Straube
B. Straughan
B.S. Thomson
R. Timoney
R.H. Torres
D. Waterman
C.E. Wayne
J. Wei
W.L. Wendland
J.S.W. Wong
J.D.M. Wright
J. Xiao
H. Zwart
Editor: R.P. Boas, 1985–1991
Founding Editor: Richard Bellman
Honorary Editors: George Leitmann William F. Ames
Available online at www.sciencedirect.com
ScienceDirect

J. Math. Anal. Appl. 379 (2011) 512–517

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# Bohr's phenomenon for analytic functions into the exterior of a compact convex body ☆

Y. Abu Muhanna [a], Rosihan M. Ali [b,*]

[a] Department of Mathematics, American University of Sharjah, Sharjah, Box 26666, United Arab Emirates
[b] School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

ARTICLE INFO

*Article history:*
Received 15 October 2010
Available online 15 January 2011
Submitted by R. Timoney

*Keywords:*
Bohr's inequality
Subordination
Covering map
Convex body

ABSTRACT

Bohr's inequality for the class of analytic functions mapping the unit disk into the exterior of a compact convex body is established. In this general case, the radius obtained is $|z| < 3 - 2\sqrt{2}$. When the compact convex body is the closed unit disk, a sharp radius of $1/3$ is obtained.



## 1. Introduction

Bohr's inequality states that if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is analytic in the unit disk $U$ and $|f(z)| < 1$ for all $z \in U$, then

$$\sum_{n=0}^{\infty} |a_n z^n| \leqslant 1 \tag{1.1}$$

for all $z \in U$ with $|z| \leqslant 1/3$. This inequality was discovered by Bohr [7] in 1914. Bohr actually obtained the inequality for $|z| \leqslant 1/6$. Wiener, Riesz and Schur, independently established the inequality for $|z| \leqslant 1/3$ and showed that the bound $1/3$ was sharp [10,15,16]. Other proofs were also given in [11–13]. Boas and Khavinson [6], and more recently Aizenberg [3–5] extended the inequality to several complex variables.

Bohr's inequality drew the attention of operator algebraists after Dixon [8] showed a connection between the inequality and the characterization of Banach algebras that satisfy von Neumann's inequality. Specifically, by using Bohr's inequality, Dixon constructed an example of a Banach algebra that satisfies von Neumann's inequality but is not isomorphic to the algebra of bounded operators on a Hilbert space. Paulsen and Singh [11] extended Bohr's inequality to Banach algebras.

A class of analytic (harmonic) functions in the unit disk $U$ is said to satisfy Bohr's phenomenon if an inequality of type (1.1) holds uniformly in $|z| < \rho_0$, for some $0 < \rho_0 \leqslant 1$, and for all functions in the class.

☆ The work presented here was supported in part by research grants from the American University of Sharjah and Universiti Sains Malaysia.

* Corresponding author.
*E-mail addresses:* ymuhanna@aus.edu (Y. Abu Muhanna), rosihan@cs.usm.my (R.M. Ali).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2011.01.023

In this article, we shall consider the space of functions subordinated to a given analytic function. For definition and details of subordination classes, see for example [9, Chapter 6] or [14, p. 35].

Let $f$ and $g$ be two analytic functions in the unit disk $U$. A function $g$ is subordinate to $f$ if there exists a Schwarz function $\varphi$, analytic in $U$ with $\varphi(0) = 0$ and $|\varphi(z)| < 1$, satisfying $g = f \circ \varphi$. In particular, when $f$ is univalent, $g$ is subordinate to $f$ when $g(U) \subset f(U)$ and $g(0) = f(0)$ ([9, p. 190], [14, p. 35]). Consequently, when $g$ is subordinate to $f$, then $|g'(0)| \leqslant |f'(0)|$.

