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ORIGINAL PAPER

Convexity of functions defined by differential inequalities and integral operators

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Abstract The convexity conditions for analytic functions defined in the open unit disk satisfying certain second-order and third-order differential inequalities are obtained. As a consequence, conditions for convexity of functions defined by integral operators are also determined.

Keywords Convex functions · Subordination · Differential inequalities · Integral operators

Mathematics Subject Classification 30C45

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1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions f defined in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$, and n a positive integer, let

$$\mathcal{H}_n(a) = \left\{ f \in \mathcal{H} : f(z) = a + \sum_{k=n}^{\infty} a_k z^k \right\},$$

and

$$\mathcal{A}_n = \left\{ f \in \mathcal{H} : f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \right\},$$

with $\mathcal{A}_1 := \mathcal{A}$. For $0 \le \beta < 1$, denote respectively by $\mathcal{S}^*(\beta)$ and $\mathcal{CV}(\beta)$ the subclasses of $\mathcal{A}$ consisting of starlike functions of order β satisfying

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta, \quad z \in U,$$

and convex functions of order β satisfying

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \quad z \in U.$$

An analytic function f is *subordinate* to analytic function g, written $f \prec g$, if there is an analytic function $w : U \to U$ with $w(0) = 0$ such that $f(z) = g(w(z))$, $z \in U$.

An important area of research in complex function theory is to determine sufficient conditions to ensure starlikeness of analytic functions. These include conditions in terms of differential inequalities, see for example [1,2,4,5,7,9,10,15–18,20]. Miller and Mocanu [14], Kuroki and Owa [12], and Ali et al. [3], determined conditions for starlikeness of functions defined by an integral operator of the form

$$f(z) = \int_0^1 W(r, z)dr,$$

or by the double integral operator

$$f(z) = \int_0^1 \int_0^1 W(r, s, z)drds.$$

More recently, Chandra et al. [8], obtained sufficient conditions to ensure starlikeness of positive order for analytic functions satisfying certain third-order differential inequalities.

In this paper, conditions that would imply convexity of positive order for functions satisfying certain second-order and third-order differential inequalities are found. As a consequence, conditions on the kernel of certain integral operators are also obtained to ensure functions defined by these operators are convex.

The following known results will be required in the sequel.

Lemma 1 [11, Theorem 1, p. 192] (see also [13, Theorem 3.1b, p. 71]) *Let h be convex in U with $h(0) = a$, $\gamma \neq 0$ and* $\operatorname{Re}\gamma \ge 0$. *If $p \in \mathcal{H}_n(a)$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1} dt.$$

The function q is convex and is the best $(a, n)-$dominant.

Lemma 2 [19] (see also [13, Theorem 3.1d, p. 76]) *Let h be a starlike function with $h(0) = 0$. If $p \in \mathcal{H}_n(a)$ satisfies*

$$zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) = a + \frac{1}{n} \int_0^z \frac{h(t)}{t} dt.$$

The function q is convex and is the best $(a, n)-$dominant.

Lemma 3 [6, Theorem 1, p. 13] (see also [[13], Lemma 8.2a, p. 383]) *Let n be a positive integer and α satisfying $0 \leq \alpha < n$. Let $q \in \mathcal{H}$ with $q(0) = 0$, $q'(0) \neq 0$ and*

$$\operatorname{Re} \frac{zq''(z)}{q'(z)} + 1 > \frac{\alpha}{n}.$$

If $p \in \mathcal{H}_n(0)$ satisfies

$$zp'(z) - \alpha p(z) \prec nzq'(z) - \alpha q(z),$$

then $p(z) \prec q(z)$ and this result is sharp.

2 Convexity of functions defined by second-order differential inequalities

Theorem 1 *Let $f \in \mathcal{A}_n$, $\delta > 0$ and $0 \leq \alpha < \delta$ with $0 \leq \beta < 1$. If*

$$\left| \delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \right| < \frac{n(1-\beta)\big[\delta(n+1) - \alpha\big]}{(n+1)(n+1-\beta)}, \tag{1}$$

then $f \in \mathcal{CV}(\beta)$.

