CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

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CONTENTS

SYMBOLS

Symbol	Description
$\mathcal{A}_p$	Class of all p-valent analytic functions of the form
	$f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k \quad (z \in U)$
$\mathcal{A} := \mathcal{A}_1$	Class of analytic functions of the form
	$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U)$
$\mathcal{S}$	Class of all normalized univalent functions of the form
	$f(z) = z + a_2 z^2 + \cdots \quad z \in U$
U	Open unit disk $\{z \in \mathcal{C} : \lvert z\rvert < 1\}$
U^*	Punctured unit disk $U \setminus \{0\}$
$U(\mathfrak{p})$	Class of meromorphic functions $f(z)$ in unit disk
	U with a simple pole at $z = \mathfrak{p}$, $\mathfrak{p} > 0$
$\mathcal{P}$	Class of functions $P(z)$ which are meromorphic in U
$K(\mathfrak{p})$	Class of functions which belong to $U(\mathfrak{p})$
	and map $\lvert z\rvert < r < \rho$ (for some $\mathfrak{p} < \rho < 1$)
	onto the complement of a convex set
$\mathcal{H}(U)$	Class of analytic functions in U
$\mathcal{C}$	Complex plane
K	Class of convex functions in U
$K(\alpha)$	Class of convex functions of order α in U
$\mathcal{S}^*$	Class of starlike functions in U
$S^*(\alpha)$	Class of starlike functions of order α in U
C	Class of close-to-convex functions in U
$\mathcal{UCV}$	Class of uniformly convex functions in U
$\mathcal{S}_p$	Class of parabolic starlike functions in U
$f * g$	Convolution or Hadamard product of functions f and g
$\prec$	Subordinate to

$k(z)$	Koebe function
$\mathcal{N}$	Set of all positive integers
$\mathcal{R}$	Set of all real numbers
$\Re$	Real part of a complex number
$\Im$	Imaginary part of a complex number
$\mathcal{Z}$	Set of all integers

KONVOLUSI DAN MASALAH PEKALI BAGI FUNGSI MULTIVALEN DITAKRIF DENGAN SUBORDINASI

ABSTRAK

Andaikan $\mathcal{C}$ satah kompleks, $U = \{z \in \mathcal{C} : |z| < 1\}$ cakera unit terbuka dalam $\mathcal{C}$ dan $\mathcal{H}(U)$ kelas fungsi analisis dalam U. Andaikan juga $\mathcal{A}$ kelas fungsi analisis f dalam U yang ternormalkan dengan $f(0) = 0$ dan $f'(0) = 1$. Fungsi $f \in \mathcal{A}$ mempunyai siri Taylor berbentuk

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).$$

Andaikan $\mathcal{A}_p$ $(p \in \mathcal{N})$ kelas fungsi analisis f berbentuk

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in U)$$

dengan $\mathcal{A} := \mathcal{A}_1$.

Pertimbangkan dua fungsi

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \quad \text{dan} \quad g(z) = z^p + b_{p+1} z^{p+1} + \cdots$$

dalam $\mathcal{A}_p$. Hasil darab Hadamard (atau konvolusi) untuk f dan g ialah fungsi $f * g$ berbentuk

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

Dalam Bab 1, kelas-kelas teritlak bak-bintang multivalen, cembung, hampir-cembung dan kuasi-cembung diperkenalkan. Kelas-kelas tersebut memberi kaedah penyatuan untuk pelbagai subkelas yang diketahui sebelum ini. Ciri-ciri konvolusi dan inklusi diterbitkan dengan menggunakan kaedah hul cembung dan subordinasi pembeza.

Dalam Bab 2, batas untuk pekali fungsian Fekete-Szegö bersekutu dengan transformasi punca ke-k $[f(z^k)]^{1/k}$ fungsi-fungsi analisis ternormalkan f tertakrif dalam U

diperoleh untuk kelas-kelas fungsi berikut:

$$R_b(\varphi) := \left\{ f \in \mathcal{A} \, : \, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\},$$

$$S^*(\alpha, \varphi) := \left\{ f \in \mathcal{A} \, : \, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\},$$

$$L(\alpha, \varphi) := \left\{ f \in \mathcal{A} \, : \, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$

$$M(\alpha, \varphi) := \left\{ f \in \mathcal{A} \, : \, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

dengan $b \in \mathcal{C} \setminus \{0\}$ dan $\alpha \geq 0$. Masalah yang serupa dikaji untuk fungsi $z/f(z)$ bagi f di dalam kelas-kelas fungsi tertentu.

Dalam Bab 3, beberapa subkelas fungsi univalen meromorfi dalam U diitlakkan. Andaikan $U(\mathfrak{p})$ kelas fungsi-fungsi univalen meromorfi ternormalkan f dalam U dengan kutub ringkas pada $z = \mathfrak{p}$, $\mathfrak{p} > 0$. Andaikan ϕ suatu fungsi dengan bahagian nyata positif dalam U, $\phi(0) = 1$, $\phi'(0) > 0$, yang memetakan U keseluruh rantau bak-bintang terhadap 1 dan simetri terhadap paksi nyata. Kelas $\sum^*(\mathfrak{p}, w_0, \phi)$ mengandungi fungsi $f \in U(\mathfrak{p})$, meromorfi bak-bintang terhadap w_0 sedemikian hingga

$$-\left(\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

Kelas $\sum(\mathfrak{p}, \phi)$ mengandungi fungsi $f \in U(\mathfrak{p})$, meromorfi cembung sedemikian hingga

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

Batas untuk w_0 dan beberapa pekali untuk f di dalam $\sum^*(\mathfrak{p}, w_0, \phi)$ dan $\sum(\mathfrak{p}, \phi)$ diperoleh.

CONVOLUTION AND COEFFICIENT PROBLEMS FOR MULTIVALENT FUNCTIONS DEFINED BY SUBORDINATION

ABSTRACT

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$ and $\mathcal{H}(U)$ be the class of analytic functions defined in U. Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $U := \{z \in \mathcal{C} : |z| < 1\}$, and normalized by $f(0) = 0$, and $f'(0) = 1$. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

Let $\mathcal{A}_p$ $(p \in \mathcal{N})$ be the class of all analytic functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

with $\mathcal{A} := \mathcal{A}_1$.

Consider two functions

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \quad \text{and} \quad g(z) = z^p + b_{p+1} z^{p+1} + \cdots$$

in $\mathcal{A}_p$. The Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

In Chapter 1, the general classes of multi-valent starlike, convex, close-to-convex and quasi-convex functions are introduced. These classes provide a unified treatment to various known subclasses. Inclusion and convolution properties are derived using the methods of convex hull and differential subordination.

In Chapter 2, bounds for the Fekete-Szegö coefficient functional associated with the k-th root transform $[f(z^k)]^{1/k}$ of normalized analytic functions f defined on U

are derived for the following classes of functions:

$$R_b(\varphi) := \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\},$$

$$S^*(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\},$$

$$L(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\},$$

$$M(\alpha, \varphi) := \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},$$

where $b \in \mathcal{C} \setminus \{0\}$ and $\alpha \geq 0$. A similar problem is investigated for functions $z/f(z)$ when f belongs to a certain class of functions.

In Chapter 3, some subclasses of meromorphic univalent functions in the unit disk U are extended. Let $U(\mathfrak{p})$ denote the class of normalized univalent meromorphic functions f in U with a simple pole at $z = \mathfrak{p}$, $\mathfrak{p} > 0$. Let ϕ be a function with positive real part on U, $\phi(0) = 1$, $\phi'(0) > 0$, which maps U onto a region starlike with respect to 1 and which is symmetric with respect to the real axis. The class $\sum^*(\mathfrak{p}, w_0, \phi)$ consists of functions $f \in U(\mathfrak{p})$ meromorphic starlike with respect to w_0 and satisfying

$$-\left(\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

The class $\sum(\mathfrak{p}, \phi)$ consists of functions $f \in U(\mathfrak{p})$ meromorphic convex and satisfying

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z).$$

The bounds for w_0 and some initial coefficients of f in $\sum^*(\mathfrak{p}, w_0, \phi)$ and $\sum(\mathfrak{p}, \phi)$ are obtained.

CHAPTER 1

INTRODUCTION

Let $\mathcal{C}$ be the complex plane and $U := \{z \in \mathcal{C} : |z| < 1\}$ be the open unit disk in $\mathcal{C}$ and $\mathcal{H}(U)$ be the class of analytic functions defined on U. Also let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $U := \{z \in \mathcal{C} : |z| < 1\}$, and normalized by $f(0) = 0$, and $f'(0) = 1$.

A function f is said to be univalent in a domain if it provides a one-to-one mapping onto its image: $f(z_1) = f(z_2) \Rightarrow z_1 = z_2$. Geometrically, this means that different points in the domain will be mapped into different points on the image domain. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of univalent functions. A function $f \in \mathcal{A}$ has the Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in U).$$

The Koebe function $k(z) = z/(1-z)^2$ maps U onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$ and is univalent. It plays a very important role in the study of the class $\mathcal{S}$. The Koebe function and its rotations $e^{-i\beta}k(e^{i\beta}z)$, for $\beta \in \mathcal{R}$, are the extremal functions for various problems in the class $\mathcal{S}$. For example, the de Branges Theorem tells that if $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is analytic and univalent in U, the coefficients satisfy $|a_n| \leq n$, $(n = 2, 3, \ldots)$ with equality if and only if f is a rotation of the Koebe function. This theorem was conjectured by Bieberbach in 1916 and was only proved in 1985 by de Branges. Since the Bieberbach conjecture was difficult to settle, several authors have considered classes defined by geometric conditions. Notable among them are the classes of convex functions, starlike functions and close-to-convex functions.

A set $\mathcal{D}$ in the complex plane is called *convex* if for every pair of points w_1 and w_2 lying in the interior of $\mathcal{D}$, the line segment joining w_1 and w_2 also lies in the interior of $\mathcal{D}$, i.e.

$$tw_1 + (1-t)w_2 \in \mathcal{D} \quad \text{for} \quad 0 \leq t \leq 1.$$

If a function $f \in \mathcal{A}$ maps U onto a convex domain, then $f(z)$ is called a convex function. The class of all convex functions in $\mathcal{A}$ is denoted by K. An analytic description of the class K is given by

$$K := \left\{ f \in \mathcal{A} : \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \right\}.$$

Let w_0 be an interior point of $\mathcal{D}$. A set $\mathcal{D}$ in the complex plane is called *starlike* with respect to w_0 if the line segment joining w_0 to every other point $w \in \mathcal{D}$ lies in the interior of $\mathcal{D}$, i.e.

$$(1-t)w + tw_0 \in \mathcal{D} \quad \text{for} \quad 0 \leq t \leq 1.$$

If a function $f \in \mathcal{A}$ maps U onto a domain starlike, then $f(z)$ is called a starlike function. The class of starlike functions with respect to origin is denoted by $\mathcal{S}^*$. Analytically,

$$\mathcal{S}^* := \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \right\}.$$

These two classes K and $\mathcal{S}^*$ and several other classes such as the classes of uniformly convex functions, starlike functions of order α, and strongly starlike functions investigated in geometric function theory are characterized by either of the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a given region in the right half-plane.

Ma and Minda [**28**] showed that many of these properties can be obtained by a unified method. For this purpose, they introduced the classes $K(\varphi)$ and $S^*(\varphi)$ of functions $f(z) \in \mathcal{A}$ for some analytic function $\varphi(z)$ with positive real part on U with $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps the unit disk U onto a region starlike with respect to 1, symmetric with respect to the real axis, satisfying

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$$

and

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (z \in U).$$

respectively. Here the symbol $\prec$ denotes subordination.

A function f is said to be subordinate to F in U, written $f(z) \prec F(z)$, if there exists a Schwarz function w, analytic in U with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = F(w(z))$. If the function F is univalent in U, then $f \prec F$ if $f(0) = F(0)$ and $f(U) \subseteq F(U)$.

CHAPTER 2

CONVOLUTION AND DIFFERENTIAL SUBORDINATION OF MULTIVALENT FUNCTIONS

2.1. MOTIVATION AND PRELIMINARIES

Let $\mathcal{A}_p$ $(p \in \mathcal{N})$ be the class of all analytic functions of the form

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots$$

with $\mathcal{A} := \mathcal{A}_1$. For two functions

$$f(z) = z^p + a_{p+1}z^{p+1} + \dots \quad \text{and} \quad g(z) = z^p + b_{p+1}z^{p+1} + \dots$$

in $\mathcal{A}_p$, the Hadamard product (or convolution) of f and g is the function $f * g$ defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

For univalent functions, the well-known Alexander theorem [**3**], states that $f \in K$ if and only if $zf'(z) \in \mathcal{S}^*$. Since $zf'(z) = f(z) * (z/(1-z)^2)$, it follows that f is convex if and only if $f * g$ is starlike for $g(z) = z/(1-z)^2$. Moreover, since $f(z) = f(z) * (z/(1-z))$, the investigation of the classes of convex and starlike functions can be unified by considering the class of functions f for which $f * g$ is starlike for a fixed function g. These ideas motivated the investigation of the class of functions f for which

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec h(z)$$

where g is a fixed function in $\mathcal{A}$ and h is a convex function with positive real part. Shanmugam [**55**] introduced this class and several other related classes, and investigated inclusion and convolution properties by using the convex hull method [**10, 54, 53**] and the method of differential subordination.