In this sequel the class of all functions $g$ subordinate to a fixed function $f$ is denoted by $S(f)$ and $f(U) = \Omega$. The class $S(f)$ is said to satisfy Bohr's phenomenon if for any $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$, there is a $\rho_0$, $0 < \rho_0 \leqslant 1$, so that

$$\sum_{n=1}^{\infty} |b_n z^n| \leqslant d\big(f(0), \partial\Omega\big) \tag{1.2}$$

for $|z| < \rho_0$. Here $d(f(0), \partial\Omega)$ denotes the Euclidean distance between $f(0)$ and the boundary of a domain $\Omega$. Obviously, when $\Omega = U$, $d(f(0), \partial\Omega) = 1 - |f(0)|$ and in this case (1.2) reduces to (1.1).

It is known that $S(f)$ has Bohr's phenomenon when $f$ is univalent. Abu-Muhanna [2] recently showed that every $g(z) = \sum_{n=0}^{\infty} b_n z^n \in S(f)$ satisfies (1.2) for $|z| \leqslant \rho_0 = 3 - 2\sqrt{2} \cong 0.17157$. The radius $\rho_0$ is sharp for the Koebe function $f(z) = z/(1-z)^2$.

In particular, when $f$ is convex, it was shown in [5] that (1.2) remains valid for $\rho_0 = 1/3$, a result which includes (1.1) when $\Omega = U$.

In this article, we shall consider the case when $\Omega$ is the exterior of a compact convex body, and $\mathcal{F}_\Omega$ is the class of all analytic functions mapping $U$ into $\Omega$. The measure that will be used in this instance is the spherical chordal measure given by

$$\lambda(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{1+|z_1|^2}\sqrt{1+|z_2|^2}}.$$

When $\Omega$ is the exterior of the closed unit disk $U$, it is shown in Theorem 2.1 that (1.2) remains valid with $d(f(0), \partial\Omega)$ replaced by $\lambda(f(0), \partial\Omega)$ and $\rho_0 = 1/3$. This radius $\rho_0$ obtained is sharp. In the general situation when $\Omega$ is the exterior of a compact convex body, it is shown in Theorem 2.2 that (1.2) holds with $d(f(0), \partial\Omega)$ replaced by $\lambda(f(0), \partial\Omega)$ and $\rho_0 = 3 - 2\sqrt{2}$. However, the $\rho_0$ obtained may not be sharp.

We shall require the following results.

**Proposition 1.1.** *(See [1].) If $F$ is an analytic univalent function mapping the unit disk $U$ onto $\Omega$, where the complement of $\Omega$ is convex, and $F(z) \neq 0$, then any analytic function $f \in S(F^n)$, $n = 1, 2, \ldots,$ can be expressed as*

$$f(z) = \int_{|x|=1} F^n(xz)\, d\mu(x),$$

*for some probability measure $\mu$ on the unit circle $|x| = 1$. Consequently,*

$$f(z) = \int_{|x|=1} \exp\big(F(xz)\big)\, d\mu(x), \tag{1.3}$$

*for every $f \in S(\exp(F))$.*

We shall also require the Koebe one-quarter distortion inequalities

$$1 \geqslant d(0, \partial\Omega) \geqslant \frac{1}{4} \tag{1.4}$$

when $f$ is univalent and normalized by $f(0) = 0$ and $f'(0) = 1$, see for example [9, pp. 32, 45] or [14, pp. 21–22].

## 2. Subordination to the complement of a compact convex body

First we consider the case when $\Omega = c\overline{U}$, where $c\overline{U}$ denotes the complement of $\overline{U}$. Then any universal covering map is given by

$$\exp\left(\frac{1+\varphi(z)}{1-\varphi(z)}\right),$$

where $\varphi(z) = (z+a)/(1-\bar{a}z)$ is a Möbius transformation.

In this case $\mathcal{F}_\Omega$ consists of all analytic functions mapping the unit disk $U$ into $|w| > 1$. Here is the main result, which generalizes Bohr's theorem from the interior of the disk $U$ to its exterior.

**Theorem 2.1.** *If $\Omega = c\overline{U} = \{w\colon |w| > 1\}$ and $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in F_{\Omega}$, then*

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda(a_0, \partial\Omega)$$

*for $|z| \leqslant 1/3$. Moreover, the bound $1/3$ is sharp.*

As a preliminary to the proof, the class $F_{c\overline{U}}$ is shown to be the union of subordination classes.