Proof The inequality (1) can be expressed in the subordination form

$$\delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \prec \frac{n(1-\beta)[\delta(n+1) - \alpha]}{(n+1)(n+1-\beta)} z, \qquad z \in U. \tag{2}$$

Let $P(z) = \delta \left(f'(z) - \frac{f(z)}{z} \right)$. Then (2) can be written as

$$\left(\frac{\delta - \alpha}{\delta} \right) P(z) + zP'(z) \prec \frac{n(1-\beta)[\delta(n+1) - \alpha]}{(n+1)(n+1-\beta)} z.$$

By Lemma 1 with $\gamma = \frac{\delta - \alpha}{\delta}$,

$$P(z) \prec \frac{\delta n(1-\beta)}{(n+1)(n+1-\beta)} z,$$

that is,

$$f'(z) - \frac{f(z)}{z} \prec \frac{n(1-\beta)}{(n+1)(n+1-\beta)} z. \tag{3}$$

This is equivalent to

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1-\beta)}{(n+1)(n+1-\beta)}, \tag{4}$$

which implies

$$\left| f'(z) \right| > \left| \frac{f(z)}{z} \right| - \frac{n(1-\beta)}{(n+1)(n+1-\beta)}. \tag{5}$$

On the other hand, if let $p(z) = f(z)/z$, then (3) can be written as

$$zp'(z) \prec \frac{n(1-\beta)}{(n+1)(n+1-\beta)} z.$$

By Lemma 2,

$$p(z) = \frac{f(z)}{z} \prec 1 + \frac{1-\beta}{(n+1)(n+1-\beta)} z,$$

which further implies

$$\left| \frac{f(z)}{z} \right| > 1 - \frac{1-\beta}{(n+1)(n+1-\beta)}. \tag{6}$$

By (5) and (6), it follows that

$$|f'(z)| > \frac{n}{n+1-\beta}. \tag{7}$$

Now we are ready to show that f is in $\mathcal{CV}(\beta)$. Let $Q(z) = 1 + \frac{zf''(z)}{f'(z)}$. Then (1) can be written as

$$\left| \delta f'(z)(Q(z)-1) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) \right| < \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)},$$

and so

$$\delta |f'(z)| |Q(z)-1| < \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)} + \alpha \left| f'(z) - \frac{f(z)}{z} \right|.$$

By (4) and (7),

$$\begin{aligned} \frac{\delta n}{n+1-\beta} |Q(z)-1| &< \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)} + \alpha \left(\frac{n(1-\beta)}{(n+1)(n+1-\beta)} \right) \\ &= \frac{\delta n(1-\beta)}{(n+1-\beta)}. \end{aligned}$$

By using the fact $\text{Re}\{w\} \leq |w|$, we have

$$|\text{Re}\,(Q(z)-1)| \leq |Q(z)-1| < 1-\beta,$$

which easily can give the conclusion $\text{Re}\, Q(z) > \beta$. □

Theorem 2 *Let* $\delta > 0$, $0 \leq \alpha < \delta$, $0 \leq \beta < 1$ *and* $g \in \mathcal{H}$. *If*

$$|g(z)| < \frac{n(1-\beta)[\delta(n+1)-\alpha]}{(n+1)(n+1-\beta)},$$

then

$$f(z) = z + \frac{z^{n+1}}{\delta} \int_0^1 \int_0^1 g(rsz) r^{(n\delta-\alpha)/\delta} s^{n-1} dr ds$$

is in $\mathcal{CV}(\beta)$.

