DIFFERENTIAL SUBORDINATION FOR FUNCTIONS ASSOCIATED WITH THE LEMNISCATE OF BERNOULLI

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Abstract. Conditions on $\beta$ are determined so that $1 + \beta z p'(z)$ subordinated to $\sqrt{1 + z}$ implies $p$ is subordinated to $\sqrt{1 + z}$. Analogous results are also obtained involving the expressions $1 + \beta z p'(z)/p(z)$ and $1 + \beta z p'(z)/p^2(z)$. These results are applied to obtain sufficient conditions for normalized analytic functions $f$ to satisfy the condition $|zf'(z)/f(z)|^2 - 1 < 1$.

1. INTRODUCTION

Let $A$ denote the class of analytic functions in the unit disk $D := \{ z \in \mathbb{C} : |z| < 1 \}$ normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Let $SL$ be the class of functions defined by

$$SL := \left\{ f \in A : \left| \frac{(zf'(z))^2}{f(z)} - 1 \right| < 1 \right\} \quad (z \in D).$$

Thus a function $f \in SL$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli given by $|w^2 - 1| < 1$. Since this region is contained in the right-half plane, functions in $SL$ are starlike functions, and in particular univalent. A starlike function is characterized by the condition $\text{Re}zf'(z)/f(z) > 0$ in $D$. For two functions $f$ and $g$ analytic in $D$, the function $f$ is said to be subordinate to $g$, written $f(z) \prec g(z)$ ($z \in D$), if there exists a function $w$ analytic in $D$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. In particular, if the function $g$ is univalent in $D$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$. In terms of subordination, the class $SL$ consists of normalized analytic functions $f$ satisfying $zf'(z)/f(z) \prec \sqrt{1 + z}$. This class $SL$ was introduced by Sokół and Stankiewicz.

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as special cases.

Gobradovic and Tuneski [11] have improved the result of Silverman [20] by showing of order \( \alpha \) to the class \( SL \) obtained in [24]. Recently, Sokol [25] determined various radii for functions belonging to the class \( SL \); these include the radii of convexity, starlikeness and strong starlikeness of order \( \alpha \). Recently, Sokol and Stankiewicz [23] determined the radius of convexity for functions in the class \( S \).

In fact, Silverman [20] derived the order of starlikeness for as

\[
\frac{zf'(z)}{f(z)} < \frac{1 + Az}{1 + Bz}, \quad (-1 \leq B < A \leq 1).
\]

Silverman [20], Obradovic and Tuneski [11] and several others (see [9, 10, 12, 16, 18]) have studied properties of functions defined in terms of the quotient \((1 + z f''(z) / f'(z)) / (zf'(z) / f(z))\). In fact, Silverman [20] derived the order of starlikeness for functions in the class \( G_b \) defined by

\[
G_b := \left\{ f \in A \mid \left| \frac{1 + z f''(z)}{zf'(z) / f(z)} - \frac{1}{b} \right| < b, \quad 0 < b \leq 1, \quad z \in \mathbb{D} \right\}.
\]

Obradovic and Tuneski [11] have improved the result of Silverman [20] by showing \( G_b \subset S^*[0, -b] \subset S^*[2/(1 + \sqrt{1 + 8b})] \). Later Tuneski [26] obtained conditions for the inclusion \( G_b \subset S^*[A, B] \) to hold. Letting \( z f''(z) / f'(z) = p(z) \), then \( G_b \subset S^*[A, B] \) becomes a special case of the differential chain

\[
1 + \beta \frac{zf'(z)}{p(z)^2} < \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) < \frac{1 + Az}{1 + Bz}.
\]

Similarly, for \( f \in A \) and \( 0 \leq \alpha < 1 \), Frasin and Darus [5] showed that

\[
\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} < \frac{(1 - \alpha)z}{2 - \alpha} \Rightarrow \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha.
\]

Again by writing \( \frac{zf'(z)}{f(z)} \) as \( p(z) \), the above implication is a particular case of

\[
1 + \beta \frac{zf'(z)}{p(z)} < \frac{1 + Dz}{1 + Ez} \Rightarrow p(z) < \frac{1 + Az}{1 + Bz}.
\]

Li and Owa [13] showed that \( f(z) \in S^* \) if \( f(z) \in A \) satisfies

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right\} > -\frac{\alpha}{2}, \quad z \in \mathbb{D}
\]
for some $\alpha$ ($\alpha \geq 0$). Related results may also be found in the works of [15, 17, 21, 22].