**Proposition 2.1.** *Any $f \in F_{c\overline{U}}$ is subordinate to some universal covering map $G : U \to c\overline{U}$, with $f(0) = G(0) = a_0$. In other words, $f = G \circ \varphi$, where $\varphi$ is analytic in $U$, $|\varphi(z)| < 1$ and $\varphi(0) = 0$.*

**Proof.** Since $\operatorname{Re}\log f(z) > 0$ in $U$, it is clear that $\log f$ maps $U$ into the right-half plane. Let

$$b = \frac{\log a_0 - 1}{\log a_0 + 1}$$

and

$$\psi(z) = \frac{z + b}{1 + \bar{b}z}.$$

Then the function

$$W(z) = \frac{1 + \psi(z)}{1 - \psi(z)} \tag{2.1}$$

maps $|z| < 1$ univalently into the right-half plane with $W(0) = \log a_0$. Thus

$$\log f = W \circ \varphi$$

for some analytic $\varphi$ in $U$, $|\varphi(z)| < 1$ and $\varphi(0) = 0$. The result now follows by letting

$$G(z) = \exp\bigl(W(z)\bigr). \qquad \Box \tag{2.2}$$

Here now is the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $G$ be as given in (2.1) and (2.2), and $f$ be subordinate to $G$. Write

$$G(z) = a_0\left(1 + \sum_{n=1}^{\infty} B_n z^n\right),$$

$$f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n.$$

Then

$$W(z) = \frac{1 + \psi(z)}{1 - \psi(z)} = (\operatorname{Re}\log a_0)\left(\frac{1+z}{1-z}\right) + i\operatorname{Im}\log a_0 = \log a_0 + \frac{\log|a_0|^2 z}{1-z},$$

and

$$G(z) = a_0\left(1 + \sum_{n=1}^{\infty} B_n z^n\right) = a_0 \exp\left(\frac{\log|a_0|^2 z}{1-z}\right). \tag{2.3}$$

It follows from (2.2) and (1.3) that

$$|a_n| \leqslant |a_0 B_n|, \quad \text{for all } n \geqslant 1,$$

and

$$\sum_{n=1}^{\infty} |a_n||z|^n \leqslant |a_0| \sum_{n=1}^{\infty} |B_n||z|^n. \tag{2.4}$$

Now

$$|a_0 B_1| = \left|G'(0)\right| = |a_0|\left|W'(0)\right| = 2|a_0|\frac{1-|b|^2}{|1-b|^2} = 2|a_0|(\operatorname{Re}\log a_0) = |a_0|\log|a_0|^2.$$

Next, we show that the sequence $B_n$ is positive and increasing. It follows from (2.3) that

$$\begin{aligned} &B_1 = \log|a_0|^2 > 0, \\ &B_2 = \frac{\log|a_0|^2}{2}(B_1+2) = \log|a_0|^2\left(\frac{1}{2}B_1+1\right) = \frac{1}{2}B_1^2 + B_1 > B_1. \end{aligned} \tag{2.5}$$

Differentiating $G$ in (2.3) yields

$$G'(z) = \frac{\log|a_0|^2}{(1-z)^2}G(z).$$

Hence

$$\left(1-2z+z^2\right)G'(z) = \log|a_0|^2 G(z).$$

This gives the recurrence relation

$$B_{n+1} = \frac{\log|a_0|^2+2n}{n+1}B_n - \frac{n-1}{n+1}B_{n-1} = \left(2+\frac{\log|a_0|^2-2}{n+1}\right)B_n - \frac{n-1}{n+1}B_{n-1}. \tag{2.6}$$

Clearly (2.5) shows that $B_2 > B_1$. Assuming that $B_n > B_{n-1}$, it follows now from (2.6) that

$$B_{n+1} - B_n = \left(1+\frac{\log|a_0|^2-2}{n+1}\right)B_n - \frac{n-1}{n+1}B_{n-1} = \frac{\log|a_0|^2}{n+1}B_n + \frac{n-1}{n+1}(B_n - B_{n-1}) > 0.$$