Motivated by the investigation of Shanmugam [**55**], Ravichandran [**45**] and Ali *et al.* [**4**] (see also [**6, 39, 38, 40**]), we introduce the following classes of multivalent functions. Throughout this chapter, the function $g \in \mathcal{A}_p$ is a fixed function and, unless otherwise mentioned, the function h is assumed to be a fixed convex univalent function with positive real part and $h(0) = 1$.

DEFINITION 2.1.1. The class $\mathcal{S}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*f)(z)}{z^p} \neq 0$ in U and satisfying the subordination

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z).$$

Similarly, $\mathcal{K}_{p,g}(h)$ is the class of functions $f \in \mathcal{A}_p$ satisfying $\frac{(g*f)'(z)}{z^{p-1}} \neq 0$ in U and

$$\frac{1}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right] \prec h(z).$$

With $g(z) = z^p/(1-z)$, the class $\mathcal{S}_{p,g}(h) =: \mathcal{S}_p^*(h)$ and $\mathcal{K}_{p,g}(h) =: \mathcal{K}_p(h)$ consists respectively of all p-valent starlike and convex functions satisfying the respective subordinations

$$\frac{1}{p}\frac{zf'(z)}{f(z)} \prec h(z), \text{ and } \quad \frac{1}{p}\left(1+\frac{zf''(z)}{f'(z)}\right) \prec h(z).$$

For these two classes, several interesting properties including distortion, growth and rotation inequalities as well as convolution properties have been investigated by Ali *et al.* [**4**]. Note that the two classes $\mathcal{S}_p^*(h)$ and $\mathcal{S}_{p,g}(h)$ are closely related; in fact, $f \in \mathcal{S}_{p,g}(h)$ if and only if $f*g \in \mathcal{S}_p^*(h)$. Similarly, $f \in \mathcal{K}_{p,g}(h)$ if and only if $f*g \in \mathcal{K}_p(h)$.

DEFINITION 2.1.2. The class $\mathcal{C}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*\psi)(z)}{z^p} \neq 0$ in U for some $\psi \in \mathcal{S}_{p,g}(h)$ and satisfying

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*\psi)(z)} \prec h(z).$$

DEFINITION 2.1.3. For any real number γ, the class $\mathcal{K}^{\gamma}_{p,g}(h)$ consists of functions $f \in \mathcal{A}_p$ such that $\frac{(g*f)(z)}{z^p} \neq 0$ and $\frac{(g*f)'(z)}{z^{p-1}} \neq 0$ in U, and satisfying the subordination

$$\frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right]+\frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right] \prec h(z).$$

DEFINITION 2.1.4. Let $\mathcal{Q}_{p,g}(h)$ denote the class of functions $f \in \mathcal{A}_p$ such that $\frac{(g*\delta)'(z)}{z^{p-1}} \neq 0$ in U for some $\delta \in \mathcal{K}_{p,g}(h)$ and satisfying the subordination

$$\frac{1}{p}\frac{[z(g*f)'(z)]'}{(g*\delta)'(z)} \prec h(z).$$

Polya-Schoenberg [**41**] conjectured that the class K of convex functions is preserved under convolution with convex functions:

$$f, g \in K \Rightarrow f*g \in K.$$

In 1973, Ruscheweyh and Sheil-Small [**54**] proved the Polya-Schoenberg conjecture. In fact, they proved that the classes of convex functions, starlike functions and close-to-convex functions are closed under convolution with convex functions. For an interesting development on these ideas, see Ruscheweyh [**53**] (and also Duren [**16**, pp. 246–258], as well as Goodman [**19**, pp. 129-130]). Using the techniques developed in Ruscheweyh [**53**], several authors [**4, 7, 8, 9, 10, 21, 23, 32, 33, 34, 39, 38, 40, 47, 45, 51, 55, 57, 58**] have proved that their classes are closed under convolution with convex (and other related) functions.

In this chapter, convolution properties as well as inclusion and related properties are investigated for the general classes of p-valent functions defined above. These classes are of course extensions of the classes of convex, starlike, close-to-convex, α-convex, and quasi-convex functions. The results obtained here extend the well-known convolution properties of p-valent functions.

The following definition and results are needed to prove our main results. For $\alpha \leq 1$, the class $\mathcal{R}_\alpha$ of *prestarlike* functions of order α consists of functions $f \in \mathcal{A}$

satisfying

$$\begin{cases} f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha), & (\alpha < 1); \\ \Re \frac{f(z)}{z} > \frac{1}{2}, & (\alpha = 1) \end{cases}$$

where $\mathcal{S}^*(\alpha)$ is the class introduced by Ma and Minda [**28**].

THEOREM 2.1.1. [**53**, Theorem 2.4] *Let $\alpha \leq 1$, $f \in \mathcal{R}_\alpha$ and $g \in S^*(\alpha)$. Then for any analytic function $H \in \mathcal{H}(U)$,*

$$\frac{f * Hg}{f * g}(U) \subset \overline{co}(H(U))$$

where $\overline{co}(H(U))$ denotes the closed convex hull of $H(U)$.

THEOREM 2.1.2. [**17, 31**] *Let $\beta, \nu \in \mathcal{C}$, and $h \in \mathcal{H}(U)$ be convex univalent in U, with $\Re(\beta h(z) + \nu) > 0$. If p is analytic in U with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z).$$

THEOREM 2.1.3. [**31**, Theorem 3.2b] *Let $h \in \mathcal{H}(U)$ be convex univalent in U with $h(0) = a$. Suppose that the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \nu} = h(z)$$

has a univalent solution q that satisfies $q(z) \prec h(z)$. If $p(z) = a + a_1 z + \cdots$ satisfies

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

THEOREM 2.1.4. [**31**, Theorem 3.1a] *Let h be convex in U and let $P : U \to \mathcal{C}$, with $\Re P(z) > 0$. If p is analytic in U, then*

$$p(z) + P(z)zp'(z) \prec h(z) \quad \Rightarrow \quad p(z) \prec h(z).$$

We will also be using the following convolution properties.

(i) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$(f * g)(z) = (g * f)(z).$$

PROOF. For f and g as given, we have

$$\begin{aligned}(f * g)(z) &= z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \\ &= z^p + \sum_{n=p+1}^{\infty} b_n a_n z^n \\ &= (g * f)(z). \qquad \square\end{aligned}$$

(ii) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$\frac{1}{p} z (g * f)'(z) = (g * \frac{1}{p} z f')(z).$$

PROOF. For f of the given form, we have

$$\frac{1}{p} z f'(z) = z^p + \sum_{n=p+1}^{\infty} \frac{1}{p} n a_n z^n$$

and hence

$$\begin{aligned}(g * \frac{1}{p} z f')(z) &= z^p + \sum_{n=p+1}^{\infty} \frac{1}{p} n a_n b_n z^n \\ &= \frac{1}{p} z \left(p z^{p-1} + \sum_{n=p+1}^{\infty} n a_n b_n z^{n-1} \right) \\ &= \frac{1}{p} z (g * f)'(z). \qquad \square\end{aligned}$$

(iii) For two functions f and g of the forms $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ and $g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n$, we have

$$\frac{(g * f)(z)}{z^{p-1}} = (\frac{g}{z^{p-1}} * \frac{f}{z^{p-1}})(z).$$

Proof. For

$$(g * f)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n,$$

we observe that

$$\begin{aligned}
\frac{(g * f)(z)}{z^{p-1}} &= \frac{1}{z^{p-1}} \left(z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n \right) \\
&= z + \frac{1}{z^{p-1}} \sum_{n=p+1}^{\infty} a_n b_n z^n \\
&= \frac{g(z)}{z^{p-1}} * \frac{f(z)}{z^{p-1}}.
\end{aligned}$$

□

2.2. INCLUSION AND CONVOLUTION THEOREMS

Every convex univalent function is starlike or equivalently $K \subset \mathcal{S}^*$, and Alexander's theorem gives $f \in K$ if and only if $zf' \in \mathcal{S}^*$. These properties remain valid even for multivalent functions.

Theorem 2.2.1. *Let g be a fixed function in $\mathcal{A}_p$ and h be a convex univalent function with positive real part and $h(0) = 1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$,

(ii) $f \in \mathcal{K}_{p,g}(h)$ *if and only if* $\frac{1}{p} zf' \in \mathcal{S}_{p,g}(h)$.

Proof. (i) Since $(f * g)(z)/z^p \neq 0$, the function q defined by

$$q(z) = \frac{1}{p} \frac{z(g * f)'(z)}{(g * f)(z)},$$

is analytic in U. By some computations we have,

$$\begin{aligned}
\frac{zq'(z)}{q(z)} &= 1 + \frac{z(g * f)''(z)}{(g * f)'(z)} - \frac{z(g * f)'(z)}{(g * f)(z)} \\
&= 1 + \frac{z(g * f)''(z)}{(g * f)'(z)} - pq(z).
\end{aligned}$$

Equivalently, we have

$$pq(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{z(g * f)''(z)}{(g * f)'(z)}$$

and satisfies

$$q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}=\frac{1}{p}\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right). \tag{2.2.1}$$

If $f\in\mathcal{K}_{p,g}(h)$, the right-hand side of (2.2.1) is subordinate to h. It follows from Theorem 2.1.2 that $q(z)\prec h(z)$, and thus $\mathcal{K}_{p,g}(h)\subseteq\mathcal{S}_{p,g}(h)$.

(ii) Since

$$\begin{aligned}\frac{1}{p}\left(1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right)&=\frac{1}{p}\left[\frac{(g*f)'(z)+z(g*f)''(z)}{(g*f)'(z)}\right]\\&=\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{(g*zf')'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{\frac{1}{p}z}{\frac{1}{p}z}\frac{(g*zf')'(z)}{(g*f)'(z)}\\&=\frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}zf')(z)},\end{aligned}$$

it follows that $f\in\mathcal{K}_{p,g}(h)$ if and only if $\frac{1}{p}zf'\in\mathcal{S}_{p,g}(h)$. □

Suppose that the differential equation

$$q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}=h(z)$$

has a univalent solution q that satisfies $q(z)\prec h(z)$. If $f\in\mathcal{K}_{p,g}(h)$, then from Theorem 2.1.3 and (2.2.1), it follows that $f\in\mathcal{S}_{p,g}(q)$, or equivalently $\mathcal{K}_{p,g}(h)\subset\mathcal{S}_{p,g}(q)$.

Theorem 2.2.2. *Let h be a convex univalent function satisfying the condition*

$$\Re h(z)>1-\frac{1-\alpha}{p}\quad(0\le\alpha<1), \tag{2.2.2}$$

and $\phi\in\mathcal{A}_p$ with $\phi/z^{p-1}\in\mathcal{R}_\alpha$. If $f\in\mathcal{S}_{p,g}(h)$, then $\phi*f\in\mathcal{S}_{p,g}(h)$.

Proof. For $f\in\mathcal{S}_{p,g}(h)$, let the function H be defined by

$$H(z):=\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

Then H is analytic in U and $H(z) \prec h(z)$. Also let $\Phi(z) := \phi(z)/z^{p-1}$ belongs to $\mathcal{R}_\alpha$. We now show that the function $G(z) := (f * g)(z)/z^{p-1}$ belongs to $\mathcal{S}^*(\alpha)$.

Since $f \in \mathcal{S}_{p,g}(h)$, and h is a convex univalent function satisfying (2.2.2), it follows that

$$\frac{1}{p}\Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) > 1 - \frac{1-\alpha}{p},$$

and hence

$$\begin{aligned}\Re\frac{zG'(z)}{G(z)} &= \Re\left(\frac{z(f*g)'(z)}{(f*g)(z)}\right) - p + 1 \\ &> p - 1 + \alpha - p + 1 \\ &> \alpha.\end{aligned}$$

Thus $G \in \mathcal{S}^*(\alpha)$. Since $\Phi \in \mathcal{R}_\alpha$, $G \in \mathcal{S}^*(\alpha)$, and h is convex, an application of Theorem 2.1.1 shows that

$$\frac{(\Phi * GH)(z)}{(\Phi * G)(z)} \prec h(z). \tag{2.2.3}$$

We have

$$\begin{aligned}\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} &= \frac{\phi(z) * \frac{1}{p}z(g*f)'(z)}{\phi(z) * (g*f)(z)} \\ &= \frac{\frac{\phi(z)}{z^{p-1}} * \frac{\frac{1}{p}z(g*f)'(z)}{z^{p-1}}}{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}} \\ &= \frac{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}} * \frac{(g*f)(z)}{z^{p-1}}} \\ &= \frac{(\Phi * GH)(z)}{(\Phi * G)(z)}.\end{aligned}$$

Thus the subordination (2.2.3) gives

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*f)(z)} \prec h(z),$$

which proves $\phi * f \in \mathcal{S}_{p,g}(h)$. □

Corollary 2.2.1. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. Then $\mathcal{S}_{p,g}(h) \subseteq \mathcal{S}_{p,\phi*g}(h)$.*

Proof. If $f \in \mathcal{S}_{p,g}(h)$, Theorem 2.2.2 yields $f * \phi \in \mathcal{S}_{p,g}(h)$, that is $f * \phi * g \in \mathcal{S}_p^*(h)$. Hence $f \in \mathcal{S}_{p,\phi * g}(h)$. $\square$

In particular, when $g(z) = z^p/(1-z)$, the following corollary is obtained.

