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Geometric Properties of Generalized Bessel Functions

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Abstract. In this work, the generalized Bessel functions with their normalization are considered. Various conditions are obtained so that these Bessel functions have certain geometric properties including close-to-convexity (univalency), starlikeness and convexity in the unit disc. Results obtained for certain classes are new and for the other classes for which similar results exist in the literature, examples are given to support that these results are better than the existing ones.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f defined in the unit disk $\mathbb{D}$ that are normalized by the condition $f(0) = 0 = f'(0) - 1$ and $\mathcal{S}$ be the subclass of functions in $\mathcal{A}$ that are univalent in the unit disk $\mathbb{D} = \{z : |z| < 1\}$. A function $f \in \mathcal{S}$ is said to be starlike or convex, if f maps $\mathbb{D}$ conformally onto domains, respectively, starlike with respect to origin or convex. The class of such functions are denoted by $\mathcal{S}^*$ and $\mathcal{C}$ respectively. Extension of these classes are $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$, $0 \leq \mu < 1$, and given by their respective analytic characterization

$$f \in \mathcal{S}^*(\mu) \Leftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \mu \qquad \text{and} \quad f \in \mathcal{C}(\mu) \Leftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu.$$

Another important class is known as close-to-convex of order μ with respect to a particular starlike function and analytically it can be represented as

$$\operatorname{Re} e^{i\eta}\left(\frac{zf'(z)}{g(z)} - \mu\right) > 0, \quad g \in \mathcal{S}^*, \quad z \in \mathbb{D},$$

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for some real $\eta \in (-\pi/2, \pi/2)$. The family of all close-to-convex functions of order μ relative to $g \in \mathcal{S}^*$ is denoted by $\mathcal{K}_g(\mu)$. For particular choice of g, we get particular class of close-to-convex functions $\mathcal{K}_g$. Note that in this work, we only consider the case where $\eta = 0$. An important fact about the class $\mathcal{K}_g$ is that $f \in \mathcal{K}_g$ implies $f \in \mathcal{S}$ in $\mathbb{D}$. More details about these classes can be found in [9] and for their generalizations, we refer the interested reader to [23].

The functions

$$z,\ \frac{z}{(1-z)},\ \frac{z}{1-z^2},\ \frac{z}{(1-z)^2} \quad \text{and} \quad \frac{z}{1-z+z^2} \tag{1.1}$$

and their particular rotations

$$\frac{z}{1+z},\ \frac{z}{1+z^2},\ \frac{z}{(1+z)^2} \quad \text{and} \quad \frac{z}{1+z+z^2}$$

are the only nine functions which are starlike univalent and have integer coefficients in $\mathbb{D}$, (see [13] for details). We note that, it is easy to give sufficient coefficient conditions for f to be close-to-convex, at least when the corresponding starlike function $g(z)$ takes one of the above forms. In this paper, we only consider z, $z/(1-z)$, $z/(1-z^2)$ and $\eta = 0$. Generalization and unification of the coefficient conditions for these classes is given in [34], by considering the starlike functions $z/(1-z)^\alpha$, $0 \leq \alpha \leq 2$.

We are also interested in another important class, introduced in [25], known as prestarlike of order μ, which is denoted as $\mathcal{R}_\mu$. A function $f \in \mathcal{A}$ is prestarlike of order μ if and only if

$$\begin{cases} \operatorname{Re} \frac{f(z)}{z} > 0, \quad z \in \mathbb{D} \quad \text{for} \quad \mu = 1, \\ \frac{z}{(1-z)^{2(1-\mu)}} * f(z) \in \mathcal{S}^*(\mu), \quad z \in \mathbb{D} \quad \text{for} \quad 0 \leq \mu < 1. \end{cases}$$

In particular $\mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$ and $\mathcal{R}_0 = \mathcal{C}$. Here $*$ is the well known Hadamard product or convolution, defined as $(f * g)(z) = z + \sum_{k=2}^\infty a_k b_k z^k$, where $f(z) = z + \sum_{k=2}^\infty a_k z^k$ and $g(z) = z + \sum_{k=2}^\infty b_k z^k$. For details about these convolution techniques and the corresponding properties related to the class $\mathcal{S}$, we refer [9, 24].

Among various results of the class $\mathcal{R}_\mu$, we list the following:

Lemma 1.1. [26]

(1) *For $f, g \in \mathcal{R}_\mu$, we have $f * g \in \mathcal{R}_\mu$.*
(2) *For $\mu \leq \beta \leq 1$, we have $\mathcal{R}_\mu \subset \mathcal{R}_\beta$.*
(3) *For $f \in \mathcal{S}^*(\mu)$, $g \in \mathcal{R}_\mu$, we have $f * g \in \mathcal{S}^*(\mu)$.*
(4) *For $\mu \leq 1/2$, $\mathcal{R}_\mu \subset \mathcal{S}$.*

In this work, we also consider a generalization of $\mathcal{R}_\mu$ given in [28]. A function $f \in \mathcal{A}$ is in $\mathcal{R}[\alpha, \mu]$, if $f * \mathcal{S}_\alpha \in \mathcal{S}^*(\mu)$ where $\mathcal{S}_\alpha = z/(1-z)^{2-2\alpha}, 0 \leq \alpha < 1$. Note that $\mathcal{R}[\mu, \mu] = \mathcal{R}_\mu$.

Finding the relation between various classes of analytic functions is an interesting research problem and has contributed many results in the past. We are interested in the following particular problem.

Problem 1.1. *For a class of analytic functions $\mathcal{F} \subset \mathcal{A}$, find sufficient conditions such that $\mathcal{F}$ is starlike, (convex or close-to-convex) in $\mathbb{D}$.*

The answer to this problem is two-fold. One way is to consider a particular class and find various technique so that $\mathcal{F}$ answers Problem 1.1. The class consisting of all hypergeometric functions $z\ {}_pF_q$, of the form,

$$z\ {}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;z)=\sum_{k=1}^{\infty}\frac{(a_1)_{k-1}\cdots(a_p)_{k-1}}{(c_1)_{k-1}\cdots(c_q)_{k-1}(1)_{k-1}}z^k,\qquad z\in\mathbb{D},$$

where none of the denominator parameters can be zero or a negative integer and $(a)_n$ is the well known Pochhammer symbol given by $(\lambda)_n=\lambda(\lambda+1)_{n-1}$, $(\lambda)_0=1$ is one such example. The search for a solution to this class, with reference to the Problem 1.1 has a long literature, for example see, [10, 21, 29, 30, 31] and references therein. Even though, this problem is far from getting completely solved for the generalized hypergeometric functions ${}_pF_q$, its particular case, $p=2$ and $q=1$ is almost solved up to starlike and convex functions (see [17, 18, 33] for details).