The implications (1.1) and (1.2) have been considered in [3]. All the results discussed above led us to consider differential implications with the superordinate function $(1 + Az)/(1 + Bz)$ replaced by the superordinate function $\sqrt{1 + z}$ that maps $D$ onto the right-half of the lemniscate of Bernoulli. Additionally, applications of our results will yield sufficient conditions for functions $f \in A$ to belong to the class $SL$.

The following results will be required.

**Lemma 1.1.** [8, Corollary 3.4h.1, p. 135]. Let $q$ be univalent in $D$, and let $\varphi$ be analytic in a domain containing $q(D)$. Let $zq'(z)\varphi(q(z))$ be starlike. If $p$ is analytic in $D$, $p(0) = q(0)$ and satisfies

$$zp'(z)\varphi(p(z)) < zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and $q$ is the best dominant.

A more general version of the above lemma is the following:

**Lemma 1.2.** [8, Theorem 3.4h, p. 132]. Let $q$ be univalent in the unit disk $D$ and $\psi$ and $\varphi$ be analytic in a domain $D$ containing $q(D)$ with $\varphi(w) \neq 0$ when $w \in q(D)$. Set $Q(z) = zq'(z)\varphi(q(z))$, $h(z) = \psi(q(z)) + Q(z)$. Suppose that

1. either $h$ is convex, or $Q$ is starlike univalent in $D$, and
2. $\text{Re} \frac{zh'(z)}{Q(z)} > 0$ for $z \in D$.

If $p$ is analytic in $D$, $p(0) = q(0)$ and satisfies

$$\psi(p(z)) + zp'(z)\varphi(p(z)) \prec \psi(q(z)) + zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$, and $q$ is the best dominant.

2. **Main Results**

We first determine a lower bound for $\beta$ so that $1 + \beta zp'(z) \prec \sqrt{1 + z}$ implies $p(z) \prec \sqrt{1 + z}$.

**Lemma 2.1.** Let $p$ be an analytic function on $D$ and $p(0) = 1$. Let $\beta_0 = 2\sqrt{2}((\sqrt{2} - 1) \approx 1.17$. If the function $p$ satisfies the subordination

$$1 + \beta zp'(z) \prec \sqrt{1 + z} \quad (\beta \geq \beta_0),$$

then $p$ also satisfies the subordination

$$p(z) \prec \sqrt{1 + z}.$$

The lower bound $\beta_0$ is best possible.
Proof. Define the function \( q : \mathbb{D} \to \mathbb{C} \) by \( q(z) = \sqrt{1 + z} \) with \( q(0) = 1 \). Since \( q(\mathbb{D}) = \{ w : |w^2 - 1| < 1 \} \) is the right-half of the lemniscate of Bernoulli, \( q(\mathbb{D}) \) is a convex set and hence \( q \) is a convex function. This shows that the function \( zq'(z) \) is starlike with respect to \( 0 \). By Lemma 1.1, it follows that the subordination
\[
1 + \beta z p'(z) < 1 + \beta z q'(z)
\]
implies \( p(z) < q(z) \). In light of this differential chain, the result is proved if it could be shown that
\[
q(z) = \sqrt{1 + z} < 1 + \beta z q'(z) = 1 + \frac{\beta z}{2\sqrt{1 + z}} =: h(z).
\]
Since \( q^{-1}(w) = w^2 - 1 \), it follows that
\[
q^{-1}(h(z)) = \left(2 + \frac{\beta z}{2\sqrt{1 + z}}\right) \frac{\beta z}{2\sqrt{1 + z}}.
\]
For \( z = e^{it}, t \in [-\pi, \pi] \), clearly
\[
|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \frac{\beta}{2\sqrt{2\cos \frac{t}{2}}} \left|2 + \frac{\beta e^{it}}{2\sqrt{2\cos \frac{t}{2}}}\right|.
\]
A calculation shows that the minimum of the above expression is attained at \( t = 0 \).
Hence
\[
|q^{-1}(h(e^{it}))| \geq \frac{\beta}{2\sqrt{2}} \left(2 + \frac{\beta}{2\sqrt{2}}\right) = \left(1 + \frac{\beta}{2\sqrt{2}}\right)^2 - 1 \geq 1
\]
provided \( \beta \geq 2\sqrt{2}(\sqrt{2} - 1) \). Hence \( q^{-1}(h(\mathbb{D})) \supset \mathbb{D} \) or \( h(\mathbb{D}) \supset q(\mathbb{D}) \). This shows that \( q(z) < h(z) \), and completes the proof. \( \blacksquare \)

Theorem 2.2. Let \( \beta_0 = 2\sqrt{2}(\sqrt{2} - 1) \approx 1.17 \) and \( f \in A \).