Hence the sequence $B_n$ is increasing. Consequently, (2.4) implies that, for $|z| \leqslant \rho$,

$$\sum_{n=0}^{\infty}|a_n||z|^n \leqslant |a_0|\sum_{n=0}^{\infty}B_n\rho^n = |a_0|\exp\left[\frac{\log|a_0|^2\rho}{1-\rho}\right] = |a_0||a_0|^{\frac{2\rho}{1-\rho}}. \tag{2.7}$$

When $\rho = 1/3$, then

$$\sum_{n=0}^{\infty}|a_n||z|^n \leqslant |a_0|^2. \tag{2.8}$$

Simple calculation shows that

$$\frac{\lambda(|a_0|, |a_0^2|)}{\lambda(|a_0|, 1)} = \frac{\sqrt{2}|a_0|}{\sqrt{1+|a_0^4|}} < 1, \tag{2.9}$$

and consequently, it follows from (2.8) and (2.9) that

$$\lambda\left(\sum_{n=0}^{\infty}|a_n||z|^n, |a_0|\right) \leqslant \lambda\left(|a_0|, \left|a_0^2\right|\right) \leqslant \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega).$$

For sharpness, assume that $\rho > 1/3$. Then by (2.3) and (2.7),

$$\left|G(\rho)\right| = |a_0|\sum_{n=0}^{\infty}B_n\rho^n = |a_0||a_0|^{\frac{2\rho}{1-\rho}} = |a_0|^{\frac{1+\rho}{1-\rho}}.$$

Note that $\frac{1+\rho}{1-\rho} = 2+\delta$ with $\delta > 0$, and $\frac{1+\rho}{1-\rho} \to 2$ as $\rho \to \frac{1}{3}$. Also note that $\frac{|a_0|^{\frac{2\rho}{1-\rho}}-1}{|a_0|-1} \to \frac{2\rho}{1-\rho} = 1+\delta$ as $|a_0| \to 1$. Hence

$$\frac{\lambda(|a_0|, |a_0|^{\frac{1+\rho}{1-\rho}})}{\lambda(|a_0|, 1)} = \frac{\sqrt{2}|a_0|\frac{|a_0|^{\frac{2\rho}{1-\rho}}-1}{|a_0|-1}}{\sqrt{1+|a_0|^{4+2\delta}}} \to (1+\delta)$$

as $|a_0| \to 1$. Consequently, for $|a_0|$ close to 1,

$$\lambda\left(|a_0|\sum_{n=0}^{\infty}B_n|z|^n, |a_0|\right) = \lambda\left(|a_0|, |a_0|^{\frac{1+\rho}{1-\rho}}\right) > \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega). \qquad \square$$

The theorem below gives a result under a more general setting than Theorem 2.1.

**Theorem 2.2.** *Let $\Delta$ be a compact convex body with $0 \in \Delta$, $1 \in \partial\Delta$, and $\Omega = c\Delta$. Suppose the universal covering map from $U$ into $\Omega$ has a univalent logarithmic branch that maps $U$ into the complement of a convex set. If $f(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n \in \mathcal{F}_\Omega$ satisfies $a_0 > 1$, then for $|z| < 3 - 2\sqrt{2} \cong 0.17157$,*

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda(a_0, \partial\Omega).$$

**Proof.** Let $F$ be the universal covering map from $U$ onto $\Omega$ with $F(0) = a_0$. Let $G(z) = \log F(z)$ be its univalent logarithmic branch. Then

$$F(z) = \exp G(z),$$
$$a_0 + \sum_{n=1}^{\infty} A_n z^n = \exp\left(\log a_0 + \sum_{n=1}^{\infty} c_n z^n\right).$$

As $G$ is univalent,

$$\frac{G(z) - \log a_0}{c_1} = g \in S,$$

where $S$ is the class consisting of normalized analytic univalent functions in $U$. For $|z| \leqslant \rho$, it follows from comparing coefficients that

$$|a_0| + \sum_{n=1}^{\infty} |A_n|\rho^n \leqslant |a_0| \exp\left(\sum_{n=1}^{\infty} |c_n|\rho^n\right).$$