Another way is to find various techniques to obtain certain properties for the general class $\mathcal{F}$ and using these properties to deduce the applications for various types of functions like ${}_pF_q$ and polylogarithms. Among various techniques used, Fejer's coefficient criterion [11], Vietoris' coefficient condition [15, 27], differential subordination [3, 4, 8, 19, 32], Jack's lemma [9, 14], and duality techniques [25] are of interest to many researchers in this field. One another way is to find the positivity conditions of certain finite sums [1, 16, 20] and using it to deduce the conditions for the geometric behaviour of the class $\mathcal{F}$. In this work, for a particular class of $\mathcal{F}$, we use the results obtained in [20], using the technique of positivity of certain finite sums.

The following result is given in [20].

Lemma 1.2. [20] *Let $\alpha\geq 0$,$\gamma\geq 1$ and $a_0,\ a_1,\ a_2,\ \ldots$ be a sequence of positive numbers such that*

$$2a_1\leq a_0,\quad (2+\alpha)^{\gamma}a_2\leq a_1,\quad (k+1+\alpha)^{\gamma}a_{k+1}\leq(k+\alpha)^{\gamma}a_k, k\geq 2.$$

Then for all $0<\phi<\pi$ and for all $k\in\mathbb{N}$, the following inequalities hold:

1. $\frac{a_0}{2}+\sum_{k=1}^{n}a_k\cos k\phi>0.$
2. $\sum_{k=1}^{n}a_k\sin k\phi>0.$

Lemma 1.2 is generalization of earlier results obtained by [1] and [5]. We also remark that Lemma 1.2 is also true, if we replace a_k by r^ka_k, $0\leq r<1$. In [20], using Lemma 1.2, a sufficient condition on a_k such that the normalized analytic function $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$ are close-to-convex with respect to starlike function z, $z/(1-z)$, $z/(1-z^2)$ are found. In what follows, together with these results, we also mention the result which gives the condition for which $f(z)$ is starlike of order μ.

Lemma 1.3. [20, Theorem 4.1] *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real number such that $a_1=1$, $a_1\geq 2a_2$. Suppose that, for $1\leq\gamma<2$ $2a_2\geq 2^{\gamma}(3a_3)$ and $k(k-1-\gamma)a_k\geq(k-1)(k+1)a_{k+1}$, $\forall k\geq 3$. Then, $f(z)=z+\sum_{n=2}^{\infty}a_kz^k$ is close-to-convex with respect to both the starlike functions z and $z/(1-z)$. Further, for the same condition f is starlike univalent.*

Corollary 1.1. *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real number such that $1=a_1\geq 2a_2\geq 6a_3$ and $k(k-2)a_k\geq(k-1)(k+1)a_{k+1}$, $\forall k\geq 3$. Then $f(z)=z+\sum_{n=2}^{\infty}a_kz^k$*

is close-to-convex with respect to both the starlike functions z and $z/(1-z)$. Further that, for the same condition f is starlike univalent.

Lemma 1.4. [20, Theorem 4.3] *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers such that $a_1 = 1$. For $0 \leq \mu < 1$, let*

(1) $(1-\mu)a_1 \geq (2-\mu)a_2 \geq 2^{(\mu+1)}(3-\mu)a_3$,

(2) $(k-1-\mu)(k-\mu)a_k \geq k(k+1-\mu)a_{k+1}, \forall k \geq 3$,

then $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in \mathcal{S}^*(\mu)$.

Lemma 1.5. [20, Theorem 4.4] *Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of positive real numbers such that $a_1 = 1$. Suppose that, $a_1 \geq 8a_2$, and $(k-1)a_k \geq (k+1)a_{k+1}, \forall k \geq 2$. Then, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ is close-to-convex with respect to the starlike function $z/(1-z^2)$.*

2. The generalized class of Bessel functions

As mentioned earlier, we are interested in finding one particular class of $\mathcal{F}$ such that it addresses Problem 1.1. In this context, many results are available in the literature regarding the generalized hypergeometric functions, polylogarithms [10, 21, 31]. Here, to differ from this usual practice, we are interested in considering certain class of functions that are related to the well known Bessel functions. Consider the differential equation

$$z^2 w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0 \tag{2.1}$$

where $b, c, p \in \mathbb{C}$. The differential equation (2.1) is known as the generalized Bessel differential equation. For a particular value of b and c, the differential equation (2.1) reduces to (i) Bessel ($b = 1 = c$), (ii) Modified Bessel ($b = 1, c = -1$) and (iii) Spherical Bessel ($b = 2, c = 1$) differential equations. A particular solution of the equation (2.1), known as generalized Bessel function of the first kind of order p, can be given as

$$w_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma(p+k+\frac{b+1}{2})} \cdot \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}. \tag{2.2}$$

The study of the geometric properties such as univalency, starlikeness, convexity of $w_p(z)$ permit us to study the geometric properties of Bessel, modified Bessel and spherical Bessel functions together. For further details, we refer the interested readers to [6, 7] and to the references therein. To study the convexity and univalency of the generalized Bessel functions, in [6, 7] $w_p(z)$ was normalized by the transformation $u_p(z) = [a_0(p)]^{-1} z^{-p/2} w_p(\sqrt{z})$. It is easy to see that the series representation of $u_p(z)$ is

$$u_p(z) = {}_0F_1\left(\kappa, -\frac{cz}{4}\right) = \sum_{k \geq 0} \frac{(-1)^k c^k}{4^k (\kappa)_k} \frac{z^k}{k!} \tag{2.3}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3 \cdots$.

Further that the function $u_p(z)$ is analytic in $\mathbb{D}$ and satisfies the differential equation

$$4z^2 u''(z) + 4\kappa z u'(z) + czu(z) = 0. \tag{2.4}$$

Now, we list few results given in [6] for the geometric properties such as univalency, starlikeness, convexity for the function u_p in $\mathbb{D}$ that are useful for further discussion.

Lemma 2.1. [6] *If $0 \leq \mu < 1/2$ and $b, p, c \in \mathbb{R}$, then the following assertions are true:*

(i) *If $4\kappa \geq (1-\mu)(1-2\mu)^{-1/2}|c| + 1$, then $\operatorname{Re} u_p(z) \geq \mu$ for all $z \in \mathbb{D}$;*
(ii) *If $4\kappa \geq (1-\mu)(1-2\mu)^{-1/2}|c|$ and $c \neq 0$, then $u_p(z)$ is close-to-convex of order μ in $\mathbb{D}$.*

Lemma 2.2. [6] *If $0 \leq \mu < 1$ and $b, p, c \in \mathbb{R}$ such that $c \neq 0$ and $4\mu^2 + (|c|-6)\mu + 2 \geq 0$, then the functions w_p and u_p have the following properties:*

(i) *If $4(1-\mu)\kappa \geq |c| + 2(1-\mu)(1-2\mu)$, then $u_p(z)$ is convex of order μ in $\mathbb{D}$;*
(ii) *If $4(1-\mu)\kappa \geq |c| + 2(1-\mu)(3-2\mu)$, then $zu_p(z)$ is starlike of order μ in $\mathbb{D}$;*
(iii) *If If $4(1-\mu)\kappa \geq |c| + 2(1-\mu)(3-2\mu)$ and $\mu \neq 0$, then $z^{(2(1-\mu)-p)/(2\mu)} w_p(z^{1/(2\mu)})$ is starlike in $\mathbb{D}$.*

For a function $f \in \mathcal{S}$, the Alexander transform is defined as $\Lambda_f(z) := \int_0^z \frac{f(t)}{t} dt$.