Corollary 2.2.2. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{S}_p^*(h)$, then $f \in \mathcal{S}_{p,\phi}^*(h)$.*

Corollary 2.2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{K}_{p,g}(h)$, then $f * \phi \in \mathcal{K}_{p,g}(h)$ and $\mathcal{K}_{p,g}(h) \subseteq \mathcal{K}_{p,\phi * g}(h)$.*

Proof. If $f \in \mathcal{K}_{p,g}(h)$, it follows from Theorem 2.2.1(ii) and Theorem 2.2.2 that $\frac{1}{p}(zf' * \phi) \in \mathcal{S}_{p,g}(h)$. Hence $f * \phi \in \mathcal{K}_{p,g}(h)$. The second part follows from Corollary 2.2.1. $\square$

Theorem 2.2.3. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{C}_{p,g}(h)$ with respect to $\psi \in \mathcal{S}_{p,g}(h)$, then $\phi * f \in \mathcal{C}_{p,g}(h)$ with respect to $\phi * \psi \in \mathcal{S}_{p,g}(h)$.*

Proof. As in the proof of Theorem 2.2.2, define the functions H, Φ and G by

$$H(z) := \frac{1}{p}\frac{z(g * f)'(z)}{(g * \psi)(z)}, \quad \Phi(z) := \frac{\phi(z)}{z^{p-1}}, \quad \text{and} \quad G(z) := \frac{(\psi * g)(z)}{z^{p-1}}.$$

Then $\Phi \in \mathcal{R}_\alpha$ and $G \in \mathcal{S}^*(\alpha)$. An application of Theorem 2.1.1 shows that the quantity $\dfrac{(\Phi * GH)(z)}{(\Phi * G)(z)}$ lies in the closed convex hull of $H(U)$. Since h is convex and $H \prec h$, it follows that

$$\frac{(\Phi * GH)(z)}{(\Phi * G)(z)} \prec h(z). \tag{2.2.4}$$

Observe that

$$\begin{aligned}
\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*\psi)(z)} &= \frac{\phi(z)*\frac{1}{p}z(g*f)'(z)}{\phi(z)*(g*\psi)(z)}\\
&= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{\frac{1}{p}z(g*f)'(z)}{z^{p-1}}}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}}\\
&= \frac{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}H(z)}{\frac{\phi(z)}{z^{p-1}}*\frac{(g*\psi)(z)}{z^{p-1}}}\\
&= \frac{(\Phi*GH)(z)}{(\Phi*G)(z)}.
\end{aligned}$$

Thus, the subordination (2.2.4) shows that $\phi*f\in\mathcal{C}_{p,g}(h)$ with respect to $\phi*\psi\in\mathcal{S}_{p,g}(h)$. ☐

COROLLARY 2.2.4. *If h and ϕ satisfy the conditions of Theorem 2.2.2, then $\mathcal{C}_{p,g}(h)\subseteq\mathcal{C}_{p,\phi*g}(h)$.*

PROOF. From Theorem 2.2.3, for a function $f\in\mathcal{C}_{p,g}(h)$ with respect to $\psi\in\mathcal{S}_{p,g}(h)$, we have

$$\frac{1}{p}\frac{z(g*\phi*f)'(z)}{(g*\phi*\psi)(z)}\prec h(z).$$

Thus, $f\in\mathcal{C}_{p,\phi*g}(h)$, and hence $\mathcal{C}_{p,g}(h)\subseteq\mathcal{C}_{p,\phi*g}(h)$. ☐

THEOREM 2.2.4. *Let h be a convex univalent function with positive real part and $h(0)=1$. Then*

(i) $\mathcal{K}^{\gamma}_{p,g}(h)\subseteq\mathcal{S}_{p,g}(h)$ *for* $\gamma>0$,

(ii) $\mathcal{K}^{\gamma}_{p,g}(h)\subseteq\mathcal{K}^{\beta}_{p,g}(h)$ *for* $\gamma>\beta\geq0$,
where $\mathcal{K}^{\gamma}_{p,g}(h)$ is defined in Definition 2.1.3.

PROOF. (i) Let

$$J_{p,g}(\gamma;f(z)):=\frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right]+\frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right]$$

and the function $q(z)$ be defined by

$$q(z) := \frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)}.$$

Note that

$$\begin{aligned} J_{p,g}(\gamma; f(z)) &= \frac{\gamma}{p}\left[1+\frac{z(g*f)''(z)}{(g*f)'(z)}\right] + \frac{(1-\gamma)}{p}\left[\frac{z(g*f)'(z)}{(g*f)(z)}\right] \\ &= \gamma\left(q(z)+\frac{1}{p}\frac{zq'(z)}{q(z)}\right) + (1-\gamma)q(z) \\ &= q(z) + \frac{\gamma z q'(z)}{pq(z)}. \end{aligned} \tag{2.2.5}$$

Let $f \in \mathcal{K}^{\gamma}_{p,g}(h)$, so $J_{p,g}(\gamma; f(z)) \prec h(z)$. Now an application of Theorem 2.1.2 shows that $q(z) \prec h(z)$. Hence $f \in \mathcal{S}_{p,g}(h)$.

(ii) The case $\beta = 0$ is contained in (i), so we assume $\beta > 0$. We define $q(z)$ as in (i), then (2.2.5) yields

$$J_{p,g}(\beta; f(z)) = q(z) + \frac{\beta}{p}\frac{zq'(z)}{q(z)}. \tag{2.2.6}$$

Since

$$J_{p,g}(\gamma; f(z)) = q(z) + \frac{\gamma z q'(z)}{pq(z)},$$

we have

$$\frac{1}{\gamma}\left(J_{p,g}(\gamma; f(z)) - q(z)\right) = \frac{1}{p}\frac{zq'(z)}{q(z)}. \tag{2.2.7}$$

Substituting (2.2.7) in (2.2.6), we have

$$J_{p,g}(\beta; f(z)) = (1-\frac{\beta}{\gamma})\frac{z(g*f)'(z)}{p(g*f)(z)} + \frac{\beta}{\gamma}J_{p,g}(\gamma; f(z))$$

From part (i),

$$\frac{1}{p}\frac{z(g*f)'(z)}{(g*f)(z)} \prec h(z)$$

and

$$J_{p,g}(\gamma; f(z)) \prec h(z).$$

Since $J_{p,g}(\beta; f(z))$ is a convex combination of points in $h(U)$, and h is convex, it follows that $J_{p,g}(\beta; f(z)) \prec h(z)$, proving that $f \in \mathcal{K}^{\beta}_{p,g}(h)$. $\square$

Theorem 2.2.5. *Let h be a convex univalent function with positive real part and $h(0)=1$. Then*

(i) $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h) \subseteq \mathcal{C}_{p,g}(h)$,

(ii) $f \in \mathcal{Q}_{p,g}(h)$ *if and only if* $\frac{1}{p} z f' \in \mathcal{C}_{p,g}(h)$.

Proof. (i) By taking $f=\delta$, it follows from the definition that $\mathcal{K}_{p,g}(h) \subseteq \mathcal{Q}_{p,g}(h)$. To prove the middle inclusion, let

$$q(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)(z)}.$$

By logarithmic differentiation and multiplication of z, we have

$$\frac{zq'(z)}{q(z)} = 1 + \frac{z(g*f)''(z)}{(g*f)'(z)} - \frac{z(g*\delta)'(z)}{(g*\delta)(z)}.$$

Rewriting the equation,

$$zq'(z) = q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z) - \frac{z(g*\delta)'(z)}{(g*\delta)(z)}q(z)$$

$$\frac{z(g*\delta)'(z)}{(g*\delta)(z)}q(z) + zq'(z) = q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z)$$

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{(g*\delta)(z)}{z(g*\delta)'(z)}q(z) + \frac{z(g*f)''(z)}{(g*f)'(z)}q(z)\cdot\frac{(g*\delta)(z)}{z(g*\delta)'(z)}.$$

Substituting $q(z) = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)(z)}$ on the right-hand side of the above equation, we have

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{1}{p}\frac{z(g*f)'(z)}{(g*\delta)'(z)} + \frac{1}{p}\frac{z(g*f)''(z)}{(g*\delta)'(z)}.$$

The above computations shows that

$$q(z) + \frac{zq'(z)}{\frac{z(g*\delta)'(z)}{(g*\delta)(z)}} = \frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\delta)'(z)}. \tag{2.2.8}$$

If $f \in \mathcal{Q}_{p,g}(h)$, then there exists a function $\delta \in \mathcal{K}_{p,g}(h)$ such that the expression on the right-hand side of (2.2.8) is subordinate to $h(z)$. Also $\delta \in \mathcal{K}_{p,g}(h) \subseteq \mathcal{S}_{p,g}(h)$ implies $\Re\frac{z(g*\delta)'(z)}{(g*\delta)} > 0$. Hence, an application of Theorem 2.1.4 to (2.2.8) yields $q(z) \prec h(z)$. This shows that $f \in \mathcal{C}_{p,g}(h)$.

(ii) It is easy to see that

$$\begin{aligned}\frac{1}{p}\frac{[z(g*f)'(z)]'(z)}{(g*\delta)'(z)} &= \frac{1}{p}\frac{(g*zf')'(z)}{(g*\delta)'(z)}\cdot\frac{\frac{1}{p}z}{\frac{1}{p}z}\\ &= \frac{1}{p}\frac{z(g*\frac{1}{p}zf')'(z)}{(g*\frac{1}{p}z\delta')(z)}. \qquad (2.2.9)\end{aligned}$$

Now if $f \in \mathcal{Q}_{p,g}(h)$ with respect to a function $\delta \in \mathcal{K}_{p,g}(h)$, then the expression on the left-hand side of (2.2.9) is subordinate to $h(z)$. Now by Theorem 2.2.1(ii) and definition of $\mathcal{C}_{p,g}(h)$, we get $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$.

Conversely, if $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$, then there exists a function $\delta_1 \in \mathcal{S}_{p,g}(h)$ such that $\frac{1}{p}z\delta' = \delta_1$. The expression on the right-hand side of (2.2.9) is subordinate to $h(z)$ and thus $f \in \mathcal{Q}_{p,g}(h)$. □

Corollary 2.2.5. *Let h and ϕ satisfy the conditions of Theorem 2.2.2. If $f \in \mathcal{Q}_{p,g}(h)$, then $\phi * f \in \mathcal{Q}_{p,g}(h)$.*

Proof. If $f \in \mathcal{Q}_{p,g}(h)$, then by Theorem 2.2.5(ii), $\frac{1}{p}zf' \in \mathcal{C}_{p,g}(h)$. Theorem 2.2.3 shows that $\frac{1}{p}z(\phi * f)' \in \mathcal{C}_{p,g}(h)$, and by Theorem 2.2.5(ii), we have $\phi * f \in \mathcal{Q}_{p,g}(h)$. □

Corollary 2.2.6. *If h and ϕ satisfy the conditions of Theorem 2.2.2, then $\mathcal{Q}_{p,g}(h) \subseteq \mathcal{Q}_{p,\phi*g}$.*

Proof. If $f \in \mathcal{Q}_{p,g}(h)$, Corollary 2.2.5 yields $f * \phi \in \mathcal{Q}_{p,g}(h)$ with respect to $\phi * \delta \in \mathcal{K}_{p,g}(h)$. The subordination

$$\frac{1}{p}\frac{[z(g*\phi*f)'(z)]'}{(g*\phi*\delta)'(z)} \prec h(z)$$

gives $f \in \mathcal{Q}_{p,g*\phi}$. Therefore, $\mathcal{Q}_{p,g}(h) \subseteq \mathcal{Q}_{p,g*\phi}$. □

A function is prestarlike of order 0 if $f(z)*(z/(1-z)^2)$ is starlike, or equivalently if f is convex. Thus, the class of prestarlike functions of order 0 is the class of convex functions, and therefore the results obtained in this chapter contains the results of Shanmugam [**55**] for the special case $p = 1$ and $\alpha = 0$.

EXAMPLE 2.2.1. Let $p = 1$, $g(z) = z/(1 - z)$, and $\alpha = 0$. For $h(z) = (1 + z)/(1 - z)$, Theorem 2.2.1 reduces to the following: $K \subseteq \mathcal{S}^*$ and $f \in K \Leftrightarrow zf' \in \mathcal{S}^*$. Also Theorem 2.2.2 reduces to $f \in \mathcal{S}^*, g \in K \Rightarrow f * g \in \mathcal{S}^*$, and Corollary 2.23 shows that the class of convex functions is closed under convolution with convex functions.

For

$$h(z) = 1 + \frac{2}{\pi^2}\left[\log\frac{1 + \sqrt{z}}{1 - \sqrt{z}}\right]^2,$$

the results obtained imply that $\mathcal{UCV} \subseteq \mathcal{S}_p$ and $f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{S}_p$, where $\mathcal{UCV}$ and $\mathcal{S}_p$ are the classes of uniformly convex functions and parabolic starlike functions [**51, 50**]. It also follows as special cases that the classes $\mathcal{S}_p$ and $\mathcal{UCV}$ are closed under convolution with convex functions.

CHAPTER 3

THE FEKETE-SZEGÖ COEFFICIENT FUNCTIONAL FOR TRANSFORMS OF ANALYTIC FUNCTIONS

3.1. MOTIVATION AND PRELIMINARIES

For a univalent function in the class $\mathcal{A}$, it is well known that the n-th coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions. For example, the bound for the second coefficient of normalized univalent functions readily yields the growth and distortion bounds for univalent functions. The Fekete-Szegö coefficient functional also arise in the investigation of univalency of analytic functions. Several authors have investigated the Fekete-Szegö functional for functions in various subclasses of univalent and multivalent functions [**1, 7, 5, 2, 11, 14, 13, 15, 22, 25, 35, 36, 44, 46, 56**], and more recently by Choi, Kim, and Sugawa [**12**].