Proof Let $f \in \mathcal{A}_n$ satisfy

$$\delta z f''(z) - \alpha \left(f'(z) - \frac{f(z)}{z} \right) = z^n g(z). \tag{8}$$

From Theorem 1, it is clear that the solution f of (8) lies in $\mathcal{CV}(\beta)$. Let

$$\phi(z) = \delta \left(f'(z) - \frac{f(z)}{z} \right).$$

Then (8) becomes

$$z\phi'(z) + \frac{(\delta - \alpha)}{\delta} \phi(z) = z^n g(z).$$

The solution ϕ is given by

$$\phi(z) = z^n \int_0^1 g(rz) r^{n-(\alpha/\delta)} dr,$$

and hence

$$f(z) = z + \frac{z^{n+1}}{\delta} \int_0^1 \int_0^1 g(rsz) r^{n-(\alpha/\delta)} s^{n-1} dr ds.$$

This completes the proof. □

Theorem 3 *Let* $f \in \mathcal{A}_n$, $\delta > 0$, $0 \le \alpha < \delta$ *and* $0 \le \beta < 1$. *If*

$$|\delta z f''(z) - \alpha(f'(z) - 1)| < \frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta}, \qquad z \in U, \tag{9}$$

then $f \in \mathcal{CV}(\beta)$.

Proof The inequality (9) can be expressed in the subordination form

$$\delta z f''(z) - \alpha \left(f'(z) - 1 \right) \prec \frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta} z, \qquad z \in U. \tag{10}$$

Let $P(z) = \delta f'(z) - (\delta + \alpha) f(z)/z$. Then (10) can be written as

$$P(z) + zP'(z) = \delta z f''(z) - \alpha f'(z) \prec \frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta} z - \alpha.$$

Lemma 1, with $\gamma = 1$, then readily yields

$$P(z) \prec \frac{1}{nz^{1/n}} \int_0^z \left(\frac{(1-\beta)(n\delta - \alpha)}{n + 1 - \beta} t - \alpha \right) t^{(1/n)-1} dt = \frac{(1-\beta)(n\delta - \alpha)}{(n+1)(n + 1 - \beta)} z - \alpha,$$

namely,

$$\delta f'(z) - (\delta + \alpha) \frac{f(z)}{z} \prec \frac{(1-\beta)(n\delta - \alpha)}{(n+1)(n + 1 - \beta)} z - \alpha. \tag{11}$$

Now consider

$$p(z) = \delta\left(\frac{f(z)}{z} - 1\right) \quad \text{and} \quad q(z) = \frac{\delta(1-\beta)}{(n+1)(n+1-\beta)}z.$$

Then $q(0) = 0$, $q'(0) \neq 0$ and $\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) = 1 > \frac{\alpha}{\delta n}$. Since

$$nzq'(z) - \frac{\alpha}{\delta}q(z) = \frac{(1-\beta)(n\delta-\alpha)}{(n+1)(n+1-\beta)}z$$

and

$$zp'(z) - \frac{\alpha}{\delta}p(z) = \delta f'(z) - (\delta+\alpha)\frac{f(z)}{z} + \alpha,$$

the subordination (11) can be written as

$$zp'(z) - \frac{\alpha}{\delta}p(z) \prec nzq'(z) - \frac{\alpha}{\delta}q(z).$$

Lemma 3 then gives $p \prec q$, which is

$$\frac{f(z)}{z} \prec 1 + \frac{1-\beta}{(n+1)(n+1-\beta)}z. \tag{12}$$

This implies

$$\left|\frac{f(z)}{z} - 1\right| < \frac{1-\beta}{(n+1)(n+1-\beta)}. \tag{13}$$

To complete the proof, we will show that inequality (9) implies inequality (1):

$$\begin{aligned}\left|\delta zf''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| &\leq \left|\delta zf''(z) - \alpha\left(f'(z) - 1\right)\right| + \alpha\left|\frac{f(z)}{z} - 1\right| \\ &< \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta} + \alpha\frac{1-\beta}{(n+1)(n+1-\beta)} \\ &= \frac{n(1-\beta)\big[\delta(n+1)-\alpha\big]}{(n+1)(n+1-\beta)}.\end{aligned}$$

Then f is in $\mathcal{CV}(\beta)$ by Theorem 1. □

Theorem 4 *Let $\delta > 0$, $0 \leq \alpha < \delta$, $0 \leq \beta < 1$, and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{(1-\beta)(n\delta-\alpha)}{n+1-\beta}, \qquad z \in U,$$

then

$$f(z) = z + \frac{z^{n+1}}{\delta}\int_0^1\int_0^1 g(rsz)r^{[(n-1)\delta-\alpha]/\delta}s^n drds$$

is in $\mathcal{CV}(\beta)$.