Lemma 2.3. [6] *Let $c < 0$ and $b, p \in \mathbb{R}$, then Λ_{U_p} is close-to-convex with respect to starlike functions z and $z/(1-z)$ if $4\kappa > -(c+2) + \sqrt{c^2/2 - 4c + 4}$. Further Λ_{U_p} is also starlike. Here U_p is given by* (2.5).

In this work we normalize $w_p(z)$ by the transformation

$$U_p(z) = z_0F_1(\kappa, -\frac{cz}{4}) = [a_0(p)]^{-1} z^{1-p/2} w_p(\sqrt{z}) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{2.5}$$

where

$$b_{k+1} = -\frac{c}{4k(\kappa + k - 1)} b_k, \quad k \geq 1.$$

Clearly, $U_p(0) = 0 = U_p'(0) - 1$ and $U_p(z) = zu_p(z)$. The reason behind the consideration of $U_p(z)$ is the fact that the geometric property of an analytic function $f(z)$ in $\mathbb{D}$ normalized by $f(0) = 1$, may not be inherited by $zf(z)$. For example, consider the function $1 + z$, which is convex but it's normalization $f(z) = z + z^2$ is not even univalent in $\mathbb{D}$ as $f'(-1/2) = 0$.

Lemma 2.4. [6] *If $b, p, c \in \mathbb{C}$ such that $\kappa = p + (b+1)/2 \neq 0, -1, -2, -3, \ldots,$ and $z \in \mathbb{C}$, then for the normalized generalized Bessel function of the first kind of order p, we have the following recurrence relation*

$$4\kappa u_p'(z) = -c u_{p+1}(z) \tag{2.6}$$

In Section 3, we find the conditions under which $U_p(z)$ and $u_p(z)$ are close-to-convex with respect to particular starlike functions. We restrict ourselves in finding only the starlikeness and convexity of $U_p(z)$, since we are interested only in the normalized case. In Section 4, we find conditions under which $U_p(z)$ is in the class of prestarlike functions. Results related to a particular integral transform is discussed in Section 5. We also provide examples in the next section to show that our result are better than the results available in the literature, at least for the case $c < 0$. Moreover, there seems to be not many results for the case of prestarlike functions related to Bessel functions in the literature. Further, since the modified Bessel functions is in fact just the Bessel function with imaginary argument, and consequently it maps the unit disk into same domain as the Bessel function, as we have better range for modified Bessel function, we can claim that our result is also better for Bessel functions.

3. Close-to-convexity, starlikeness and convexity of generalized Bessel functions

We give one of our main results that answers Problem 1.1, whose proof is given in Section 6.

Theorem 3.1. *Let $c < 0$, $0 \leq \mu < 1$ and $p, b \in \mathbb{R}$. Further, if for $\alpha \geq 0$, $[(2+\alpha)^{\mu+1}(1-\mu) - 2]c + 8(1-\mu) \geq 0$. Then the following are true.*

(1) $\operatorname{Re} u_{p,n}(z) > \mu$ *in* $\mathbb{D}$ *for* $4(1-\mu)\kappa \geq -c$.
(2) $u_p(z)$ *is close-to-convex of order* μ *in* $\mathbb{D}$ *for* $4(1-\mu)\kappa \geq -c - 4(1-\mu)$.

where $u_{p,n}(z) = \sum_{k=0}^{n} \frac{(-c)^k}{4^k (\kappa)_k} \frac{z^k}{k!}$.

Since for $\alpha = 0$, we have $[(2+\alpha)^{\mu+1}(1-\mu) - 2] < 0$, the following results are immediate.

Corollary 3.1. *Let $c < 0$ and $p, b \in \mathbb{R}$.*

(1) $\operatorname{Re} u_{p,n}(z) > \mu$ *in* $\mathbb{D}$ *for* $4(1-\mu)\kappa \geq -c$.
(2) $u_p(z)$ *is close-to-convex of order* μ *in* $\mathbb{D}$ *for* $4(1-\mu)\kappa \geq -c - 4(1-\mu)$.

Remark 3.1. By Lemma 2.1, for $0 \leq \mu < 1/2$, if $4\kappa \geq (1-\mu)(1-2\mu)^{-1/2}|c|$ and $c \neq 0$, then we have $u_p(z)$ is close-to-convex of order μ. Lemma 2.1 does not say anything when $\mu \geq 1/2$. Whereas Corollary 3.1 implies that $u_p(z)$ is close-to-convex of order μ, for $0 \leq \mu < 1$ if $\kappa \geq -\frac{1}{4(1-\mu)}c - 1$ and $c < 0$.

Now for $0 \leq \mu < 1/2$ and $c < 0$

$$\left(-\frac{(1-\mu)}{(1-2\mu)^{1/2}}c\right) - \left(-\frac{1}{4(1-\mu)}c - 1\right) = -\left[\frac{(1-\mu)}{4(1-2\mu)^{1/2}} - \frac{1}{4(1-\mu)}\right]c + 1 \geq 0,$$

as

$$(1-\mu)^2 - (1-2\mu)^{1/2} = (1-2\mu)^2 + 2\mu(1-2\mu) + \mu^2 - (1-2\mu)^{1/2} \geq 0.$$

Therefore, we have

$$\left(-\frac{(1-\mu)}{(1-2\mu)^{1/2}}c\right) \geq \left(-\frac{1}{4(1-\mu)}c - 1\right)$$

and hence Theorem 3.1(Corollary 3.1) is better than the Lemma 2.1 when $c < 0$, in the sense that Theorem 3.1 gives better range of κ.

Corollary 3.2. *Let $c < 0$ and $b \in \mathbb{R}$. Then for $p \geq p_1$, $u_p(z)$ is close-to-convex of order μ in $\mathbb{D}$, where $p_1 = -((b+3)/2 - c/4(1-\mu))$.*

Similar to class u_p, results for the class U_p can be obtained and we state this as a theorem, whereas its proof is given in Section 6.

Theorem 3.2. *Let $c < 0$ and $b, p \in \mathbb{R}$ such that $\kappa \geq -c/2$. Then $U_p(z)$ is close-to-convex with respect to starlike function z and $z/(1-z)$.*

Further $U_p(z)$ is also starlike under the same condition.

Remark 3.2. In [6, Theorem 4.1], using a result given in [22], it has been proved that $U_P(z)$ is close-to-convex with respect to $z/(1-z)$ with the condition $\kappa > -c/2$. Theorem 3.2 extends this result for starlike functions also.