Ma and Minda [**28**] gave a unified treatment of various subclasses consisting of starlike and convex functions for which either the quantity $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. In fact, they considered the analytic function φ with positive real part in the unit disk U, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis.

The unified class $S^*(\varphi)$ introduced by Ma and Minda [**28**] consists of starlike functions and they also investigated the corresponding class $K(\varphi)$ of convex functions, for $f \in \mathcal{A}$ satisfying

$$\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad (z \in U)$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)$$

respectively. Ma and Minda [**28**] obtained subordination results, distortion, growth and rotation theorems. They also obtained estimates for the first few coefficients and determined bounds for the associated Fekete-Szegö functional. A function $f \in S^*(\varphi)$ is said to be starlike function with respect to φ, and a function $f \in K(\varphi)$ is a convex function with respect to φ.

The unified treatment of various subclasses of starlike and convex functions by Ma and Minda [**28**] motivates one to consider similar classes defined by subordination. In this chapter, we consider the following classes of functions which have been defined earlier by several authors in [**27**], [**37**], [**42**] [**43**], [**48**].

$$\begin{aligned}
R_b(\varphi) &:= \left\{ f \in \mathcal{A} \,:\, 1 + \frac{1}{b}\left(f'(z) - 1\right) \prec \varphi(z) \right\}, \\
S^*(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} \prec \varphi(z) \right\}, \\
L(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, \left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec \varphi(z) \right\}, \\
M(\alpha, \varphi) &:= \left\{ f \in \mathcal{A} \,:\, (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \right\},
\end{aligned}$$

where $b \in \mathcal{C} \setminus \{0\}$, and $\alpha \geq 0$. Some coefficient problems for functions f belonging to certain classes of p-valent functions were investigated in [**5**].

For a univalent function $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{3.1.1}$$

the k-th root transform is defined by

$$F(z) := [f(z^k)]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{3.1.2}$$

In Section 3.2, sharp bounds for the Fekete-Szegö coefficient functional $\left|b_{2k+1} - \mu b_{k+1}^2\right|$ associated with the k-th root transform of the function f belonging to the above mentioned classes are derived. In Section 3.3, a similar problem is investigated for functions G where $G(z) := z/f(z)$ and the function f belongs to the above mentioned classes.

Let Ω be the class of analytic functions w, normalized by $w(0) = 0$, and satisfying the condition $|w(z)| < 1$. The following two lemmas regarding the coefficients of functions in Ω are needed to prove our main results. Lemma 3.1.1 is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [**28**].

LEMMA 3.1.1. [**5**] *If $w \in \Omega$ and*

$$w(z) := w_1 z + w_2 z^2 + \cdots \quad (z \in U), \tag{3.1.3}$$

then

$$|w_2 - t w_1^2| \leq \begin{cases} -t & \text{if } t \leq -1 \\ 1 & \text{if } -1 \leq t \leq 1 \\ t & \text{if } t \geq 1. \end{cases}$$

When $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z\frac{\lambda+z}{1+\lambda z}(0 \leq \lambda \leq 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $w(z) = -z\frac{\lambda+z}{1+\lambda z}$ $(0 \leq \lambda \leq 1)$ or one of its rotations.

LEMMA 3.1.2. [**22**] *If $w \in \Omega$, then*

$$|w_2 - t w_1^2| \leq \max\{1;\ |t|\},$$

for any complex number t. The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

3.2. COEFFICIENT BOUNDS FOR THE k-TH ROOT TRANSFORMATION

In the first theorem below, the bound for the coefficient functional $|b_{2k+1} - \mu b_{k+1}^2|$ corresponding to the k-th root transformation of starlike functions with respect to φ is given. Notice that the classes $S^*(\alpha,\varphi)$, $L(\alpha,\varphi)$ and $M(\alpha,\varphi)$ reduce to the class $S^*(\varphi)$ for appropriate choice of the parameters. Although Theorem 3.2.1 is contained in the corresponding results for the classes $S^*(\alpha,\varphi)$, $L(\alpha,\varphi)$ and $M(\alpha,\varphi)$, it is stated and proved separately here because of its importance in its own right as well as to illustrate the main ideas.

THEOREM 3.2.1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ and*

$$\sigma_1 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right], \quad \sigma_2 := \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right].$$

If f given by (3.1.1) belongs to $S^*(\varphi)$, and F is the k-th root transformation of f given by (3.1.2), then

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \begin{cases} \dfrac{B_1^2}{2k^2}(1-2\mu) + \dfrac{B_2}{2k}, & \text{if } \mu \le \sigma_1, \\ \dfrac{B_1}{2k}, & \text{if } \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{B_1^2}{2k^2}(1-2\mu) - \dfrac{B_2}{2k}, & \text{if } \mu \ge \sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \le \frac{B_1}{2k} \max\left\{1;\ \left|\frac{B_1}{k}(1-2\mu) + \frac{B_2}{B_1}\right|\right\}.$$

PROOF. If $f \in S^*(\varphi)$, then there is an analytic function $w \in \Omega$ of the form (3.1.3) such that

$$\frac{zf'(z)}{f(z)} = \varphi(w(z)). \tag{3.2.1}$$

Since

$$\frac{zf'(z)}{f(z)} = \frac{z(1+2a_2z+3a_3z^2+\dots)}{z+a_2z^2+a_3z^3+\dots} = \frac{1+2a_2z+3a_3z^2+\dots}{1+a_2z+a_3z^2+\dots}$$
$$= (1+2a_2z+3a_3z^2+\dots)[1-(a_2z+a_3z^2+\dots)+(a_2z+a_3z^2+\dots)^2-\dots]$$
$$= 1+a_2z+(-a_2^2+2a_3)z^2+(3a_4-3a_2a_3+a_2^3)z^3+\dots$$

and

$$\varphi(w(z)) = 1+B_1(w_1z+w_2z^2+\dots)+B_2(w_1z+w_2z^2+\dots)^2+\dots$$
$$= 1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\dots,$$

it follows from (3.2.1) that

$$a_2 = B_1w_1 \tag{3.2.2}$$

and

$$a_3 = \frac{1}{2}[B_1w_2+(B_2+B_1^2)w_1^2]. \tag{3.2.3}$$

For a function f given by (3.1.1), we have

$$[f(z^k)]^{1/k} = [z^k+a_2z^{2k}+a_3z^{3k}+\dots]^{1/k}$$
$$= [z^k(1+a_2z^k+a_3z^{2k}+\dots)]^{1/k}$$
$$= z\left[1+\frac{1}{k}(a_2z^k+a_3z^{2k}+\dots)+\frac{1-k}{2k^2}(a_2z^k+a_3z^{2k}+\dots)^2+\dots\right]$$
$$= z+\frac{1}{k}a_2z^{k+1}+\left(\frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1}+\cdots. \tag{3.2.4}$$

The equations (3.1.2) and (3.2.4) yield

$$b_{k+1} = \frac{1}{k}a_2, \tag{3.2.5}$$

and

$$b_{2k+1} = \frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{3.2.6}$$

On substituting for a_2 and a_3 in (3.2.5) and (3.2.6) from (3.2.2) and (3.2.3), it follows that

$$b_{k+1} = \frac{B_1 w_1}{k}$$

and

$$b_{2k+1} = \frac{1}{2k}\left[B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{k}\right],$$

and hence

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k}\left\{w_2 - \left[-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

The first half of the result is established by an application of Lemma 3.1.1.

If $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq -1$, then

$$\mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \quad (\mu \leq \sigma_1),$$

and Lemma 3.1.1 gives

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1^2}{2k^2}(1-2\mu) + \frac{B_2}{2k}.$$

For $-1 \leq -\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \leq 1$, we have

$$\frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} - 1\right) + 1\right] \leq \mu \leq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\sigma_1 \leq \mu \leq \sigma_2),$$

and Lemma 3.1.1 yields

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k}.$$

For $-\frac{B_1}{k}(1-2\mu) - \frac{B_2}{B_1} \geq 1$, we have

$$\mu \geq \frac{1}{2}\left[\frac{k}{B_1}\left(\frac{B_2}{B_1} + 1\right) + 1\right] \quad (\mu \geq \sigma_2),$$

and it follows from Lemma 3.1.1 that

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq -\frac{B_1^2}{2k^2}(1-2\mu) - \frac{B_2}{2k}.$$

The second half of the result follows by an application of Lemma 3.1.2:

$$\begin{aligned}\left|b_{2k+1}-\mu b_{k+1}^2\right| &= \frac{B_1}{2k}\left|w_2-\left[-\frac{B_1}{k}(1-2\mu)-\frac{B_2}{B_1}\right]w_1^2\right| \\ &\leq \frac{B_1}{2k}\max\left\{1;\ \left|\frac{B_1}{k}(1-2\mu)+\frac{B_2}{B_1}\right|\right\}. \qquad \square\end{aligned}$$

REMARK 3.2.1.

(1) In view of the Alexander result **[3]** that $f \in K(\varphi)$ if and only if $zf' \in S^*(\varphi)$, the estimate for $\left|b_{2k+1}-\mu b_{k+1}^2\right|$ for a function in $K(\varphi)$ can be obtained from the corresponding estimates in Theorem 3.2.1 for functions in $S^*(\varphi)$.

(2) For $k=1$, the k-th root transformation of f reduces to the given function f itself. Thus, the estimate given in equation (3.2.1) of Theorem 3.2.1 is an extension of the corresponding result for the Fekete-Szegö coefficient functional corresponding to functions starlike with respect to φ. Similar remark applies to the other results in this section.

The well-known Noshiro-Warschawski theorem states that a function $f \in \mathcal{A}$ with positive derivative in U is univalent. The class $R_b(\varphi)$ of functions defined in terms of the subordination of the derivative f' is closely associated with the class of functions with positive real part. The bound for the Fekete-Szegö functional corresponding to the k-th root transformation of functions in $R_b(\varphi)$ is given in Theorem 3.2.2.

THEOREM 3.2.2. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *If* f *given by (3.1.1) belongs to* $R_b(\varphi)$, *and* F *is the* k*-th root transformation of* f *given by (3.1.2), then*

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{|b|B_1}{3k}\max\left\{1; \left|\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right|\right\}.$$

PROOF. If $f \in R_b(\varphi)$, then there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ such that

$$1+\frac{1}{b}\left(f'(z)-1\right) = \varphi(w(z)). \tag{3.2.7}$$

Since

$$1+\frac{1}{b}\left(f'(z)-1\right)=1+\frac{1}{b}\left[2a_2z+3a_3z^2+\dots\right]$$
$$=1+\frac{2}{b}a_2z+\frac{3}{b}a_3z^2+\dots$$

and

$$\varphi(w(z))=1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\dots,$$

it follows from (3.2.7) that

$$a_2=\frac{bB_1w_1}{2} \tag{3.2.8}$$

and

$$a_3=\frac{b}{3}(B_1w_2+B_2w_1^2). \tag{3.2.9}$$

By using (3.2.8) and (3.2.9) in (3.2.5) and (3.2.6), it follows that

$$b_{k+1}=\frac{bB_1w_1}{2k}$$

and

$$b_{2k+1}=\frac{bB_1w_2}{3k}+\frac{bB_2w_1^2}{3k}-\frac{b^2B_1^2w_1^2}{8k}+\frac{b^2B_1^2w_1^2}{8k^2},$$

and hence

$$b_{2k+1}-\mu b_{k+1}^2=\frac{bB_1}{3k}\left\{w_2-\left[\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right]w_1^2\right\}. \tag{3.2.10}$$

Applying Lemma 3.1.2 yields

$$\left|b_{2k+1}-\mu b_{k+1}^2\right|=\frac{|b|B_1}{3k}\left|w_2-\left[\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right]w_1^2\right|$$
$$\leq\frac{|b|B_1}{3k}\max\left\{1;\left|\frac{3bB_1}{4}\left(\frac{1}{2}-\frac{1}{2k}+\frac{\mu}{k}\right)-\frac{B_2}{B_1}\right|\right\}. \qquad \square$$

Remark 3.2.2. When $k=1$ and

$$\varphi(z)=\frac{1+Az}{1+Bz}\quad(-1\leq B\leq A\leq 1),$$

Theorem 3.2.2 reduces to a result in [**15**, Theorem 4, p. 894].

We consider the special case of functions $f \in \mathcal{A}$ having positive derivatives, in particular, functions f satisfying the subordination $f'(z) \prec \varphi(z)$. The class of such functions is the special case of the class $R_b(\varphi)$ when $b = 1$. In fact, when $b = 1$, equation (3.2.10) becomes

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{3k}\left\{ w_2 - \left[\frac{3B_1}{4}\left(\frac{1}{2} - \frac{1}{2k} + \frac{\mu}{k}\right) - \frac{B_2}{B_1}\right] w_1^2 \right\}.$$

Lemma 3.1.1 now yields the following.

Corollary 3.2.1. *If $f \in \mathcal{A}$ satisfies $f'(z) \prec \varphi(z)$, then*

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} \dfrac{-B_1^2}{4k}\left(\dfrac{1}{2} - \dfrac{1}{2k} + \dfrac{\mu}{k}\right) + \dfrac{B_2}{3k}, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{3k}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1^2}{4k}\left(\dfrac{1}{2} - \dfrac{1}{2k} + \dfrac{\mu}{k}\right) - \dfrac{B_2}{3k}, & \text{if} \quad \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 := \frac{4kB_2}{3B_1^2} - \frac{4k}{3B_1} - \frac{k}{2} + \frac{1}{2} \quad \text{and} \quad \sigma_2 := \frac{4kB_2}{3B_1^2} + \frac{4k}{3B_1} - \frac{k}{2} + \frac{1}{2}.$$

The following result gives the bounds for the Fekete-Szegö coefficient functional corresponding to the k-th root transformation of functions in the class $S^*(\alpha, \varphi)$.