1. If \( f \) satisfies the subordination
\[
1 + \beta z f''(z) f'(z) \left(1 + \frac{z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \sqrt{1 + z} \quad (\beta \geq \beta_0),
\]
then \( f \in SC \).

2. If \( 1 + \beta z f''(z) < \sqrt{1 + z} \quad (\beta \geq \beta_0) \), then \( f'(z) < \sqrt{1 + z} \).

Proof. Define the function \( p : \mathbb{D} \to \mathbb{C} \) by
\[
p(z) = \frac{zf'(z)}{f(z)}.
\]
Then \( p \) is analytic in \( \mathbb{D} \) and \( p(0) = 1 \). A calculation shows that
\[
zp'(z) = \frac{zf'(z)}{f(z)} \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right).
\]
Applying Lemma 2.1 to this function \( p \) yields the first part of the theorem. The second part follows by taking \( p(z) = f'(z) \) in Lemma 2.1.

**Lemma 2.3.** Let \( \beta_0 = 4(\sqrt{2} - 1) \approx 1.65 \). If

\[
1 + \frac{\beta z p'(z)}{p(z)} < \sqrt{1 + z} \quad (\beta \geq \beta_0),
\]

then

\[
p(z) < \sqrt{1 + z}.
\]

The lower bound \( \beta_0 \) is best possible.

**Proof.** Let \( q \) be the convex function given by \( q(z) = \sqrt{1 + z} \), and consider the subordination

\[
1 + \frac{\beta z p'(z)}{p(z)} < 1 + \frac{\beta z q'(z)}{q(z)}.
\]

A calculation shows that

\[
\frac{\beta z q'(z)}{q(z)} = \frac{\beta z}{2(1 + z)}
\]

is convex in \( \mathbb{D} \) (and hence starlike). Thus, in view of Lemma 1.1, it follows that \( p(z) < q(z) \). To complete the proof, it is left to show that

\[
q(z) = \sqrt{1 + z} < 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta z}{2(1 + z)} =: h(z).
\]

Since \( h(\mathbb{D}) = \{ w : \text{Re}w < 1 + \beta/4 \} \), and \( q(\mathbb{D}) = \{ w : |w^2 - 1| < 1 \} \subset \{ w : \text{Re}w < \sqrt{2} \} \), it follows that \( q(\mathbb{D}) \subset h(\mathbb{D}) \) if \( \sqrt{2} \leq 1 + \beta/4 \). Thus \( q(z) < h(z) \) for \( \beta \geq 4(\sqrt{2} - 1) \), and this completes the proof.

**Theorem 2.4.** Let \( \beta_0 = 4(\sqrt{2} - 1) \approx 1.65 \) and \( f \in \mathcal{A} \).

1. If \( f \) satisfies

\[
1 + \beta \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) < \sqrt{1 + z} \quad (\beta \geq \beta_0),
\]

then \( f \in \mathcal{S} \mathcal{L} \).

2. If \( f \) satisfies

\[
1 + \beta \left( \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \right) < \sqrt{1 + z} \quad (\beta \geq \beta_0),
\]

then

\[
\frac{zf'(z)}{f^2(z)} < \sqrt{1 + z}.
\]
It follows that \( \beta \) for \( q \)

Since \( \beta \) is defined by \( Q \)

Thus the function \( Q \) implies

The lower bound \( \beta_0 \) is best possible.

Proof. With \( q \) being the convex function \( q(z) = \sqrt{1 + z} \), consider the function \( Q \) defined by

\[
Q(z) := \frac{q'(z)}{q(z)} = \frac{z}{2(1 + z)^{\frac{3}{2}}},
\]

Since

\[
\text{Re} \left( \frac{1 + (1 - 2\alpha)z}{1 - z} \right) > \alpha \quad (0 \leq \alpha < 1),
\]

it follows that

\[
\text{Re} \left( \frac{zQ'(z)}{Q(z)} \right) = \text{Re} \left( \frac{2 - z}{2(1 + z)} \right) > \frac{1}{4} > 0.
\]

Thus the function \( Q \) is starlike and Lemma 1.1 shows that the subordination

\[
1 + \frac{\beta z p'(z)}{p(z)} < 1 + \frac{\beta z q'(z)}{q(z)}
\]

implies \( p(z) < q(z) \). We next show that

\[
q(z) = \sqrt{1 + z} < 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta z}{2(1 + z)^{\frac{3}{2}}} =: h(z).
\]

Since \( q^{-1}(w) = w^2 - 1 \), then

\[
q^{-1}(h(z)) = \left( 2 + \frac{\beta z}{2(1 + z)^{\frac{3}{2}}} \right) \frac{\beta z}{2(1 + z)^{\frac{3}{2}}}
\]