In the case of close-to-convexity of $U_p(z)$ with respect to $z/(1-z^2)$, consider $U_p(z)$ as $U_p(z) = z + \sum_{k=2}^{\infty} a_k z^k$, where

$$a_1 = 1, \quad a_2 = -\frac{c}{4\kappa} \quad \text{and} \quad a_{k+1} = -\frac{c}{4k(\kappa + k - 1)} a_k, \quad \forall k \geq 2. \tag{3.1}$$

Since, $a_1 - 8a_2 = 1 + 2c/\kappa \geq 0$ and it s easy to verify that, for $k \geq 2$, $(k-1)a_k - (k+1)a_{k+1} \geq 0$, the following result is a consequence of Lemma 1.5. We omit the details of the proof.

Theorem 3.3. *Let $c < 0$ and $b, p \in \mathbb{R}$ such that $\kappa \geq -2c$. Then $U_P(z)$ is close-to-convex w.r.to starlike function $z/(1-z^2)$*

We answer, the remaining part of Problem 1.1, concerning the starlikeness and convexity of $U_p(z)$, in the following results.

Theorem 3.4. *Let $c < 0$, $0 \leq \mu < 1$ and $p, b \in \mathbb{R}$. If $4(1-\mu)\kappa \geq -(2-\mu)c$, then $U_p(z)$ is starlike of order μ in $\mathbb{D}$.*

Proof. It is enough to verify that $U_p(z)$ satisfies conditions given in Lemma 1.4. As before, consider $U_p(z) = z + \sum_{k=2}^{\infty} a_k z^k$, where $\{a_k\}$ satisfies (3.1). By a simple calculation, we observe that

$$4(1-\mu)\kappa \geq -(2-\mu)c \quad \text{implies} \quad (1-\mu)a_1 \geq (2-\mu)a_2,$$

and

$$\begin{aligned}(2-\mu)a_2 - 2^{(\mu+1)}(3-\mu)a_3 &= \frac{a_2}{8(\kappa+1)}\left(8(2-\mu)(\kappa+1) + 2^{(\mu+1)}(3-\mu)c\right)\\ &\geq \frac{a_2}{8(1-\mu)(\kappa+1)}\left(-2(2-\mu)^2 + 2^{(\mu+1)}(3-\mu)(1-\mu)\right)c\\ &\geq 0.\end{aligned}$$

Now, let $(k-1-\mu)(k-\mu)a_k - k(k+1-\mu)a_{k+1} = A(k)M(k)$, where

$$A(k) = \frac{a_k}{4k(\kappa+k-1)}$$

$$M(k) = 4k(\kappa+k-1)(k-1-\mu)(k-\mu) + ck(k+1-\mu) = \sum_{i=1}^{5} T_i(k-3)^i$$

where $T_1 = 4$ and $T_2 = (40 + 4\kappa - 8\mu) > 0$,

$$T_3 = 60(1-\mu) + 8(1-\mu)\kappa + c + 24\kappa + 88 + 4\mu^2 \geq 0,$$

$$T_4 = 148(1-\mu) + 44(1-\mu)\kappa + (7-\mu)c + 40\kappa + 4\mu^2(5+\kappa) + 92 \geq 0,$$

$$T_5 = 120(1-\mu) + 60(1-\mu)\kappa + (12-3\mu)c + 12\kappa + 12\mu^2(2+\kappa) + 24 \geq 0,$$

$M(k)$ is an increasing function in $k \geq 3$. Further that $M(3) > 0$ implies that $(k-1-\mu)(k-\mu)a_k \geq k(k+1-\mu)a_{k+1}, \quad \forall k \geq 3$. This verifies the fact that $\{a_k\}$ satisfies the hypothesis of Lemma 1.4, and the proof is complete. ∎

By applying Alexander type theorem, which gives $U_p(z) \in \mathcal{C}(\mu)$ if and only if $zU_p'(z) \in \mathcal{S}^*(\mu)$, and using Theorem 3.4, we have the following result.

Theorem 3.5. *Let $c < 0$, $0 \leq \mu < 1$ and $p, b \in \mathbb{R}$. If $2(1-\mu)\kappa \geq -(2-\mu)c$, then $U_p(z)$ is convex of order μ in $\mathbb{D}$.*

With the failure of Mandelbrojt-Schiffer conjecture [2], namely $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$, the proof of Pólya-Schoenberg conjecture and its extension [24], took the center stage of the study of univalent functions, by which the following result is immediate.

Corollary 3.3. *Assume the hypothesis of Theorem* 3.4 (Theorem 3.5)*. Then for any $f(z) \in \mathcal{C}(\mu)$, $f(z) * U_p(z) \in \mathcal{S}^*(\mu)$ or $\mathcal{C}(\mu)$.*

Corollary 3.4. *Let $c < 0$, $0 \leq \mu < 1$ and $b \in \mathbb{R}$. If*

$$p \geq p_1 = -\frac{(2-\mu)c}{4(1-\mu)} - \frac{1}{2}(b+1),$$

and

$$p \geq p_2 = -\frac{(2-\mu)c}{2(1-\mu)} - \frac{1}{2}(b+1)$$

then $U_p(z)$ is respectively starlike of order μ and convex of order μ in $\mathbb{D}$.

For $b = 1, c = -1$, the generalized Bessel differential equation reduces to the Modified Bessel differential equation and it's solution is known as the modified Bessel function. Modified Bessel function of the first kind of order p is denoted as $I_p(z)$, which is given as

$$I_p(z) = \sum_{k=1}^{\infty} \frac{1}{k!\,\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p}.$$

Example 3.1. Denote $\mathcal{I}_p(z) = 2^p\Gamma(p+1)z^{1-p}I_p(\sqrt{z})$, the normalized modified Bessel functions of first kind of order p, then by Theorem 3.4 and Theorem 3.5, $\mathcal{I}_p(z)$ is starlike and convex of order μ when $p \geq (3\mu-2)/4(1-\mu)$ and $p \geq \mu/(2(1-\mu))$ respectively.

Remark 3.3. Theorem 3.4 asserts that $U_p(z)$ is starlike of order μ, if $\kappa \geq -(2-\mu)c/4(1-\mu)$ and $c < 0$ while by Lemma 2.2(ii), $\kappa \geq |c|/4(1-\mu) + (3-2\mu)/2, c \neq 0$. Since for $c < 0$,

$$\begin{aligned}\left(-\frac{1}{4(1-\mu)}c + \frac{(3-2\mu)}{2}\right) - \left(-\frac{(2-\mu)}{4(1-\mu)}c\right) &= \left[\frac{(2-\mu)}{4(1-\mu)} - \frac{1}{4(1-\mu)}\right]c + \frac{(3-2\mu)}{2} \\ &= \frac{1}{4}c + \frac{(3-2\mu)}{2} \geq 0\end{aligned}$$

if $c \geq -2(3-2\mu)$.