Theorem 3.2.3. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. Further define σ_1, σ_2 and υ by*

$$\sigma_1 := \frac{k(1+2\alpha)^2}{2B_1(1+3\alpha)}\left[\frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1} - \frac{B_1(k-1)(1+3\alpha)}{(1+2\alpha)^2} - 1\right],$$

$$\sigma_2 := \frac{k(1+2\alpha)^2}{2B_1(1+3\alpha)}\left[\frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1} - \frac{B_1(k-1)(1+3\alpha)}{(1+2\alpha)^2} + 1\right],$$

$$\upsilon := \frac{B_1}{(1+2\alpha)}\left[\frac{(k-1)(1+3\alpha)}{k(1+2\alpha)} + \frac{2\mu(1+3\alpha)}{k(1+2\alpha)} - 1\right] - \frac{B_2}{B_1}.$$

If f given by (3.1.1) belongs to $S^*(\alpha,\varphi)$, and F is k-th root transformation of f given by (3.1.2), then

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} -\dfrac{B_1}{2k(1+3\alpha)}\upsilon, & \textit{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k(1+3\alpha)}, & \textit{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1}{2k(1+3\alpha)}\upsilon, & \textit{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+3\alpha)} \max\{1; |\upsilon|\}.$$

Proof. If $f \in S^*(\alpha,\varphi)$, then there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ such that

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = \varphi(w(z)).$$

Since

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z(1+2a_2z+3a_3z^2+\ldots)}{z+a_2z^2+a_3z^3+\ldots} = \frac{1+2a_2z+3a_3z^2+\ldots}{1+a_2z+a_3z^2+\ldots} \\ &= (1+2a_2z+3a_3z^2+\ldots)[1-(a_2z+a_3z^2+\ldots)+(a_2z+a_3z^2+\ldots)^2-\ldots],\end{aligned}$$

we get

$$\frac{zf'(z)}{f(z)} = 1 + a_2z + (-a_2^2+2a_3)z^2 + (3a_4 - 3a_2a_3 + a_2^3)z^3 + \ldots \tag{3.2.11}$$

and since

$$\begin{aligned}\frac{\alpha z^2 f''(z)}{f(z)} &= \frac{\alpha z^2[2a_2+6a_3z+\ldots]}{z+a_2z^2+a_3z^3+\ldots} \\ &= [2a_2\alpha+6a_3\alpha z+\ldots][1-(a_2z+a_3z^2+\ldots)+(a_2z+a_3z^2+\ldots)^2-\ldots],\end{aligned}$$

we get

$$\frac{\alpha z^2 f''(z)}{f(z)} = 2a_2\alpha z - (2a_2^2\alpha - 6a_3\alpha)z^2. \tag{3.2.12}$$

Then equations (3.2.11) and (3.2.12) yield

$$\frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f(z)} = 1 + a_2(1+2\alpha)z + [2(1+3\alpha)a_3 - (1+2\alpha)a_2^2]z^2 + \cdots. \tag{3.2.13}$$

Since

$$\varphi(w(z)) = 1 + B_1 w_1 z + (B_1 w_2 + B_2 w_1^2) z^2 + \ldots,$$

equation (3.2.13) gives

$$a_2 = \frac{B_1 w_1}{(1+2\alpha)} \tag{3.2.14}$$

and

$$a_3 = \frac{1}{2(1+3\alpha)} \left[B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{(1+2\alpha)} \right]. \tag{3.2.15}$$

Using (3.2.14) and (3.2.15) in (3.2.5) and (3.2.6), we get

$$b_{k+1} = \frac{1}{k} \frac{B_1 w_1}{(1+2\alpha)},$$

and

$$b_{2k+1} = \frac{1}{2k(1+3\alpha)} \left(B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{(1+2\alpha)} \right) - \frac{B_1^2 w_1^2}{2k^2(1+2\alpha)^2}(k-1),$$

and hence

$$\begin{aligned} b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k(1+3\alpha)} \Bigg\{ w_2 - \Bigg[\frac{B_1}{(1+2\alpha)} \Bigg(\frac{(k-1)(1+3\alpha)}{k(1+2\alpha)} \\ + \frac{2\mu(1+3\alpha)}{k(1+2\alpha)} - 1 \Bigg) - \frac{B_2}{B_1} \Bigg] w_1^2 \Bigg\}. \end{aligned}$$

The first part of the result is established by applying Lemma 3.1.1.

If $\upsilon \leq -1$, then $\mu \leq \sigma_1$ and hence Lemma 3.1.1 yields

$$\left| b_{2k+1} - \mu b_{k+1}^2 \right| \leq -\frac{B_1}{2k(1+3\alpha)} \upsilon.$$

For $-1 \leq \upsilon \leq 1$, then $\sigma_1 \leq \mu \leq \sigma_2$ and hence Lemma 3.1.1 yields

$$\left| b_{2k+1} - \mu b_{k+1}^2 \right| \leq \frac{B_1}{2k(1+3\alpha)}.$$

For $\upsilon \geq 1$, then $\mu \geq \sigma_2$ and hence Lemma 3.1.1 yields

$$\left| b_{2k+1} - \mu b_{k+1}^2 \right| \leq \frac{B_1}{2k(1+3\alpha)} \upsilon.$$

The second result follows by an application of Lemma 3.1.2. $\square$

Observe that Theorem 3.2.3 reduces to Theorem 3.2.1 when $\alpha = 0$.

THEOREM 3.2.4. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$ *and* $\beta = (1-\alpha)$. *Further let*

$$\sigma_1 := \frac{k}{2(\alpha+3\beta)}\left[\frac{(\alpha+2\beta)^2}{B_1}\left(\frac{B_2}{B_1}-1\right) - \frac{(\alpha+2\beta)^2 - 3(\alpha+4\beta)}{2} - \frac{(k-1)(\alpha+3\beta)}{k}\right],$$

$$\sigma_2 := \frac{k}{2(\alpha+3\beta)}\left[\frac{(\alpha+2\beta)^2}{B_1}\left(\frac{B_2}{B_1}+1\right) - \frac{(\alpha+2\beta)^2 - 3(\alpha+4\beta)}{2} - \frac{(k-1)(\alpha+3\beta)}{k}\right],$$

$$\upsilon := \frac{B_1}{(\alpha+2\beta)^2}\left[\frac{(\alpha+2\beta)^2 - 3(\alpha+4\beta)}{2} + \frac{1}{k}(k-1)(\alpha+3\beta) + \frac{2\mu}{k}(\alpha+3\beta)\right] - \frac{B_2}{B_1}.$$

If f given by (3.1.1) belongs to $L(\alpha,\varphi)$, and F is k-th root transformation of f given by (3.1.2), then

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \begin{cases} -\dfrac{B_1}{2k(\alpha+3\beta)}\upsilon, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{2k(\alpha+3\beta)}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \dfrac{B_1}{2k(\alpha+3\beta)}\upsilon, & \text{if} \quad \mu \geq \sigma_2, \end{cases} \tag{3.2.16}$$

and for μ complex,

$$\left|b_{2k+1} - \mu b_{k+1}^2\right| \leq \frac{B_1}{2k(\alpha+3\beta)} \max\{1; |\upsilon|\}.$$

PROOF. If $f \in L(\alpha,\varphi)$, then there is an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ such that

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{\beta} = \varphi(w(z)). \tag{3.2.17}$$

We have

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_3 - a_2^2)z^2 + (3a_4 + a_2^3 - 3a_3 a_2)z^3 + \cdots$$

and therefore

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} = 1 + \alpha a_2 z + \left(2\alpha a_3 + \frac{\alpha^2 - 3\alpha}{2}a_2^2\right)z^2 + \cdots. \tag{3.2.18}$$

Similarly

$$\begin{aligned}1+\frac{zf''(z)}{f'(z)} &= 1+\frac{z(2a_2+6a_3z+\ldots)}{(1+2a_2z+3a_3z^2+\ldots)}\\ &= 1+[2a_2+6a_3z+\ldots][1-(2a_2z+3a_3z^2+\ldots)+(2a_2z+3a_3z^2+\ldots)^2-\ldots]\\ &= 1+2a_2z+(6a_3-4a_2^2)z^2+\cdots\end{aligned}$$

and therefore

$$\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = 1+2\beta a_2z+\left(6\beta a_3+2(\beta^2-3\beta)a_2^2\right)z^2+\cdots. \tag{3.2.19}$$

Thus, from (3.2.18) and (3.2.19),

$$\begin{aligned}\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta} = 1+(\alpha+2\beta)a_2z+\Big[&2(\alpha+3\beta)a_3\\ &+\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}a_2^2\Big]z^2+\cdots.\end{aligned}$$

Since

$$\varphi(w(z)) = 1+B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\ldots,$$

it follows from (3.2.17) that

$$a_2=\frac{B_1w_1}{(\alpha+2\beta)} \tag{3.2.20}$$

and

$$a_3=\frac{B_1w_2+B_2w_1^2}{2(\alpha+3\beta)}-\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]B_1^2w_1^2}{4(\alpha+2\beta)^2(\alpha+3\beta)}. \tag{3.2.21}$$

Using (3.2.20) and (3.2.21) in (3.2.5) and (3.2.6), we get

$$b_{k+1}=\frac{1}{k}\frac{B_1w_1}{k(\alpha+2\beta)},$$

and

$$b_{2k+1}=\frac{B_1w_2}{2k(\alpha+3\beta)}+\frac{B_2w_1^2}{2k(\alpha+3\beta)}-\frac{[(\alpha+2\beta)^2-3(\alpha+4\beta)]B_1^2w_1^2}{4k(\alpha+2\beta)^2(\alpha+3\beta)}-\frac{B_1^2w_1^2(k-1)}{2k^2(\alpha+2\beta)^2},$$

and hence

$$b_{2k+1}-\mu b_{k+1}^2=\frac{B_1}{2k(\alpha+3\beta)}\left\{w_2-\sigma w_1^2\right\},$$

where

$$\sigma := \frac{B_1}{(\alpha+2\beta)^2}\left[\frac{(\alpha+2\beta)^2-3(\alpha+4\beta)}{2}+\frac{1}{k}(k-1)(\alpha+3\beta)+\frac{2\mu}{k}(\alpha+3\beta)\right]-\frac{B_2}{B_1}.$$

The results now follow by using Lemma 3.1.1 and Lemma 3.1.2. $\square$

REMARK 3.2.3. We note that if $k=1$, inequality (3.2.16) is the result established in [**44**, Theorem 2.1, p.3].

For the class $M(\alpha,\varphi)$, we now get the following coefficient bounds:

THEOREM 3.2.5. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. *Further let*

$$\sigma_1 := \frac{k}{2(1+2\alpha)}\left[\frac{(1+\alpha)^2}{B_1}\left(\frac{B_2}{B_1}-1\right)+(1+3\alpha)-(k-1)(1+2\alpha)\right],$$

$$\sigma_2 := \frac{k}{2(1+2\alpha)}\left[\frac{(1+\alpha)^2}{B_1}\left(\frac{B_2}{B_1}+1\right)+(1+3\alpha)-(k-1)(1+2\alpha)\right],$$

$$\upsilon := \frac{B_1}{(1+\alpha)^2}\left[(k-1)(1+2\alpha)+\frac{2\mu}{k}(1+2\alpha)-(1+3\alpha)\right]-\frac{B_2}{B_1}.$$

If f given by (3.1.1) belongs to $M(\alpha,\varphi)$, and F is k-th root transformation of f given by (3.1.2), then

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \begin{cases} -\dfrac{B_1}{2k(1+2\alpha)}\upsilon, & \text{if} \quad \mu\leq\sigma_1, \\ \dfrac{B_1}{2k(1+2\alpha)}, & \text{if} \quad \sigma_1\leq\mu\leq\sigma_2, \\ \dfrac{B_1}{2k(1+2\alpha)}\upsilon, & \text{if} \quad \mu\geq\sigma_2, \end{cases}$$

and for μ complex,

$$\left|b_{2k+1}-\mu b_{k+1}^2\right| \leq \frac{B_1}{2k(1+2\alpha)}\max\{1;|\upsilon|\}.$$

PROOF. If $f\in M(\alpha,\varphi)$, then there is an analytic function $w(z)=w_1z+w_2z^2+\cdots\in\Omega$ such that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)=\varphi(w(z)). \tag{3.2.22}$$

Since

$$(3.2.23) \qquad (1-\alpha)\frac{zf'(z)}{f(z)} = (1-\alpha) + a_2(1-\alpha)z + (1-\alpha)(-a_2^2 + 2a_3)z^2 + \cdots$$

and

$$(3.2.24) \qquad \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \alpha + 2a_2\alpha z + 2\alpha(3a_3 - 2a_2^2)z^2,$$

from equations (3.2.23) and (3.2.24), it follows that

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1+(1+\alpha)a_2z+[-(1+3\alpha)a_2^2+2(1+2\alpha)a_3]z^2+\dots.$$

Since

$$\varphi(w(z)) = 1 + B_1w_1z + (B_1w_2 + B_2w_1^2)z^2 + \dots,$$

it follows from equation (3.2.22) that

$$(3.2.25) \qquad a_2 = \frac{B_1w_1}{(1+\alpha)}$$

and

$$(3.2.26) \qquad a_3 = \frac{1}{2(1+2\alpha)}\left[B_1w_2 + B_2w_1^2 + \frac{(1+3\alpha)B_1^2w_1^2}{(1+\alpha)^2}\right].$$

By using, (3.2.25) and (3.2.26) in (3.2.5) and (3.2.6), it follows that

$$b_{k+1} = \frac{B_1w_1}{k(1+\alpha)},$$

and

$$b_{2k+1} = \frac{1}{2k(1+2\alpha)}\left[B_1w_2 + B_2w_1^2 + \frac{(1+3\alpha)B_1^2w_1^2}{(1+\alpha)^2}\right] - \frac{B_1^2w_1^2(k-1)}{2k^2(1+\alpha)^2},$$

and hence

$$b_{2k+1} - \mu b_{k+1}^2 = \frac{B_1}{2k(1+\alpha)}\{w_2 - \sigma w_1^2\}$$

where

$$\sigma := \frac{B_1}{(1+\alpha)^2}\left[(k-1)(1+2\alpha) + \frac{2\mu}{k}(1+2\alpha) - (1+3\alpha)\right] - \frac{B_2}{B_1}.$$

The results follow from Lemma 3.1.1 and Lemma 3.1.2. $\square$

REMARK 3.2.4. When $k = 1$ and $\alpha = 1$, Theorem 3.2.5 reduces to a result in [**28**, Theorem 3, p. 164].