Thus with \( z = e^{it}, \ t \in [-\pi, \pi] \), yields

\[
|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \frac{\beta}{2(2 \cos \frac{4}{3})^{\frac{3}{2}}} \left| 2 + \frac{\beta e^{\frac{4}{3}it}}{2(2 \cos \frac{4}{3})^{\frac{3}{2}}} \right|.
\]

A computation shows that the minimum of the above expression is attained at \( t = 0 \). Hence

\[
|q^{-1}(h(e^{it}))| \geq \frac{\beta}{4\sqrt{2}} \left( 2 + \frac{\beta}{4\sqrt{2}} \right) = \left( 1 + \frac{\beta}{4\sqrt{2}} \right)^2 - 1 \geq 1
\]

for \( \beta \geq 4\sqrt{2}(\sqrt{2} - 1) \). Hence \( q(z) < h(z) \).

By taking \( p(z) = \frac{z f'(z)}{f(z)} \) in Lemma 2.5, we obtain the following theorem.
Theorem 2.6. Let $\beta_0 = 4\sqrt{2}(\sqrt{2} - 1) \approx 2.34$ and $f \in A$. Then $f \in SL$ if

$$1 - \beta + \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} < \sqrt{1 + z} \quad (\beta \geq \beta_0).$$

Lemma 2.7. Let $0 < \alpha \leq 1$. If $p \in A$ satisfies

$$(1 - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z) < \sqrt{1 + z},$$

then $p(z) < \sqrt{1 + z}$.

Proof. Define the function $q$ by $q(z) = \sqrt{1 + z}$. We first show that $p(z) < q(z)$ if $p$ satisfies

$$(1 - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z) < (1 - \alpha)q(z) + \alpha q^2(z) + \alpha z q'(z).$$

For this purpose, let the functions $\vartheta$ and $\varphi$ be defined by $\vartheta(w) := (1 - \alpha)w + \alpha w^2$ and $\varphi(w) := \alpha$. Clearly the functions $\vartheta$ and $\varphi$ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Also let $Q$ and $h$ be the functions defined by

$$Q(z) := zq'(z)\varphi(q(z)) = \alpha z q'(z)$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = (1 - \alpha)q(z) + \alpha q^2(z) + \alpha z q'(z).$$

Since $q$ is convex, the function $zq'(z)$ is starlike, and therefore $Q$ is starlike univalent in $\mathbb{D}$. In view of the fact that $Req(z) > 0$, it follows that

$$Re \frac{zh'(z)}{Q(z)} = \frac{1}{\alpha} Re \left[ (1 - \alpha) + 2\alpha q(z) + \alpha \left( 1 + \frac{zq''(z)}{q'(z)} \right) \right] > 0 \quad (z \in \mathbb{D})$$

for $0 < \alpha \leq 1$. By Lemma 1.2, it follows that $p < q = \sqrt{1 + z}$. To complete the proof, we seek conditions on $\alpha$ so that $q(z) < h(z)$, or equivalently $|h(e^{it})|^2 - 1| \geq 1$ for all $t \in [-\pi, \pi]$. Now

$$h(z) = \frac{\alpha z + 2(1 - \alpha)(1 + z) + 2\alpha(1 + z)^{3/2}}{2\sqrt{1 + z}},$$

and a calculation shows that $|h(e^{it})|^2 - 1|$ attains its minimum at $t = 0$. Thus $|h(e^{it})|^2 - 1| \geq |(h(1))^2 - 1| > 1$ if $h(1) = \frac{8 - 3\sqrt{2}}{4\sqrt{2}} \alpha + \sqrt{2} > \sqrt{2}$ and this holds for $\alpha > 0$. Hence we conclude that $(1 - \alpha)p(z) + \alpha p^2(z) + \alpha z p'(z) < \sqrt{1 + z}$ implies $p(z) < \sqrt{1 + z}$. \hfill \blacksquare

Theorem 2.8. If $f \in A$ satisfies

$$\frac{zf'(z)}{f(z)} \left( 1 + \alpha \frac{zf''(z)}{f'(z)} \right) < \sqrt{1 + z} \quad (0 < \alpha \leq 1),$$

then $\frac{zf'(z)}{f(z)} < \sqrt{1 + z}$, or equivalently $f \in SL$. 

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**Proof.** With \( p(z) = \frac{zf'(z)}{f(z)} \), a computation shows that

\[
p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.
\]

Evidently

\[
\frac{zf'(z)}{f(z)} \left( 1 + \alpha \frac{zf''(z)}{f'(z)} \right) = \frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f(z)} = (1 - \alpha)p(z) + \alpha p^2(z) + \alpha zp'(z).
\]

The result now follows from Lemma 2.7.

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