Hence Theorem 3.4 is better than the Lemma 2.2(ii) for $c \in [-2(3-2\mu), 0]$. Now in particular for $b = 1$, $c = -1$, we have the modified Bessel function $\mathcal{I}_p(z)$. Hence by taking $\kappa = p + (b+1)/2$ Theorem 3.4 gives the modified Bessel function of order $p \geq (3\mu-2)/4(1-\mu)$ is starlike of order μ, while by Lemma 2.2(ii), $\mathcal{I}_p(z)$ is starlike of order μ if $p \geq (2+(1-\mu))/(1-2\mu)4(1-\mu) \geq (3\mu-2)/4(1-\mu)$.

4. Prestarlikeness of generalized Bessel functions

Due to the fact that, results related to prestarlike functions are very much limited in the literature, we extend the question of Problem 1.1 to the class of prestarlike functions also.

Theorem 4.1. *Let $c<0$ and $p,b\in\mathbb{R}$. Then $U_p(z)\in\mathcal{R}[\alpha,\mu]$ if for $0\le\mu<1$,*

$$4(\kappa+1)\ge\begin{cases} T_1(\alpha,\mu)c+4, & 0\le\alpha\le\alpha_1(\mu),\\ \\ \max\Big\{T_1(\alpha,\mu)c+4,\ T_2(\alpha,\mu)c,\ T_3(\alpha,\mu)c-4\Big\}, & \alpha_1(\mu)\le\alpha<1.\end{cases}$$

where,

$$T_1(\alpha,\mu)=-\frac{2(1-\alpha)(2-\mu)}{1-\mu},\quad T_2(\alpha,\mu)=-\frac{2^{\mu-1}(3-\mu)(3-2\alpha)}{2-\mu}.$$
$$T_3(\alpha,\mu)=-\frac{2(4-\mu)(2-\alpha)}{3(2-\mu)(3-\mu)},\quad \alpha_1(\mu)=1-\frac{2^{\mu}(1-\mu)(3-\mu)}{4(2-\mu)^2-2.2^{\mu}(1-\mu)(3-\mu)}.$$

Proof. Consider the function $g(z)=z+\sum_{k=2}^{\infty}b_kz^k$, where b_k is given as

$$b_1=1,\qquad b_{k+1}=-\frac{c(k+1-2\alpha)}{4k^2(\kappa+k-1)}b_k,\quad \forall k\ge1.$$

Let for $c<0$, $0\le\mu,\ \alpha<1$ and $p,b\in\mathbb{R}$

$$4(\kappa+1)\ge\max\left\{T_1(\alpha,\mu)c+4,\ T_2(\alpha,\mu)c,\ T_3(\alpha,\mu)c-4\right\},\tag{4.1}$$

Clearly $4(\kappa+1)\ge T_1(\alpha,\mu)c+4$, which is equivalent to $2(1-\mu)\kappa\ge-(1-\alpha)(2-\mu)c$. Hence, $(1-\mu)b_1-(2-\mu)b_2=\frac{1}{2\kappa}\Big[2(1-\mu)\kappa+(1-\alpha)(2-\mu)c\Big]\ge0$. Again

$$\begin{aligned}(2-\mu)b_2-2^{\mu+1}(3-\mu)b_3&=\frac{b_2}{4(\kappa+1)}\Big[4(2-\mu)(\kappa+1)+2^{\mu-1}(3-\mu)(3-2\alpha)c\Big]\\&=\frac{b_2(2-\mu)}{4(\kappa+1)}\Big[4(\kappa+1)-T_2(\alpha,\mu)c\Big]\ge0.\end{aligned}$$

Let us consider

$$A(\alpha,\mu)=8(4-\mu)(\kappa+1)+c+4(\mu^2-13\mu+29)\tag{4.2}$$
$$B(\alpha,\mu)=4(\mu^2-11\mu+21)(\kappa+1)+(8-2\alpha-\mu)c+4(4\mu^2-26\mu+39)\tag{4.3}$$
$$D(\alpha,\mu)=12(2-\mu)(3-\mu)(\kappa+1)+2(2-\alpha)(4-\mu)c+12(2-\mu)(3-\mu)\tag{4.4}$$

Now if $4(\kappa+1)\ge T_3(\alpha,\mu)c-4$, then clearly $D(\alpha,\mu)\ge0$ and

$$\begin{aligned}&3(2-\mu)(3-\mu)A(\alpha,\mu)\\&=24(2-\mu)(3-\mu)(4-\mu)(\kappa+1)+3(2-\mu)(3-\mu)\Big[c+4(\mu^2-13\mu+29)\Big]\\&\ge\Big[3(2-\mu)(3-\mu)-2(2-\alpha)(4-\mu)^2\Big]c+12(2-\mu)(3-\mu)\Big[(\mu^2-13\mu+29)-1\Big],\\&>\Big[3(2-\mu)(3-\mu)-2(4-\mu)^2\Big]c-2\Big[(1-\alpha)(4-\mu)^2\Big]c\ge0,\end{aligned}$$

as $c < 0$ and for all $0 \leq \mu < 1$, $[3(2-\mu)(3-\mu) - 2(4-\mu)^2] < 0$. This gives $A(\alpha, \mu) \geq 0$. Similarly,

$$\begin{aligned}
&3(2-\mu)(3-\mu)B(\alpha,\mu)\\
&= 12(2-\mu)(3-\mu)(\mu^2 - 11\mu + 21)(\kappa+1)\\
&\quad + 3(2-\mu)(3-\mu)\Big[(8-2\alpha-\mu)c + 4(4\mu^2 - 26\mu + 39)\Big]\\
&= \Big[3(2-\mu)(3-\mu)(8-2\alpha-\mu) - 2(\mu^2 - 11\mu + 21)(2-\alpha)(4-\mu)\Big]c\\
&\quad + 12(2-\mu)(3-\mu)\Big[(4\mu^2 - 26\mu + 39) - 1\Big]\\
&> \Big[3(2-\mu)(3-\mu)(8-2\alpha-\mu) - 2(\mu^2 - 11\mu + 21)(2-\alpha)(4-\mu)\Big]c\\
&= \Big[3(8-\mu)(2-\mu)(3-\mu) - 2(\mu^2 - 11\mu + 21)(4-\mu)\Big]c\\
&\quad - 2(\mu^2 - 11\mu + 21)(1-\alpha)(4-\mu)c - 6\alpha(2-\mu)(3-\mu)c \geq 0,
\end{aligned}$$

which implies $B(\alpha, \mu) \geq 0$.