Proof For $g \in \mathcal{H}$, suppose that $f \in \mathcal{A}_n$ satisfy the differential equation

$$\delta zf''(z) - \alpha(f'(z) - 1) = z^n g(z). \tag{14}$$

From Theorem 3, it is clear that the solution f of (14) is in $\mathcal{CV}(\beta)$.

To solve Eq. (14), let $\phi(z) = f'(z) - 1$. Then the equation becomes

$$\delta z\phi'(z) - \alpha\phi(z) = z^n g(z).$$

Solving this gives

$$\phi(z) = \frac{z^n}{\delta}\int_0^1 g(rz)r^{n-1-(\alpha/\delta)}dr.$$

Since $\phi(z) = f'(z) - 1$, it follows that

$$f(z) = z + \frac{z^{n+1}}{\delta}\int_0^1\int_0^1 g(rsz)r^{[(n-1)\delta-\alpha]/\delta}s^n drds.$$

This completes the proof. □

3 Convexity of functions defined by third-order differential inequalities

Theorem 5 *Let $f \in \mathcal{A}_n$, $\delta > 0$, $\gamma > 0$ and $0 \le \beta < 1$. Further let $0 \le \alpha < (1-\mu)\nu$, where $0 < \mu < 1$ and $\nu > 0$ satisfying*

$$\nu - \frac{\alpha\mu}{1-\mu} = \delta - \gamma, \quad \nu\mu = \gamma. \tag{15}$$

If

$$\left|\gamma z^2 f'''(z) + \delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(1-\beta)(1+n\mu)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)}, \tag{16}$$

then $f \in \mathcal{CV}(\beta)$.

Proof Let

$$p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right).$$

Then

$$\begin{aligned} p(z) + \mu z p'(z) &= \mu\nu z^2 f'''(z) + \left(\mu\nu - \frac{\alpha\mu}{1-\mu} + \nu\right) z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) \\ &= \gamma z^2 f'''(z) + \delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right). \end{aligned}$$

Hence Eq. (16) can be written as

$$p(z) + \mu z p'(z) \prec \frac{n(1-\beta)(1+n\mu)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)} z.$$

It follows from Lemma 1 that

$$\begin{aligned} p(z) &\prec \frac{1}{\mu n z^{\frac{1}{n\mu}}}\int_0^z \frac{n(1+n\mu)(1-\beta)[(n+1)(\nu-\gamma)-\alpha]}{(n+1-\beta)(n+1)(1-\mu)} t^{\frac{1}{n\mu}} dt \\ &= \frac{n(1-\beta)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)} z, \end{aligned}$$

which then implies

$$\begin{aligned} \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right) &\prec \frac{n(1-\beta)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)} z \\ &= \frac{n(1-\beta)}{(n+1)(n+1-\beta)}\left[\frac{\nu(n+1)(1-\mu)}{(1-\mu)} - \frac{\alpha}{(1-\mu)}\right] z. \end{aligned}$$

Hence

$$\left|\nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right)\right| < \frac{n(1-\beta)\left(\nu(n+1) - \frac{\alpha}{1-\mu}\right)}{(n+1)(n+1-\beta)}.$$

By using Theorem 1, $f \in \mathcal{CV}(\beta)$. □

Theorem 6 *Let $\delta > 0$, $\gamma > 0$, $0 \le \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{n(1-\beta)(1+n\mu)[(n+1)(\nu-\gamma)-\alpha]}{(1-\mu)(n+1)(n+1-\beta)},$$

where μ and ν are given by (15) and $0 \le \alpha < (1-\mu)\nu$, then

$$f(z) = z + \frac{z^{n+1}}{\gamma}\int_0^1\int_0^1\int_0^1 g(rstz) r^{n-1+\frac{1}{\mu}} s^{n-\frac{\alpha}{\nu(1-\mu)}} t^{n-1} drdsdt$$

is in $\mathcal{CV}(\beta)$.