3.3. THE FEKETE-SZEGÖ FUNCTIONAL ASSOCIATED WITH $z/f(z)$

In this section, bounds for the Fekete-Szegö coefficient functional associated with the function G defined by

$$G(z) = \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} d_n z^n, \tag{3.3.1}$$

where f belongs to one of the classes $S^*(\varphi)$, $R_b(\varphi)$, $S^*(\alpha, \varphi)$, $L(\alpha, \varphi)$ and $M(\alpha, \varphi)$ are investigated. Proofs of the results obtained here are similar to those proofs in Section 3.2, and hence detailed proofs are omitted.

The following result is for functions belonging to the class $S^*(\varphi)$.

THEOREM 3.3.1. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$, *and*

$$\sigma_1 := \frac{1}{2} - \frac{2}{B_1^3} - \frac{B_2}{2B_1^2}, \quad \sigma_2 := \frac{1}{2} + \frac{2}{B_1^3} - \frac{B_2}{2B_1^2}.$$

If f given by (3.1.1) belongs to $S^*(\varphi)$, and G is a function given by (3.3.1), then

$$\left|d_2 - \mu d_1^2\right| \leq \begin{cases} -\frac{1}{4}B_1 B_2 - \frac{1}{4}B_1^3(2\mu - 1), & \text{if} \quad \mu \leq \sigma_1, \\ \frac{1}{2}B_1, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ \frac{1}{4}B_1 B_2 + \frac{1}{4}B_1^3(2\mu - 1), & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|d_2 - \mu d_1^2\right| \leq \frac{1}{2}B_1 \max\left\{1; \ \left|(1-\mu)B_1^2 - \frac{1}{2}(B_2 + B_1^2)\right|\right\}.$$

PROOF. Observe that

$$\frac{z}{f(z)} = \frac{z}{z + a_2 z^2 + a_3 z^3}$$
$$= 1 - (a_2 z + a_3 z^2 + \ldots) + (a_2 z + a_3 z^2 + \ldots)^2 - \ldots$$
$$= 1 - a_2 z + (a_2^2 - a_3) z^2 + \cdots . \tag{3.3.2}$$

Comparing coefficients of (3.3.1), and (3.3.2) yields

$$d_1 = -a_2 \tag{3.3.3}$$

and

$$d_2 = a_2^2 - a_3. \tag{3.3.4}$$

By using (3.2.2) and (3.2.3) in (3.3.3) and (3.3.4), it follows that

$$d_1 = -B_1 w_1$$

and

$$d_2 = B_1^2 w_1^2 - \frac{1}{2}[B_1 w_2 + (B_2 + B_1^2) w_1^2],$$

and hence

$$d_2 - \mu d_1^2 = -\frac{1}{2} B_1 \left\{ w_2 - \left[(1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right] w_1^2 \right\}.$$

The result is established by an application of Lemma 3.1.1. The second result follows from Lemma 3.1.2:

$$\left| d_2 - \mu d_1^2 \right| = \frac{1}{2} B_1 \left| w_2 - \left[(1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right] w_1^2 \right|$$
$$\leq \frac{1}{2} B_1 \max \left\{ 1; \ \left| (1-\mu) B_1^2 - \frac{1}{2}(B_2 + B_1^2) \right| \right\}. \qquad \square$$

For the class $R_b(\varphi)$, the following coefficient bound is obtained:

Theorem 3.3.2. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. If f given by (3.1.1) belongs to $R_b(\varphi)$, and G is a function given by (3.3.1), then*

$$\left|d_2 - \mu d_1^2\right| \leq \frac{|b|B_1}{3} \max\left\{1; \left|\frac{3}{4}(1-\mu)bB_1 - \frac{B_2}{B_1}\right|\right\}.$$

Proof. Using (3.2.8) and (3.2.9) in (3.3.3) and (3.3.4), it follows that

$$d_1 = -\frac{bB_1 w_1}{2}$$

and

$$d_2 = \frac{1}{4} b^2 B_1^2 w_1^2 - \frac{b}{3}(B_1 w_2 + B_2 w_1^2),$$

and hence

$$d_2 - \mu d_1^2 = -\frac{bB_1}{3}\left\{w_2 - \left[\frac{3}{4}(1-\mu)bB_1 - \frac{B_2}{B_1}\right] w_1^2\right\}.$$

Lemma 3.1.2 gives

$$\begin{aligned} \left|d_2 - \mu d_1^2\right| &= \frac{|b|B_1}{3}\left|w_2 - \left[\frac{3}{4}(1-\mu)bB_1 - \frac{B_2}{B_1}\right] w_1^2\right| \\ &\leq \frac{|b|B_1}{3k} \max\left\{1; \left|\frac{3}{4}(1-\mu)bB_1 - \frac{B_2}{B_1}\right|\right\}. \end{aligned}$$ □

For functions with positive derivative, the above theorem becomes

Corollary 3.3.1. *If $f \in \mathcal{A}$ satisfies $f'(z) \prec \varphi(z)$, then*

$$\left|d_2 - \mu d_1^2\right| \leq \begin{cases} \frac{1}{4}(1-\mu)B_1^2 - \frac{B_2}{3}, & \textit{if} \quad \mu \leq \sigma_1, \\ \frac{B_1}{3}, & \textit{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{4}(1-\mu)B_1^2 + \frac{B_2}{3}, & \textit{if} \quad \mu \geq \sigma_2 \end{cases}$$

where

$$\sigma_1 := 1 - \frac{4}{3B_1} - \frac{4B_2}{3B_1^2} \quad \textit{and} \quad \sigma_2 := 1 + \frac{4}{3B_1} - \frac{4B_2}{3B_1^2}.$$

The following result gives the coefficient bounds for the class $S^*(\alpha, \varphi)$.

THEOREM 3.3.3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots$. *Further define* σ_1, σ_2, υ *and* γ *by*

$$\sigma_1 := (1+2\alpha)\left[1 + \frac{(1+2\alpha)B_2}{2(1+3\alpha)B_1^2} + \frac{1}{2(1+3\alpha)} - \frac{(1+2\alpha)}{2(1+3\alpha)B_1}\right]$$

$$\sigma_2 := (1+2\alpha)\left[1 + \frac{(1+2\alpha)B_2}{2(1+3\alpha)B_1^2} + \frac{1}{2(1+3\alpha)} + \frac{(1+2\alpha)}{2(1+3\alpha)B_1}\right],$$

$$\upsilon := \frac{2B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2} + \frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1}, \quad \gamma := -\frac{B_1}{2(1+3\alpha)}.$$

If f *given by (3.1.1) belongs to* $S^*(\alpha,\varphi)$, *and* G *is a function given by (3.3.1), then*

$$\left|d_2 - \mu d_1^2\right| \le \begin{cases} \dfrac{B_1}{2(1+3\alpha)}\upsilon, & \text{if} \quad \mu \le \sigma_1, \\ \dfrac{B_1}{2(1+3\alpha)}, & \text{if} \quad \sigma_1 \le \mu \le \sigma_2, \\ -\dfrac{B_1}{2(1+3\alpha)}\upsilon, & \text{if} \quad \mu \ge \sigma_2, \end{cases}$$

and for μ *complex,*

$$\left|d_2 - \mu {d_1}^2\right| \le \frac{B_1}{2(1+3\alpha)} \max\{1; |\upsilon|\}.$$

PROOF. By using the relations (3.2.14) and (3.2.15) in (3.3.3) and (3.3.4), it follows that

$$d_1 = -\frac{B_1 w_1}{(1+2\alpha)},$$

and

$$d_2 = \frac{B_1^2 w_1^2}{(1+2\alpha)^2} - \frac{1}{2(1+3\alpha)}\left(B_1 w_2 + B_2 w_1^2 + \frac{B_1^2 w_1^2}{(1+2\alpha)}\right),$$

and hence

$$d_2 - \mu d_1^2 = -\frac{B_1}{2(1+3\alpha)}\left\{w_2 - \left[\frac{2B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2} + \frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1}\right] w_1^2\right\}.$$

The result is established by an application of Lemma 3.1.1. The second result follows by an application of Lemma 3.1.2:

$$\begin{aligned} \left|d_2 - \mu d_1^2\right| &= \frac{B_1}{2(1+3\alpha)}\left|w_2 - \left[\frac{2B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{2\mu(1+3\alpha)B_1}{(1+2\alpha)^2} + \frac{B_1}{(1+2\alpha)} + \frac{B_2}{B_1}\right] w_1^2\right| \\ &\le \frac{B_1}{2(1+3\alpha)} \max\{1; |\upsilon|\}. \end{aligned}$$

□

For the class $L(\alpha,\varphi)$, we now get the following coefficient bounds:

THEOREM 3.3.4. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \cdots$ and $\beta = (1-\alpha)$. Further let*

$$\sigma_1 := 1 - \frac{(\alpha+2\beta)^2B_2}{2(\alpha+3\beta)B_1^2} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)](\alpha+2\beta)}{4(\alpha+3\beta)} - \frac{1}{B_1},$$

$$\sigma_2 := 1 - \frac{(\alpha+2\beta)^2B_2}{2(\alpha+3\beta)B_1^2} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)](\alpha+2\beta)}{4(\alpha+3\beta)} + \frac{1}{B_1},$$

$$\upsilon := B_1 - \frac{(\alpha+2\beta)^2B_2}{2(\alpha+3\beta)B_1} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1(\alpha+2\beta)}{4(\alpha+3\beta)} - \mu B_1.$$

If f given by (3.1.1) belongs to $L(\alpha,\varphi)$, and G is a function given by (3.3.1), then

$$\left|d_2 - \mu d_1^2\right| \leq \begin{cases} \dfrac{B_1\upsilon}{(\alpha+2\beta)^2}, & \text{if} \quad \mu \leq \sigma_1, \\ \dfrac{B_1}{(\alpha+2\beta)^2}, & \text{if} \quad \sigma_1 \leq \mu \leq \sigma_2, \\ -\dfrac{B_1\upsilon}{(\alpha+2\beta)^2}, & \text{if} \quad \mu \geq \sigma_2, \end{cases}$$

and for μ complex,

$$\left|d_2 - \mu {d_1}^2\right| \leq \frac{B_1}{(\alpha+2\beta)^2} \max\left\{1; |\upsilon|\right\}.$$

PROOF. Using (3.2.20) and (3.2.21) in (3.3.3) and (3.3.4), it follows that

$$d_1 = -\frac{B_1w_1}{(\alpha+2\beta)},$$

and

$$d_2 = \frac{B_1^2w_1^2}{(\alpha+2\beta)^2} - \frac{B_2w_1 + B_2w_1^2}{2(\alpha+3\beta)} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1^2w_1^2}{4(\alpha+2\beta)^2(\alpha+3\beta)}.$$

Hence

$$d_2 - \mu d_1^2 = -\frac{B_1}{(\alpha+2\beta)^2}\left\{w_2 - \sigma w_1^2\right\},$$

where

$$\sigma := B_1 - \frac{(\alpha+2\beta)^2B_2}{2(\alpha+3\beta)B_1} + \frac{[(\alpha+2\beta)^2 - 3(\alpha+4\beta)]B_1(\alpha+2\beta)}{4(\alpha+3\beta)} - \mu B_1.$$

The result now follows from Lemma 3.1.1. The second result follows by an application of Lemma 3.1.2:

$$\begin{aligned}\left|d_2-\mu d_1^2\right| &= \frac{B_1}{(\alpha+2\beta)^2}\left|w_2-\sigma w_1^2\right| \\ &\le \frac{B_1}{(\alpha+2\beta)^2}\max\{1;|\sigma|\}. \qquad \square\end{aligned}$$

Finally for the class $M(\alpha,\varphi)$, the following coefficient bounds are obtained:

THEOREM 3.3.5. *Let* $\varphi(z)=1+B_1z+B_2z^2+B_3z^3+\cdots$. *Further let*

$$\sigma_1 := 1-\frac{B_2(1+\alpha)^2}{2B_1^2(1+2\alpha)}-(1+3\alpha)-\frac{(1+\alpha)^2}{2B_1(1+2\alpha)},$$

$$\sigma_2 := 1-\frac{B_2(1+\alpha)^2}{2B_1^2(1+2\alpha)}-(1+3\alpha)+\frac{(1+\alpha)^2}{2B_1(1+2\alpha)},$$

$$\upsilon := \frac{2(1+2\alpha)B_1}{(1+\alpha)^2}-\frac{B_2}{B_1}-\frac{2(1+2\alpha)(1+3\alpha)B_1}{(1+\alpha)^2}-\frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}.$$