Now for $k \geq 3$, consider $(k-1-\mu)(k-\mu)b_k - k(k+1-\mu)b_{k+1} = A(k)M(k)$, where

$$A(k) = \frac{b_k}{4k(\kappa + k - 1)}$$

and

$$\begin{aligned}
M(k) &= 4k(\kappa + k - 1)(k-1-\mu)(k-\mu) + c(k+1-\mu)(k+1-2\alpha)\\
&= 4(k-3)^4 + 4(\kappa - 2\mu + 20)(k-3)^3 + A(\alpha,\mu)(k-3)^2\\
&\quad + B(\alpha,\mu)(k-3) + D(\alpha,\mu).
\end{aligned}$$

Here $A(\alpha,\mu), B(\alpha,\mu), D(\alpha,\mu)$, are non-negative expressions as given in (4.2), (4.3), (4.4) respectively. Since each coefficient of $(k-3)$ and the constant term $D(\alpha,\mu)$ in the expression on $M(k)$ are non-negative, we have $M(k)$ as an increasing function for $k \geq 3$. Since $M(3) > 0$, we have $(k-1-\mu)(k-\mu)b_k \geq k(k+1-\mu)b_{k+1}$.

Thus b_k satisfies the hypothesis of Lemma 1.4, and hence $g(z) \in \mathcal{S}^*(\mu)$. By a simple calculation one can observe that

$$g(z) = U_p(z) * \frac{z}{(1-z)^{2-2\alpha}}.$$

Therefore by definition of $\mathcal{R}[\alpha,\mu]$, we have $U_p(z) \in \mathcal{R}[\alpha,\mu]$. Now

$$\begin{aligned}
T_1(\alpha,\mu) - T_3(\alpha,\mu) &= \frac{2(4-\mu)(2-\alpha)}{3(2-\mu)(3-\mu)} - \frac{2(1-\alpha)(2-\mu)}{(1-\mu)}\\
&= \frac{2(4-\mu)(1-\mu)(2-\alpha) - 6(2-\mu)^2(3-\mu)(1-\alpha)}{3(1-\mu)(2-\mu)(3-\mu)}.
\end{aligned}$$

One can easily verify that for $0 \leq \alpha \leq \alpha_0(\mu)$, the numerator is negative for all μ and hence $T_1(\alpha,\mu) \leq T_3(\alpha,\mu)$. Similarly if $0 \leq \alpha \leq \alpha_1(\mu)$, $T_1(\alpha,\mu) \leq T_2(\alpha,\mu)$ for all μ. Here,

$$\alpha_0(\mu) = 1 - \frac{(4-\mu)(1-\mu)}{3(2-\mu)^2(3-\mu) - (4-\mu)(1-\mu)},$$
$$\alpha_1(\mu) = 1 - \frac{2^\mu(1-\mu)(3-\mu)}{4(2-\mu)^2 - 2.2^\mu(1-\mu)(3-\mu)}.$$

Clearly, we can conclude that, for $0 \le \alpha \le \min\{\alpha_0(\mu), \alpha_1(\mu)\}$,

$$\min_{i=1,2,3}\{T_i(\alpha,\mu)\} = T_1(\alpha,\mu) \quad \text{implies} \quad \max_{i=1,2,3}\{T_i(\alpha,\mu)c\} = T_1(\alpha,\mu)c, \quad \forall c < 0.$$

To complete the proof we only need to check that $\min\{\alpha_0(\mu), \alpha_1(\mu)\} = \alpha_1(\mu)$. Since

$$\begin{aligned}\alpha_1 - \alpha_0 &= \frac{(4-\mu)(1-\mu)}{3(2-\mu)^2(3-\mu) - (4-\mu)(1-\mu)} - \frac{2^\mu(1-\mu)(3-\mu)}{4(2-\mu)^2 - 2.2^\mu(1-\mu)(3-\mu)} \\ &= \frac{N(\mu)}{(3(2-\mu)^2(3-\mu) - (4-\mu)(1-\mu))\,(4(2-\mu)^2 - 2.2^\mu(1-\mu)(3-\mu))}\end{aligned}$$

where

$$\begin{aligned}N(\mu) &= 4(2-\mu)^2(4-\mu)(1-\mu) - 2^\mu(1-\mu)^2(3-\mu)(4-\mu) \\ &\quad - 32^\mu(1-\mu)(2-\mu)^2(3-\mu)^2 \\ &< 4(2-\mu)^2(1-\mu)\left[(4-\mu) - 32^\mu(3-\mu)^2\right] < 0.\end{aligned}$$

Therefore, $\alpha_1(\mu) = \min\{\alpha_0(\mu), \alpha_1(\mu)\}$, and the proof is complete. ∎

Theorem 4.2. *Let $c < 0$, $0 \le \mu < 1$ and $p, b \in \mathbb{R}$. If $2\kappa \ge -(2-\mu)c$, then $U_p(z)$ is prestarlike of order μ in $\mathbb{D}$.*

Proof. Consider $T_i(\alpha,\mu), i = 1,2,3$, as given in the hypothesis of Theorem 4.1. Now for $\alpha = \mu$, we have $T_1(\mu) = -2(2-\mu)$,

$$T_2(\mu) = -\frac{2^{\mu-1}(3-\mu)(3-2\mu)}{(2-\mu)} \quad \text{and} \quad T_3(\mu) = -\frac{2(4-\mu)}{3(3-\mu)}.$$

Note that for $0 \le \mu < 1$,

$$T_2(\mu) = -\frac{2^{\mu-1}(3-\mu)(3-2\mu)}{(2-\mu)} > -\frac{(3-\mu)(3-2\mu)}{2(2-\mu)}$$

and hence

$$\begin{aligned}T_2(\mu) - T_1(\mu) &> -\frac{(3-\mu)(3-2\mu)}{2(2-\mu)} + 2(2-\mu) \\ &= \frac{2\mu^2 - 7\mu + 7}{2(2-\mu)} > 0.\end{aligned}$$

Similarly,

$$\begin{aligned}T_3(\mu) - T_1(\mu) &= -\frac{2(4-\mu)}{3(3-\mu)} + 2(2-\mu) \\ &= \frac{6\mu^2 - 28\mu + 28}{3(3-\mu)} > 0.\end{aligned}$$

Therefore, $T_1(\mu)$ is the minimum one. Hence for all $c < 0$,

$$4(\kappa+1) \geq \max\{T_1(\mu)c+4, T_2(\mu)c, T_3(\mu)c-1\} = T_1(\mu)c+4.$$

which is equivalent to $2\kappa \geq -(2-\mu)c$. ∎

The following results are immediate consequences of Lemma 1.1.