Proof Let $f \in \mathcal{A}_n$ satisfy

$$\gamma z^2 f'''(z) + \delta z f''(z) - \alpha\left(f'(z) - \frac{f(z)}{z}\right) = z^n g(z). \tag{17}$$

It follows from Theorem 5 that the solution of Eq. (17) is convex of order β. Now to solve this Eq. (17), note that it can be written as

$$p(z) + \mu z p'(z) = z^n g(z), \tag{18}$$

where

$$p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right).$$

Equation (18) has a solution

$$p(z) = z^n \phi(z),$$

where

$$\phi(z) = \frac{1}{\mu}\int_0^1 g(rz) r^{n+\frac{1}{\mu}-1} dr.$$

Now similarly as in the proof of Theorem 2, the equation

$$p(z) = \nu z f''(z) - \frac{\alpha}{1-\mu}\left(f'(z) - \frac{f(z)}{z}\right) = z^n \phi(z)$$

gives the solution

$$f(z) = z + \frac{z^{n+1}}{\nu}\int_0^1\int_0^1 \phi(stz) s^{n-\frac{\alpha}{\nu(1-\mu)}} t^{n-1} dsdt. \tag{19}$$

Substituting $\phi(rsz)$ into (19) yields

$$\begin{aligned} f(z) &= z + \frac{z^{n+1}}{\nu}\int_0^1\int_0^1\left[\frac{1}{\mu}\int_0^1 g(rstz) r^{n+\frac{1}{\mu}-1} dr\right] s^{n-\frac{\alpha}{\nu(1-\mu)}} t^{n-1} dsdt \\ &= z + \frac{z^{n+1}}{\gamma}\int_0^1\int_0^1\int_0^1 g(rstz) r^{n-1+\frac{1}{\mu}} s^{n-\frac{\alpha}{\nu(1-\mu)}} t^{n-1} drdsdt. \end{aligned}$$

This completes the proof. □

Theorem 7 *Let $f \in \mathcal{A}_n$, $\delta > 0$, $\gamma > 0$ and $0 \le \beta < 1$. Further, let $0 < \alpha < \nu$, and $\mu > 0$ and $\nu > 0$ are satisfying*

$$\nu - \alpha\mu = \delta - \gamma, \quad \nu\mu = \gamma. \tag{20}$$

If

$$|\gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1)| < \frac{(1+n\mu)(1-\beta)(n\nu - \alpha)}{n+1-\beta}, \tag{21}$$

then $f \in \mathcal{CV}(\beta)$.

Proof Let

$$p(z) = \nu z f''(z) - \alpha(f'(z) - 1).$$

Then

$$\begin{aligned} p(z) + \mu z p'(z) &= \mu\nu z^2 f'''(z) + [\nu - \mu\alpha + \mu\nu] z f''(z) - \alpha(f'(z) - 1) \\ &= \gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1). \end{aligned}$$

Hence, (21) can be written as

$$p(z) + \mu z p'(z) \prec \frac{(1+n\mu)(1-\beta)(n\nu - \alpha)}{n+1-\beta} z.$$

Applying Lemma 1 gives

$$\begin{aligned} p(z) &\prec \frac{1}{\mu n z^{1/(n\mu)}} \int_0^z \frac{(1+n\mu)(1-\beta)(n\nu - \alpha)}{n+1-\beta} t^{1/(n\mu)} dt \\ &= \frac{(1-\beta)(n\nu - \alpha)}{n+1-\beta} z, \end{aligned}$$

which further yields

$$\left|\nu z f''(z) - \alpha(f'(z) - 1)\right| < \frac{(1-\beta)(n\nu - \alpha)}{n+1-\beta}.$$