If f given by (3.1.1) belongs to $M(\alpha,\varphi)$, and G is a function given by (3.3.1), then

$$\left|d_2-\mu d_1^2\right| \le \begin{cases} \dfrac{B_1}{2(1+2\alpha)}\upsilon, & \text{if} \quad \mu\le\sigma_1, \\ \dfrac{B_1}{2(1+2\alpha)}, & \text{if} \quad \sigma_1\le\mu\le\sigma_2, \\ -\dfrac{B_1}{2(1+2\alpha)}\upsilon, & \text{if} \quad \mu\ge\sigma_2, \end{cases}$$

and for μ complex,

$$\left|d_2-\mu {d_1}^2\right| \le \frac{B_1}{2(1+2\alpha)}\max\{1;|\upsilon|\}.$$

PROOF. Putting (3.2.25) and (3.2.26) in (3.3.3) and (3.3.4), yield

$$d_1 = -\frac{B_1w_1}{(1+\alpha)},$$

and

$$d_2 = \frac{B_1^2w_1^2}{(1+\alpha)^2}-\frac{1}{2(1+2\alpha)}\left[B_1w_2+B_2w_1^2+\frac{(1+3\alpha)B_1^2w_1^2}{(1+\alpha)^2}\right],$$

and hence

$$d_2-\mu d_1^2 = -\frac{B_1}{2(1+2\alpha)^2}\left\{w_2-\sigma w_1^2\right\},$$

where

$$\sigma := \frac{2(1+2\alpha)B_1}{(1+\alpha)^2} - \frac{B_2}{B_1} - \frac{2(1+2\alpha)(1+3\alpha)B_1}{(1+\alpha)^2} - \frac{2\mu(1+2\alpha)B_1}{(1+\alpha)^2}.$$

The results follows from Lemma 3.1.1. The second result follows by an application of Lemma 3.1.2:

$$\begin{aligned} \left|d_2 - \mu d_1^2\right| &= \frac{B_1}{2(1+2\alpha)^2}\left|w_2 - \sigma w_1^2\right| \\ &\leq \frac{B_1}{2(1+2\alpha)} \max\{1; |\sigma|\}. \end{aligned}$$ $\square$

CHAPTER 4

COEFFICIENT BOUNDS FOR MEROMORPHIC STARLIKE AND CONVEX FUNCTIONS

4.1. INTRODUCTION AND PRELIMINARIES

Let $U(\mathfrak{p})$ denote the class of univalent meromorphic functions $f(z)$ in the unit disk U with a simple pole at $z = \mathfrak{p}$, $\mathfrak{p} > 0$ and with the normalization $f(0) = 0$ and $f'(0) = 1$. Let $U^*(\mathfrak{p}, w_0)$ be the subclass of $U(\mathfrak{p})$ such that $f(z) \in U^*(\mathfrak{p}, w_0)$ if and only if there is a ρ, $0 < \rho < 1$, with the property that $\Re\frac{zf'(z)}{f(z)-w_0} < 0$ for $\rho < |z| < 1$. The functions in $U^*(\mathfrak{p}, w_0)$ map $|z| < r < \rho$ (for some ρ, $\mathfrak{p} < \rho < 1$) onto the complement of a set which is starlike with respect to w_0. Further the functions in $U^*(\mathfrak{p}, w_0)$ all omit the value w_0. This class of starlike meromorphic functions is developed from Robertson's concept of star center points [**49**]. Let $\mathcal{P}$ denote the class of functions $P(z)$ which are meromorphic in U and satisfy $P(0) = 1$ and $\Re\{P(z)\} \geq 0$ for all $z \in U$.

We consider the function

$$A(z) \equiv Q_{ST_{(w_0)}} + S(z) \equiv z\frac{f'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}.$$

By taking $f(z) = a_{-1}/(z - \mathfrak{p}) + \sum_{n=0}^{\infty} a_n(z - \mathfrak{p})^n$, we have

$$\begin{aligned} Q_{ST_{(w_0)}} &= z\frac{f'(z)}{f(z) - w_0} \\ &= \frac{-a_{-1}z(z-\mathfrak{p})^{-2} + z\sum_{n=0}^{\infty} na_n(z-\mathfrak{p})^{n-1}}{a_{-1}(z-\mathfrak{p})^{-1} + \sum_{n=0}^{\infty} a_n(z-\mathfrak{p})^n - w_0} \\ &\equiv \frac{-\mathfrak{p}}{(z-\mathfrak{p})} + \sum_{n=0}^{\infty} B_n(z-\mathfrak{p})^n \end{aligned}$$

for some coefficient B_n, $(n = 0, 1, \ldots)$.

On the other hand,

$$\begin{aligned} S(z) &= \frac{\mathfrak{p}}{z-\mathfrak{p}} - \frac{\mathfrak{p}z}{1-\mathfrak{p}z} \\ &= \frac{\mathfrak{p}}{z-\mathfrak{p}} - \frac{\mathfrak{p}(z-\mathfrak{p})+\mathfrak{p}^2}{-\mathfrak{p}(z-\mathfrak{p})+(1-\mathfrak{p}^2)} \\ &\equiv \frac{\mathfrak{p}}{z-\mathfrak{p}} - \sum_{n=0}^{\infty} C_n(z-\mathfrak{p})^n \end{aligned}$$

for some coefficient C_n, $(n = 0, 1, \ldots)$.

The function A is analytic in U because both $Q_{ST_{(w_0)}}$ and $S(z)$ have only one simple pole at $z = \mathfrak{p}$ and the residues are $-\mathfrak{p}$ and $\mathfrak{p}$ respectively. Since $\Re S(e^{i\theta}) = 0$, the function S maps the boundary of the unit disk to the imaginary axis. So the interior of the unit disk has to be mapped either to the left half plane or the right half plane. Since $A(0) = -1$, we conclude that $\Re A(z) < 0$, which shows that $-A(z)$ is in $P(z)$.

For $f(z) \in U^*(\mathfrak{p}, w_0)$, there is a function $P(z) \in \mathcal{P}$ such that

$$z\frac{f'(z)}{f(z)-w_0} + \frac{\mathfrak{p}}{z-\mathfrak{p}} - \frac{\mathfrak{p}z}{1-\mathfrak{p}z} = -P(z) \tag{4.1.1}$$

for all $z \in U$. Let $\sum^*(\mathfrak{p}, w_0)$ denote the class of functions $f(z)$ which satisfy (4.1.1) and the condition $f(0) = 0, f'(0) = 1$. Then $U^*(\mathfrak{p}, w_0)$ is a subset of $\sum^*(\mathfrak{p}, w_0)$. Miller **[29]** proved that $U^*(\mathfrak{p}, w_0) = \sum^*(\mathfrak{p}, w_0)$ for $\mathfrak{p} < 2 - \sqrt{3}$.

Let $K(\mathfrak{p})$ denote the class of functions which belong to $U(\mathfrak{p})$ and map $|z| < r < \rho$ (for some $\mathfrak{p} < \rho < 1$) onto the complement of a convex set. If $f \in K(\mathfrak{p})$, then there is a $\mathfrak{p} < \rho < 1$, such that for each z, $\rho < |z| < 1$

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \leq 0.$$

If $f \in K(\mathfrak{p})$, then for each z in U,

$$\Re\left\{1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z-\mathfrak{p}} - \frac{2\mathfrak{p}z}{1-\mathfrak{p}z}\right\} \leq 0. \tag{4.1.2}$$

Let $\sum(\mathfrak{p})$ denote the class of functions f which satisfy (4.1.2) and the conditions $f(0) = 0$ and $f'(0) = 1$. The class $K(\mathfrak{p})$ is contained in $\sum(\mathfrak{p})$. Royster [**52**] showed that for $0 < \mathfrak{p} < 2 - \sqrt{3}$, if $f \in \sum(\mathfrak{p})$ and is meromorphic, then $f(z) \in K(\mathfrak{p})$. Also, for each function $f \in \sum(\mathfrak{p})$, there is a function $P \in \mathcal{P}$ such that

$$1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z} = -P(z).$$

The class $U(\mathfrak{p})$ and related classes have been studied in [**18**],[**20**],[**24**] and [**26**].

Several subclasses of univalent functions are characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying often in a region in the right-half plane. Ma and Minda [**28**] gave a unified presentation of various subclasses of convex and starlike functions. For an analytic function $\phi(z)$ with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps the unit disk U onto a region starlike with respect to 1, symmetric with respect to the real axis, they considered the class $S^*(\phi)$ consisting of functions $f \in \mathcal{A}$ for which $zf'(z)/f(z) \prec \phi(z) \quad (z \in U)$. As mentioned in Chapter 2, they also investigated a corresponding class $K(\phi)$ of functions $f \in \mathcal{A}$ satisfying $1 + zf''(z)/f'(z) \prec \phi(z) \quad (z \in U)$. In the following definition, we consider the corresponding extension for meromorphic univalent functions.

Definition 4.1.1. Let $\phi(z)$ be a function with positive real part on U with $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps U onto a region starlike with respect to 1, symmetric with respect to the real axis. The class $\sum^*(\mathfrak{p}, w_0, \phi)$ consists of functions $f \in U(\mathfrak{p})$ satisfying

$$-\left(\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z) \quad (z \in U).$$

The class $\sum(\mathfrak{p}, \phi)$ consists of functions $f \in U(\mathfrak{p})$ satisfying

$$-\left(1 + z\frac{f''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right) \prec \phi(z) \quad (z \in U).$$

In this chapter, the bounds on $|w_0|$ will be determined. Also the bounds for some coefficients of f in $\sum^*(\mathfrak{p}, w_0, \phi)$ and $\sum(\mathfrak{p}, \phi)$ will be obtained.

To prove our main result, we need the following:

LEMMA 4.2.1. [28] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in U, then*

$$|c_2 - vc_1^2| \leq \begin{cases} -4v+2 & \text{if } v \leq 0 \\ 2 & \text{if } 0 \leq v \leq 1 \\ 4v-2 & \text{if } v \geq 1. \end{cases}$$

When $v < 0$ or $v > 1$, equality holds if and only if $p_1(z)$ is $(1+z)/(1-z)$ or one of its rotations. If $0 < v < 1$, then equality holds if and only if $p_1(z)$ is $(1+z^2)/(1-z^2)$ or one of its rotations. If $v = 0$, equality holds if and only if

$$p_1(z) = \left(\frac{1}{2} + \frac{1}{2}\lambda\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\lambda\right)\frac{1-z}{1+z} \quad (0 \leq \lambda \leq 1)$$

or one of its rotations. If $v = 1$, equality holds if and only if p_1 is the reciprocal of one of the functions such that equality holds in the case of $v = 0$.

THEOREM 4.2.1. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ and $f(z) = z + a_2 z^2 + \ldots$ in $|z| < \mathfrak{p}$. If $f \in \sum^*(\mathfrak{p}, w_0, \phi)$, then*

$$w_0 = \frac{2\mathfrak{p}}{\mathfrak{p}B_1 c_1 - 2\mathfrak{p}^2 - 2}$$

and

$$\frac{\mathfrak{p}}{\mathfrak{p}^2 + B_1\mathfrak{p} + 1} \leq |w_0| \leq \frac{\mathfrak{p}}{\mathfrak{p}^2 - B_1\mathfrak{p} + 1}. \tag{4.2.1}$$

Also we have

$$\left| a_2 + \frac{w_0}{2}\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2} + \frac{1}{w_0^2}\right)\right| \leq \begin{cases} \frac{|w_0||B_2|}{2} & if |B_2| \geq B_1, \\ \frac{|w_0|B_1}{2} & if |B_2| \leq B_1. \end{cases} \tag{4.2.2}$$

PROOF. Let $p(z)$ be defined by

$$p(z) := -\left[\frac{zf'(z)}{f(z) - w_0} + \frac{\mathfrak{p}}{z - \mathfrak{p}} - \frac{\mathfrak{p}z}{1 - \mathfrak{p}z}\right] = 1 + b_1 z + b_2 z^2 + \ldots.$$

We have

$$
\begin{aligned}
&-\left[\frac{zf'(z)}{f(z)-w_0}+\frac{\mathfrak{p}}{z-\mathfrak{p}}-\frac{\mathfrak{p}z}{1-\mathfrak{p}z}\right]\\
&=-\left[\frac{zf'(z)}{f(z)-w_0}-\frac{1}{1-\frac{1}{\mathfrak{p}}z}-\frac{\mathfrak{p}z}{1-\mathfrak{p}z}\right]\\
&=\frac{zf'(z)}{w_0[1-\frac{1}{w_0}f(z)]}+\frac{1}{1-\frac{1}{\mathfrak{p}}z}+\frac{\mathfrak{p}z}{1-\mathfrak{p}z}\\
&=\frac{1}{w_0}(z+2a_2z^2+3a_3z^3+\dots)\left[1+\frac{1}{w_0}(z+a_2z^2+a_3z^3+\dots)\right.\\
&\left.+\frac{1}{w_0^2}(z+a_2z^2+a_3z^3+\dots)^2+\dots\right]+\left[1+\frac{1}{\mathfrak{p}}z+\frac{1}{\mathfrak{p}^2}z^2+\frac{1}{\mathfrak{p}^3}z^3+\dots\right]\\
&+\mathfrak{p}z\left[1+\mathfrak{p}z+\mathfrak{p}^2z^2+\mathfrak{p}^3z^3+\dots\right]\\
&=\frac{1}{w_0}\left[z+(2a_2+\frac{1}{w_0})z^2+(3a_3+\frac{2a_2}{w_0}+\frac{a_2}{w_0}+\frac{1}{w_0^2})z^3+\dots\right]\\
&+\left[1+\frac{1}{\mathfrak{p}}z+\frac{1}{\mathfrak{p}^2}z^2+\frac{1}{\mathfrak{p}^3}z^3+\dots\right]+(\mathfrak{p}z+\mathfrak{p}^2z^2+\mathfrak{p}^3z^3+\dots)\\
&=1+\left(\frac{1}{w_0}+\frac{1}{\mathfrak{p}}+\mathfrak{p}\right)z+\left[\frac{1}{w_0}(2a_2+\frac{1}{w_0})+\frac{1}{\mathfrak{p}_2}+\mathfrak{p}^2\right]z^2+\dots.
\end{aligned}
$$