Corollary 4.1. *Assume the hypothesis of Theorem 4.2, then for any $f \in \mathcal{S}^*(\mu)$, we have $f * U_p(z) \in \mathcal{S}^*(\mu)$.*

Corollary 4.2. *Let $c < 0$, $p, b \in \mathbb{R}$,*

(1) *$U_p(z) \in \mathcal{S}^*(1/2)$ if $\kappa \geq -\frac{3}{4}c$.*
(2) *$U_p(z) \in \mathcal{C}$ if $\kappa \geq -c$.*

Corollary 4.3. *Let $c < 0$, $b \in \mathbb{R}$, $0 \leq \mu < 1$. Then $U_p(z)$ is prestarlike of order μ if $p \geq p_1$, where $p_1 = -(1-\frac{\mu}{2})c-(b+1)$. In particular, $\mathcal{I}_p$ is prestarlike of order μ for $p \geq -\frac{\mu}{2}-1$.*

5. Alexander transform of generalized Bessel functions

The Alexander transform of a function $f(z) \in \mathcal{S}$ is defined as $\Lambda_f(z) \equiv \int_0^z \frac{f(t)}{t}dt$. It is easy to find [9, p. 257] that there exist functions $f \in \mathcal{S}$ for which the Alexander transform $\Lambda_f(z)$ is not univalent in $\mathbb{D}$. On the other hand, many results available in the literature for the starlikeness of the Alexander transform of non-univalent functions. For example,

$$\operatorname{Re} f'(z) > -\delta \Longrightarrow \Lambda_f(z) \in \mathcal{S}^*$$

with the best possible value of δ is $\delta = \frac{1-2\log 2}{2-2\log 2}$, is given in [12]. Hence, it will be interesting to find the conditions under which the Alexander transform of the generalized Bessel function has the geometric properties under consideration.

Since $\Lambda_{U_P}(z) = z + \sum_{k=2}^{\infty} b_k z^k$ with $b_1 = 1$, $a_k = kb_k$, $\forall k \geq 2$, where a_k as given in (3.1). For $\Lambda_{U_P}(z)$ to be close-to-convex with respect to z and $z/(1-z)$, it is enough to verify that $\{b_k\}$ satisfies the hypothesis of Corollary 1.1. This follows from an easy and direct computation and we state the result as:

Theorem 5.1. *Let $c < 0$ and $b, p \in \mathbb{R}$, then the Alexander transform $\Lambda_{U_P}(z)$ is close-to-convex with respect to starlike function z and $z/(1-z)$ if $\kappa > -c/4$. Further $\Lambda_{U_P}(z)$ is also starlike.*

Remark 5.1. Since for $c < 0$, $-(c+2)+\sqrt{c^2/2-4c+4} > -c$. Hence Theorem 5.1 gives better range of κ than the Lemma 2.3.

Corollary 5.1. *Let $c < 0$, $b \in \mathbb{R}$. Then the Alexander transform of $U_p(z)$ is starlike univalent for $p \geq p_1$ where $p_1 = -c/4-(b+1)/2$. In particular the Alexander transform of normalized modified Bessel function $\mathcal{I}_p(z)$ is starlike univalent for $p \geq -3/4$.*

6. Proofs of Theorems 3.1 and 3.2

6.1. Proof of Theorem 3.1.

Let $\gamma = \mu + 1$, then clearly $1 \le \gamma < 2$. Consider, for $0 \le r < 1$ and $0 \le \theta \le 2\pi$,

$$\operatorname{Re} \frac{u_{p,n}(z) - \mu}{1-\mu} = \frac{a_0}{2} + \sum_{k=1}^{n} r^k a_k \cos k\theta,$$

where

$$a_0 = 2, \quad a_1 = \frac{-c}{4(1-\mu)\kappa}, \qquad \text{and} \quad a_{k+1} = \frac{-c}{4(k+1)(\kappa+k)} a_k, \quad \forall k \ge 1.$$

Let, $4(1-\mu)\kappa \ge -c$, then clearly $a_0 \ge 2a_1$ and

$$\begin{aligned}(1-\mu)\big[a_1 - (2+\alpha)^\gamma a_2\big] &= a_1(1-\mu)\left[1 + (2+\alpha)^\gamma \frac{c}{8(\kappa+1)}\right] \\ &= \frac{a_1}{8(\kappa+1)}\left[8(1-\mu)(\kappa+1) + (1-\mu)(2+\alpha)^\gamma c\right] \\ &\ge \frac{a_1}{8(\kappa+1)}\left[8(1-\mu) + \Big((1-\mu)(2+\alpha)^\gamma - 2\Big)c\right] \ge 0.\end{aligned}$$

By a simple calculation, we have

$$\left(1 + \frac{1}{k+\alpha}\right)^{-\gamma} \ge \left[1 - \frac{\gamma}{k+\alpha}\right], \quad \forall k \ge 2.$$

Hence for all $k \ge 2$,

$$\begin{aligned}&(k+\alpha)^\gamma a_k - (k+1+\alpha)^\gamma a_{k+1} \\ &\ge (k+1+\alpha)^\gamma a_k \left[\left(1 - \frac{\gamma}{k+\alpha}\right) + \frac{c}{4(k+1)(\kappa+1)}\right] = A(k)M(k),\end{aligned}$$

where

$$A(k) = \frac{(k+1+\alpha)^\gamma a_k}{4(k+1)(k+\alpha)(\kappa+1)} \quad \text{and}$$

$$M(k) = 4(k+1)(\kappa+k)(k+\alpha-\gamma) + c(k+\alpha) = \sum_{i=1}^{4} T_i (k-2)^i$$

with

$$\begin{aligned}&T_1 = 4, \quad T_2 = (40 + 4\kappa + 4\alpha - 4\gamma) > 0, \\ &T_3 = 28(2-\gamma) + 4\kappa(2-\gamma) + c + 20\kappa + 4\kappa\alpha + 28\alpha + 76 \ge 0 \quad \text{and} \\ &T_4 = 48(3-\gamma) + 16\kappa(2-\gamma) + 3c + (48 + 16A + c)\alpha \ge 0.\end{aligned}$$

Hence for $k \ge 2$, $M(k)$ is increasing and $M(2) \ge 0$, which implies that $(k+\alpha)^\gamma a_k \ge (k+1+\alpha)^\gamma a_{k+1}, \quad \forall k \ge 2$.

Therefore $\{a_k\}$ satisfies the hypothesis of Lemma 1.2. By the fact $\cos k(2\pi - \theta) = \cos k\theta$, $0 \le \theta \le 2\pi$ and the minimum principle for harmonic functions, we have

$$\operatorname{Re} \frac{u_{p,\,n}(z) - \mu}{1-\mu} > 0 \quad \text{implies} \quad \operatorname{Re} u_{p,\,n}(z) > \mu.$$

By the first hypothesis of the theorem, $\operatorname{Re} u_{p+1,\,n}(z) > 0$, if $4(1-\mu)\kappa \geq -c - 4(1-\mu)$. Therefore by using relation (2.6), we have

$$\operatorname{Re}\left(-\frac{4\kappa}{c} u'_{p,\,n}(z)\right) = \operatorname{Re} u_{p+1,\,n}(z) > 0.$$

By definition of close-to-convexity, $u_{p,\,n}(z)$ is close-to-convex with respect to starlike function $-\frac{c}{4\kappa}z$. Due to the fact that the family of all close-to-convex function with respect to a particular starlike function is normal, $u_p(z) = \lim_{n\to\infty} u_{p,\,n}(z)$ is also close-to-convex with respect to starlike function $-cz/4\kappa$ and the proof is complete.