It follows from Theorem 3 that $f \in \mathcal{CV}(\beta)$. □

Example 1 Consider the function

$$f(z) = z + \frac{1-\beta}{(n+1)(n+1-\beta)} z^{n+1}, \quad z \in U \quad 0 \le \beta < 1.$$

Then for $\delta, \gamma, \alpha, \mu$ and ν satisfying conditions as in Theorem 7,

$$\begin{aligned} |\gamma z^2 f'''(z) + \delta z f''(z) - \alpha(f'(z) - 1)| &= \left| \frac{\gamma n(n-1)(1-\beta) z^n}{n+1-\beta} + \frac{\delta n(1-\beta) z^n}{n+1-\beta} - \frac{\alpha(1-\beta) z^n}{n+1-\beta} \right| \\ &= \left| \frac{(1-\beta)(1+n\mu)(n\nu - \alpha)}{n+1-\beta} \right| |z|^n \\ &< \frac{(1-\beta)(1+n\mu)(n\nu - \alpha)}{n+1-\beta}. \end{aligned}$$

Hence by Theorem 7, $f \in \mathcal{CV}(\beta)$. Indeed,

$$\begin{aligned} \operatorname{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right) &= \operatorname{Re}\left(\frac{1 + ((n+1)(1-\beta)/(n+1-\beta)) z^n}{1 + ((1-\beta)/(n+1-\beta)) z^n}\right) \\ &> \frac{1 - ((n+1)(1-\beta)/(n+1-\beta))}{1 - ((1-\beta)/(n+1-\beta))} = \beta. \end{aligned}$$

Theorem 8 *Let $\delta > 0$, $\gamma > 0$, $0 \le \beta < 1$ and $g \in \mathcal{H}$. If*

$$|g(z)| < \frac{(1+n\mu)(1-\beta)(n\nu-\alpha)}{n+1-\beta},$$

where μ and ν are given by (20) and $0 < \alpha < \nu$, then

$$f(z) = z + \frac{z^{n+1}}{\gamma} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-1+\frac{1}{\mu}} s^{n-1-\frac{\alpha}{\nu}} t^n \, dr ds dt$$

is in $\mathcal{CV}(\beta)$.

Proof Let $f \in \mathcal{A}_n$ satisfy

$$\gamma z^2 f'''(z) + \delta z f''(z) - \alpha \left(f'(z) - 1\right) = z^n g(z). \tag{22}$$

From Theorem 7, the solution of the differential equation (22) is a convex function of order β. Let $p(z) = \nu z f''(z) - \alpha(f'(z) - 1)$. Then (22) reduces to

$$p(z) + \mu z p'(z) = z^n g(z),$$

which has the solution

$$p(z) = \frac{z^n}{\mu} \int_0^1 g(rz) r^{n+\frac{1}{\mu}-1} dr. \tag{23}$$

By writing $\phi(z) = \frac{1}{\mu} \int_0^1 g(rz) r^{n+\frac{1}{\mu}-1} dr$, Eq. (23) becomes

$$\nu z f''(z) - \alpha(f'(z) - 1) = z^n \phi(z).$$

Comparing this with Eq. (14) in the proof of Theorem 4, the solution f is given by

$$f(z) = z + \frac{z^{n+1}}{\nu} \int_0^1 \int_0^1 \phi(stz) s^{n-\frac{\alpha}{\nu}-1} t^n ds dt.$$

Then

$$\begin{aligned} f(z) &= z + \frac{z^{n+1}}{\nu} \int_0^1 \int_0^1 \left[\frac{1}{\mu} \int_0^1 g(rstz) r^{n+\frac{1}{\mu}-1} dr \right] s^{n-\frac{\alpha}{\nu}-1} t^n ds dt \\ &= z + \frac{z^{n+1}}{\gamma} \int_0^1 \int_0^1 \int_0^1 g(rstz) r^{n-1+\frac{1}{\mu}} s^{n-1-\frac{\alpha}{\nu}} t^n dr ds dt, \end{aligned}$$

and this completes the proof. □

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