Further, since $g \in S$, then $|c_n| \leqslant n|c_1|$ for each $n$, and

$$\sum_{n=1}^{\infty} |c_n|\rho^n \leqslant |c_1| \frac{\rho}{(1-\rho)^2}.$$

Hence for $|z| \leqslant \rho$, it follows that

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0| \exp\left(\sum_{n=1}^{\infty} |c_n||z|^n\right) \leqslant |a_0| \exp\left(|c_1| \frac{\rho}{(1-\rho)^2}\right). \tag{2.10}$$

Since $0 \notin G(U)$, then $-\log a_0/c_1 \notin g(U)$. Thus the Koebe one-quarter distortion result (1.4) implies that

$$|c_1| \leqslant 4|\log a_0|,$$

and (2.10) yields

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0| \exp\left(4|\log a_0| \frac{\rho}{(1-\rho)^2}\right).$$

If $a_0 > 1$, then

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0|^{1+\frac{4\rho}{(1-\rho)^2}}. \tag{2.11}$$

Simple calculations show that when $\rho \leqslant 3 - 2\sqrt{2}$, then $4\rho/(1-\rho)^2 \leqslant 1$. Hence (2.11) becomes

$$|a_0| + \sum_{n=1}^{\infty} \left|A_n z^n\right| \leqslant |a_0|^2,$$

and (1.3) yields

$$\lambda\left(\sum_{n=0}^{\infty} |a_n||z|^n, |a_0|\right) \leqslant \lambda\left(|a_0|, \left|a_0^2\right|\right) \leqslant \lambda\left(|a_0|, 1\right) = \lambda(a_0, \partial\Omega). \qquad \square$$

## Acknowledgment

The authors are thankful to the referee for the several suggestions that helped to improve the presentation of this manuscript.

## References

[1] Y. Abu-Muhanna, D.J. Hallenbeck, A class of analytic functions with integral representations, Complex Var. Theory Appl. 19 (4) (1992) 271–278.
[2] Y. Abu-Muhanna, Bohr's phenomenon in subordination and bounded harmonic classes, Complex Var. Elliptic Equ. (2010) 1–8 (iFirst).
[3] L. Aizenberg, Multidimensional analogues of Bohr's theorem on power series, Proc. Amer. Math. Soc. 128 (4) (2000) 1147–1155.
[4] L. Aizenberg, N. Tarkhanov, A Bohr phenomenon for elliptic equations, Proc. Lond. Math. Soc. (3) 82 (2) (2001) 385–401.
[5] L. Aizenberg, Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series, in: Selected Topics in Complex Analysis, in: Oper. Theory Adv. Appl., vol. 158, Birkhäuser, Basel, 2005, pp. 87–94.
[6] H.P. Boas, D. Khavinson, Bohr's power series theorem in several variables, Proc. Amer. Math. Soc. 125 (10) (1997) 2975–2979.
[7] H. Bohr, A theorem concerning power series, Proc. Lond. Math. Soc. (3) 13 (1914) 1–5.
[8] P.G. Dixon, Banach algebras satisfying the non-unital von Neumann inequality, Bull. Lond. Math. Soc. 27 (4) (1995) 359–362.
[9] P.L. Duren, Univalent Functions, Springer, New York, 1983.
[10] V.I. Paulsen, G. Popescu, D. Singh, On Bohr's inequality, Proc. Lond. Math. Soc. (3) 85 (2) (2002) 493–512.
[11] V.I. Paulsen, D. Singh, Bohr's inequality for uniform algebras, Proc. Amer. Math. Soc. 132 (12) (2004) 3577–3579 (electronic).
[12] V.I. Paulsen, D. Singh, Extensions of Bohr's inequality, Bull. Lond. Math. Soc. 38 (6) (2006) 991–999.
[13] V.I. Paulsen, D. Singh, A simple proof of Bohr's inequality, preprint.
[14] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.
[15] S. Sidon, Über einen Satz von Herrn Bohr, Math. Z. 26 (1) (1927) 731–732.
[16] M. Tomić, Sur un théorème de H. Bohr, Math. Scand. 11 (1962) 103–106.