6.2. Proof of Theorem 3.2.

Since $U_p(z) = z + \sum_{k=2}^{\infty} a_k z^k$ with $\{a_k\}$ satisfying (3.1), it is enough to prove that a_k satisfies the hypothesis of Lemma 1.3. Clearly, for $\kappa \geq -\frac{c}{2}$, $a_1 \geq 2a_2$ and

$$\begin{aligned} 2a_2 - 6a_3 &= \frac{a_2}{8(\kappa+1)}\left[16(\kappa+1) + 6c\right] \\ &\geq \frac{a_2}{8(\kappa+1)}(-8c + 6c) = -\frac{a_2 c}{4(\kappa+1)} > 0. \end{aligned}$$

Again for $k \geq 3$, consider

$$k(k-2)a_k - (k-1)(k+1)a_{k+1} = A(k)M(k)$$

where $A(k) = a_k/4k(\kappa + k - 1)$ and

$$\begin{aligned} M(k) &= 4k^2(\kappa + k - 1)(k-2) + c(k-1)(k+1) \\ &\geq 2k^2(2(k-1) - c)(k-2) + c(k-1)(k+1) \\ &= 4(k-3)^4 + (36 - 2c)(k-3)^3 + (116 - 13c)(k-3)^2 \\ &\quad + (56 - 12c)(k-3) + (72 - 10c). \end{aligned} \tag{6.1}$$

One can easily observe that all the coefficients of $(k-3)$ and the constant term in (6.1) are non-negative for $c < 0$. Hence $\{a_k\}_{k=1}^{\infty}$ satisfies the hypothesis of Corollary 1.1 and we have the conclusion.

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References

[1] A. P. Acharya, Univalence criteria for analytic funtions and applications to hypergeometric functions, Ph.D Diss., University of Würzburg, 1997.
[2] H. S. Al-Amiri and D. H. Bshouty, A constraint coefficient problem with an application to a convolution problem, *Complex Variables Theory Appl.* **22** (1993), no. 3–4, 241–246.
[3] R. M. Ali, V. Ravichandran and N. Seenivasagan, Subordination and superordination of the Liu-Srivastava linear operator on meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **31** (2008), no. 2, 193–207.
[4] R. M. Ali, V. Ravichandran and N. Seenivasagan, On subordination and superordination of the multiplier transformation for meromorphic functions, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 2, 311–324.
[5] G. Brown and S. Koumandos, On a monotonic trigonometric sum, *Monatsh. Math.* **123** (1997), no. 2, 109–119.

[6] Á. Baricz, Geometric properties of generalized Bessel functions, *Publ. Math. Debrecen* **73** (2008), no. 1–2, 155–178.

[7] Á. Baricz, Geometric properties of generalized Bessel functions of complex order, *Mathematica* **48(71)** (2006), no. 1, 13–18.

[8] N. E. Cho and O. S. Kwon, A class of integral operators preserving subordination and superordination, *Bull. Malays. Math. Sci. Soc. (2)* **33** (2010), no. 3, 429–437.

[9] P. L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, 259, Springer, New York, 1983.

[10] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transforms Spec. Funct.* **14** (2003), no. 1, 7–18.

[11] L. Fejér, Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koeffizientenfolge, *Trans. Amer. Math. Soc.* **39** (1936), no. 1, 18–59.

[12] R. Fournier and S. Ruscheweyh, On two extremal problems related to univalent functions, *Rocky Mountain J. Math.* **24** (1994), no. 2, 529–538.

[13] B. Friedman, Two theorems on schlicht functions, *Duke Math. J.* **13** (1946), 171–177.

[14] I. S. Jack, Functions starlike and convex of order α, *J. London Math. Soc. (2)* **3** (1971), 469–474.

[15] S. Koumandos, An extension of Vietoris's inequalities, *Ramanujan J.* **14** (2007), no. 1, 1–38.

[16] S. Koumandos and S. Ruscheweyh, On a conjecture for trigonometric sums and starlike functions, *J. Approx. Theory* **149** (2007), no. 1, 42–58.

[17] R. Küstner, Mapping properties of hypergeometric functions and convolutions of starlike or convex functions of order α, *Comput. Methods Funct. Theory* **2** (2002), no. 2, 597–610.

[18] R. Küstner, On the order of starlikeness of the shifted Gauss hypergeometric function, *J. Math. Anal. Appl.* **334** (2007), no. 2, 1363–1385.

[19] S. S. Miller and P. T. Mocanu, *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Dekker, New York, 2000.

[20] S. R. Mondal and A. Swaminathan, On the positivity of certain trignometric sums and their applications, *Comput. Math. Appl.* **62** (2011), no. 10, 3871–3883.

[21] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39** (1987), no. 5, 1057–1077.

[22] S. Ozaki, On the theory of multivalent functions, *Sci. Rep. Tokyo Bunrika Daigaku. Sect. A.* **2** (1935), 167–188.

[23] M. I. S. Robertson, On the theory of univalent functions, *Ann. of Math. (2)* **37** (1936), no. 2, 374–408.

[24] S. Ruscheweyh and T. Sheil-Small, Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture, *Comment. Math. Helv.* **48** (1973), 119–135.

[25] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Séminaire de Mathématiques Supérieures, 83, Presses Univ. Montréal, Montreal, QC, 1982.

[26] S. Ruscheweyh, Linear operators between classes of prestarlike functions, *Comment. Math. Helv.* **52** (1977), no. 4, 497–509.

[27] S. Ruscheweyh and L. Salinas, Stable functions and Vietoris' theorem, *J. Math. Anal. Appl.* **291** (2004), no. 2, 596–604.

[28] T. Sheil-Small, H. Silverman and E. Silvia, Convolution multipliers and starlike functions, *J. Analyse Math.* **41** (1982), 181–192.

[29] H. Silverman, Starlike and convexity properties for hypergeometric functions, *J. Math. Anal. Appl.* **172** (1993), no. 2, 574–581.

[30] H. M. Srivastava, Some families of fractional derivative and other linear operators associated with analytic, univalent, and multivalent functions, in *Analysis and its Applications* (*Chennai, 2000*), 209–243, Allied Publ., New Delhi.

[31] H. M. Srivastava, Generalized hypergeometric functions and associated families of k-uniformly convex and k-starlike functions, *Gen. Math.* **15** (2007), no. 3, 201–226.

[32] S. Supramaniam, R. M. Ali, S. K. Lee and V. Ravichandran, Convolution and differential subordination for multivalent functions, *Bull. Malays. Math. Sci. Soc. (2)* **32** (2009), no. 3, 351–360.

[33] A. Swaminathan, Convexity of the incomplete beta functions, *Integral Transforms Spec. Funct.* **18** (2007), no. 7–8, 521–528.

[34] A. Swaminathan, Univalent polynomials and fractional order differences of their coefficients, *J. Math. Anal. Appl.* **353** (2009), no. 1, 232–238.