Then it follows that

$$
b_1=\mathfrak{p}+\frac{1}{\mathfrak{p}}+\frac{1}{w_0}, \tag{4.2.3}
$$

and

$$
b_2=\mathfrak{p}^2+\frac{1}{\mathfrak{p}^2}+\frac{1}{w_0^2}+\frac{2a_2}{w_0}. \tag{4.2.4}
$$

Since $\phi(z)$ is univalent and $p(z)\prec\phi(z)$, the function

$$
p_1(z)=\frac{1+\phi^{-1}(p(z))}{1-\phi^{-1}(p(z))}=1+c_1z+c_2z^2+\cdots
$$

is analytic and has positive real part in U. Also we have

$$
p(z)=\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right). \tag{4.2.5}
$$

We have

$$\frac{p_1(z)-1}{p_1(z)+1} = \frac{c_1z+c_2z^2+c_3z^3+\dots}{2+c_1z+c_2z^2+c_3z^3+\dots}$$

$$= \frac{1}{2}(c_1z+c_2z^2+c_3z^3+\dots)\left[1-\frac{1}{2}(c_1z+c_2z^2+c_3z^3+\dots)\right.$$

$$\left.+\frac{1}{4}(c_1z+c_2z^2+c_3z^3+\dots)^2+\dots\right]$$

$$= \frac{1}{2}\left[c_1z+(c_2-\frac{1}{2}c_1^2)z^2+(c_3-c_1c_2+\frac{1}{4}c_1^3)z^3+\dots\right]$$

which yields

$$\phi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1+B_1\left[\frac{1}{2}c_1z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+\dots\right]$$

$$+B_2\left[\frac{1}{2}c_1z+\frac{1}{2}(c_2-\frac{1}{2}c_1^2)z^2+\dots\right]^2+\dots$$

$$= 1+\left(\frac{1}{2}B_1c_1\right)z+\left[\frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2\right]z^2+\dots.$$

From equation (4.2.5), we obtain

(4.2.6) $$b_1 = \frac{1}{2}B_1c_1$$

and

(4.2.7) $$b_2 = \frac{1}{2}B_1(c_2-\frac{1}{2}c_1^2)+\frac{1}{4}B_2c_1^2.$$

From (4.2.3) and (4.2.6), we get

(4.2.8) $$w_0 = \frac{2\mathfrak{p}}{\mathfrak{p}B_1c_1-2\mathfrak{p}^2-2}$$

and from (4.2.4) and (4.2.7), we obtain

(4.2.9) $$a_2 = \frac{w_0}{8}(2B_1c_2-B_1c_1^2+B_2c_1^2)-\frac{\mathfrak{p}^2w_0}{2}-\frac{w_0}{2\mathfrak{p}^2}-\frac{1}{2w_0}.$$

From (4.2.3) and (4.2.6), we see that

$$\mathfrak{p}+\frac{1}{\mathfrak{p}}+\frac{1}{w_0} = \frac{1}{2}B_1c_1$$

and since $|c_1| \leq 2$ for a function with positive real part, we have

$$\left|\mathfrak{p} + \frac{1}{\mathfrak{p}} - \frac{1}{|w_0|}\right| \leq \left|\mathfrak{p} + \frac{1}{\mathfrak{p}} + \frac{1}{w_0}\right| \leq \frac{1}{2} B_1 |c_1| \leq B_1$$

or

$$-B_1 \leq \mathfrak{p} + \frac{1}{\mathfrak{p}} - \frac{1}{|w_0|} \leq B_1.$$

Rewriting the inequality we obtain

$$\frac{\mathfrak{p}}{\mathfrak{p}^2 + B_1\mathfrak{p} + 1} \leq |w_0| \leq \frac{\mathfrak{p}}{\mathfrak{p}^2 - B_1\mathfrak{p} + 1}.$$

To prove the second part of the results, from (4.2.9), we have

$$\begin{aligned}\left|a_2 + \frac{w_0}{2}\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2} + \frac{1}{w_0^2}\right)\right| &= \left|\frac{w_0}{8}\left(2B_1c_2 - B_1c_1^2 + B_2c_1^2\right)\right| \\ &= \frac{|w_0|B_1}{4}\left|c_2 - \left(\frac{B_1 - B_2}{2B_1}\right)c_1^2\right|.\end{aligned}$$

The result now follows from Lemma 4.2.1. □

The classes $\sum^*(\mathfrak{p}, w_0, \phi)$ and $\sum(\mathfrak{p}, \phi)$ are indeed more general classes of functions as can be seen from the following corollaries.

Corollary 4.2.1. [**30**, inequality 4, p. 447] *If $f \in \sum^*(\mathfrak{p}, w_0)$, then*

$$\frac{\mathfrak{p}}{(1+\mathfrak{p})^2} \leq |w_0| \leq \frac{\mathfrak{p}}{(1-\mathfrak{p})^2}.$$

Proof. Let $B_1 = 2$ in (4.2.1) of Theorem 4.2.1. □

Corollary 4.2.2. [**30**, Theorem 1, p. 447] *Let $f \in \sum^*(\mathfrak{p}, w_0)$ and $f(z) = z + a_2z^2 + \ldots$ in $|z| < \mathfrak{p}$. Then the second coefficient a_2 is given by*

$$a_2 = \frac{1}{2}w_0\left(b_2 - \mathfrak{p}^2 - \frac{1}{\mathfrak{p}^2} - \frac{1}{w_0^2}\right)$$

where the region of variability for a_2 is contained in the disk

$$\left|a_2 + \frac{1}{2}w_0\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2} + \frac{1}{w_0^2}\right)\right| \leq |w_0|.$$

Proof. Let $B_1 = 2$ in (4.2.2) in Theorem 4.2.1. □

The next theorem is for convex meromorphic functions.

THEOREM 4.2.2. *Let* $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ *and* $f(z) = z + a_2 z^2 + \dots$ *in* $|z| < \mathfrak{p}$. *If* $f \in \sum(\mathfrak{p}, \phi)$, *then*

$$\frac{2\mathfrak{p}^2 - B_1\mathfrak{p} + 2}{2\mathfrak{p}} \leq |a_2| \leq \frac{2\mathfrak{p}^2 + B_1\mathfrak{p} + 2}{2\mathfrak{p}}.$$

And

$$\left| a_3 - \frac{1}{3}\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2}\right) - \frac{2}{3}a_2^2 - \mu\left(a_2 - \mathfrak{p} - \frac{1}{\mathfrak{p}}\right)^2 \right| \leq \begin{cases} \frac{|2B_2 + 3\mu B_1^2|}{12} & if |\frac{2B_2}{B_1} + 3\mu B_1| \geq 2, \\ \frac{B_1}{6} & if |\frac{2B_2}{B_1} + 3\mu B_1| \leq 2. \end{cases}$$

PROOF. Let $p(z)$ be defined by

$$p(z) = -\left[1 + \frac{zf''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right] = 1 + b_1 z + b_2 z^2 + \dots$$

and $p_1(z)$ be defined as in the proof of Theorem 4.2.1. We have

$$\begin{aligned}
&-\left[1 + \frac{zf''(z)}{f'(z)} + \frac{2\mathfrak{p}}{z - \mathfrak{p}} - \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z}\right] \\
&= -1 - \frac{zf''(z)}{f'(z)} + \frac{2}{1 - \frac{1}{\mathfrak{p}}z} + \frac{2\mathfrak{p}z}{1 - \mathfrak{p}z} \\
&= -1 - (2a_2 z + 6a_3 z^2 + \dots)\left[1 - (2a_2 z + 3a_3 z^2 + \dots) + (2a_2 z + 3a_3 z^2 + \dots)^2 + \dots\right] \\
&+ 2\left[1 + \frac{1}{\mathfrak{p}}z + \frac{1}{\mathfrak{p}^2}z^2 + \dots\right] + 2\mathfrak{p}z\left[1 + \mathfrak{p}z + \mathfrak{p}^2 z^2 + \dots\right] \\
&= -1 + [-2a_2 z + (4a_2^2 - +6a_3)z^2 + \dots] + 2\left[1 + \frac{1}{\mathfrak{p}}z + \frac{1}{\mathfrak{p}^2}z^2 + \dots\right] \\
&+ 2\mathfrak{p}z\left[1 + \mathfrak{p}z + \mathfrak{p}^2 z^2 + \dots\right] \\
&= 1 + 2\left(\mathfrak{p} + \frac{1}{\mathfrak{p}} - a_2\right)z + 2\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2} + 2a_2^2 - 3a_3\right)z^2 + \dots
\end{aligned}$$

which yield

$$b_1 = 2\left(\mathfrak{p} + \frac{1}{\mathfrak{p}} - a_2\right) \tag{4.2.10}$$

and

$$b_2 = 2\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2} + 2a_2^2 - 3a_3\right). \tag{4.2.11}$$

From (4.2.6) and (4.2.10), we have

$$a_2 = \mathfrak{p} + \frac{1}{\mathfrak{p}} - \frac{B_1 c_1}{4}. \tag{4.2.12}$$

From (4.2.7) and (4.2.11), we have

$$a_3 = \frac{1}{24}\left(8\mathfrak{p}^2 + \frac{8}{\mathfrak{p}^2} + 16a_2^2 - 2B_1c_2 + B_1c_1^2 - B_2c_1^2\right). \tag{4.2.13}$$

Rewriting equation (4.2.12), we have

$$2\mathfrak{p} + \frac{2}{\mathfrak{p}} - 2a_2 = \frac{1}{2}B_1c_1$$

or

$$\left|2\mathfrak{p} + \frac{2}{\mathfrak{p}} - 2|a_2|\right| \leq |2\mathfrak{p} + \frac{2}{\mathfrak{p}} - 2a_2| \leq \frac{1}{2}B_1|c_1| \leq B_1$$

Thus we have

$$-B_1 \leq 2\mathfrak{p} + (2/\mathfrak{p}) - 2|a_2| \leq B_1$$

or

$$\frac{2\mathfrak{p}^2 - B_1\mathfrak{p} + 2}{2\mathfrak{p}} \leq |a_2| \leq \frac{2\mathfrak{p}^2 + B_1\mathfrak{p} + 2}{2\mathfrak{p}}.$$

From (4.2.12) and (4.2.13), we obtain

$$a_2 - \mathfrak{p} - \frac{1}{\mathfrak{p}} = -\frac{B_1c_1}{4}$$

$$a_3 - \frac{1}{3}\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2}\right) - \frac{2}{3}a_2^2 = \frac{1}{24}\left(-2B_1c_2 + B_1c_1^2 - B_2c_1^2\right).$$

Hence, we have

$$\begin{aligned}
&\left|a_3 - \frac{1}{3}\left(\mathfrak{p}^2 + \frac{1}{\mathfrak{p}^2}\right) - \frac{2}{3}a_2^2 - \mu\left(a_2 - \mathfrak{p} - \frac{1}{\mathfrak{p}}\right)^2\right| \\
&= \left|\frac{1}{24}\left(-2B_1c_2 + B_1c_1^2 - B_2c_1^2\right) - \mu\left(\frac{B_1^2c_1^2}{16}\right)\right| \\
&= \left|-\frac{B_1c_2}{12} + \frac{B_1c_1^2}{24} - \frac{B_2c_1^2}{24} - \frac{\mu B_1^2c_1^2}{16}\right| \\
&= \left|-\frac{B_1}{12}\left[c_2 - \frac{c_1^2}{2} + \frac{B_2c_1^2}{2B_1} + \frac{3\mu B_1c_1^2}{4}\right]\right| \\
&= \frac{B_1}{12}\left|c_2 - \left(\frac{1}{2} - \frac{B_2}{2B_1} - \frac{3\mu B_1}{4}\right)c_1^2\right|
\end{aligned}$$

The result now follows from Lemma 4.2.1. □

CONCLUSION

The work is devoted to the study of certain subclasses of univalent and multivalent functions in the unit disk U.

Certain general classes of multivalent starlike, convex, close-to-convex and quasi-convex functions are introduced. These classes provide a unified treatment to various known subclasses. Inclusion and convolution properties are derived using the methods of convex hull and differential subordination.

Bounds for the Fekete-Szegö coefficient functional associated with the k-th root transform $[f(z^k)]^{1/k}$ of normalized analytic functions f defined on the open unit disk in the complex plane are derived. A similar problem is investigated for functions $z/f(z)$ when f belongs to a certain class of functions.

Also, some subclasses of meromorphic univalent functions in the unit disk U are extended. The bounds for some initial coefficients are obtained.

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