SUBORDINATION AND CONVOLUTION OF ANALYTIC, MEROMORPHIC AND HARMONIC FUNCTIONS

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SUBORDINATION AND CONVOLUTION OF ANALYTIC, MEROMORPHIC AND HARMONIC FUNCTIONS

by

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SYMBOLS

Symbol	Description	Page
$\mathcal{A}_p$	Class of all p-valent analytic functions f of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{D})$	2
$\mathcal{A}$	$\mathcal{A}_1$	2
$(a)_n$	Pochhammer symbol defined by $(a)_n := \begin{cases} 1, & n = 0; \\ a(a+1)(a+2)\ldots(a+n-1), & n \in \mathbb{N}. \end{cases}$	18
arg	Argument of a complex number	4
$\mathbb{C}$	Complex plane	1
$\mathcal{C}_{\mathcal{S}}(\alpha)$	$\{f \in \mathcal{A} : zf' \in \mathcal{S}_{\mathcal{S}}(\alpha)\}$	44
$\mathcal{CC}$	Class of close-to-convex functions in $\mathcal{A}$	6
$\mathcal{CC}^n(h)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{\phi_n(z)} \prec h(z),\, \phi \in \mathcal{ST}^n(h),\, \frac{\phi_n(z)}{z} \neq 0 \text{ in } \mathbb{D}\right\}$	38
$\mathcal{CC}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{CC}^n(h)\}$	38
$\overline{co}(D)$	The closed convex hull of a set D	34
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	4
$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}[A,B]$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \in \mathcal{P}[A,B]\right\}$	12
$\mathcal{CV}(\varphi)$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z)\right\}$	14
$\mathcal{CV}_H$, $\mathcal{CV}_H^0$	Classes of harmonic convex functions	27
$\mathcal{CV}^n(h)$	$\left\{f \in \mathcal{A} : \frac{(zf')'(z)}{f_n'(z)} \prec h(z),\; f_n'(z) \neq 0 \text{ in } \mathbb{D}\right\}$	37

$\mathcal{CV}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{CV}^n(h)\}$	37
$\mathcal{CVC}^n(h)$	$\left\{f \in \mathcal{A} : \frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'}(\bar{z})} \prec h(z),\ f_n'(z)+\overline{f_n'}(\bar{z}) \neq 0 \text{ in } \mathbb{D}\right\}$	40
$\mathcal{CVC}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{CVC}^n(h)\}$	40
$\mathcal{CVS}^n(h)$	$\left\{f \in \mathcal{A} : \frac{2(zf')'(z)}{f_n'(z)+f_n'(-z)} \prec h(z),\ f_n'(z)+f_n'(-z) \neq 0 \text{ in } \mathbb{D}\right\}$	39
$\mathcal{CVS}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{CVS}^n(h)\}$	39
$\mathcal{CVSC}^n(h)$	$\left\{f \in \mathcal{A} : \frac{2(zf')'(z)}{f_n'(z)+\overline{f_n'}(-\bar{z})} \prec h(z),\ f_n'(z)+\overline{f_n'}(-\bar{z}) \neq 0 \text{ in } \mathbb{D}\right\}$	40
$\mathcal{CVSC}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{CVSC}^n(h)\}$	40
$\mathbb{D}$	Open unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$	1
$\overline{\mathbb{D}}$	Closed unit disk $\{z \in \mathbb{C} : \|z\| \leq 1\}$	31
$\mathbb{D}^*$	Open punctured unit disk $\{z \in \mathbb{C} : 0 < \|z\| < 1\}$	23
$\partial\mathbb{D}$	Boundary of unit disk $\mathbb{D}$	31
$\mathcal{D}^\lambda$	Ruscheweyh derivative operator	22
$E(q)$	$\left\{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\right\}$	31
$f * g$	Convolution or Hadamard product of functions f and g	17
$\mathcal{F}(\alpha, \beta, \gamma)$	Hohlov linear operator	22
$F(a, b, c; z)$	Gaussian hypergeometric function	19
F_μ	Bernardi integral operator	108
${}_lF_m$	Generalized hypergeometric function	20
$\mathcal{G}_b$	$\left\{f \in \mathcal{A} : \left\|\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f(z)}{zf'(z)}\right)-1\right\| < b, \quad (0 < b \leq 1)\right\}$	109
$G_H([\alpha_1], \gamma)$	Subclass of all starlike harmonic functions $f \in \mathcal{S}_H$	141
$\mathcal{H}(\mathbb{D})$	Class of all analytic functions in $\mathbb{D}$	2
$\mathcal{H}[a, n]$	Class of all analytic functions f in $\mathbb{D}$ of the form	2

$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$

$\mathcal{H}_0 := \mathcal{H}[0,1]$	Class of analytic functions f in $\mathbb{D}$ of the form	2
	$f(z) = a_1 z + a_2 z^2 + \cdots$	
$\mathcal{H} := \mathcal{H}[1,1]$	Class of analytic functions f in $\mathbb{D}$ of the form	2
	$f(z) = 1 + a_1 z + a_2 z^2 + \cdots$	
$H_p^{l,m}[\alpha_1]$	Dziok-Srivastava linear operator with respect to α_1	**??**
$\widetilde{H}_p^{l,m}[\beta_1]$	Dziok-Srivastava linear operator with respect to β_1	62
$\widetilde{H^*}_p^{l,m}[\beta_1]$	Liu-Srivastava linear operator	24
$HP(\alpha,\beta)$	$\{f = h + \bar{g} \in \mathcal{S}_H :$	140
	$\operatorname{Re}\left(\alpha z(h''(z) + g''(z)) + (h'(z) + g'(z))\right) > \beta\}$	
$HT(\alpha,\beta)$	$\{f \in \mathcal{S}_H : HP(\alpha,\beta) \bigcap \mathcal{T}_H\}$	140
Im	Imaginary part of a complex number	10
$J_f(z)$	Jacobian of the function $f = u + iv$	26
k	Koebe function $k(z) = z/(1-z)^2$	2
k_0	Harmonic Koebe function	28
	$k_0(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \frac{\frac{1}{2}\bar{z}^2 + \frac{1}{6}\bar{z}^3}{(1-\bar{z})^3}$	
$\mathcal{L}(\alpha,\gamma)$	Carlson and Shaffer linear operator	22
$\mathcal{M}_p$	Class of all meromorphic p-valent functions f of the form	41
	$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \geq 1,\ z \in \mathbb{D}^*)$	
$\mathcal{M}$	$\mathcal{M}_1$	41
$\mathcal{MCC}_p^n(h)$	$\Big\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{zf'(z)}{\phi_n(z)} \prec h(z),$	48

$\phi \in \mathcal{MST}_p^n(h), \phi_n(z) \neq 0 \text{ in } \mathbb{D}^*\}$

$\mathcal{MCC}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MCC}_p^n(h)\}$	48
$\mathcal{MCV}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{(zf'(z))'}{f_n'(z)} \prec h(z),\ f_n'(z) \neq 0 \text{ in } \mathbb{D}^*\right\}$	43
$\mathcal{MCV}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MCV}_p^n(h)\}$	43
$\mathcal{MCVC}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2\big(zf'\big)'(z)}{f_n'(z)+\overline{f_n'(\bar{z})}} \prec h(z), f_n'(z) + \overline{f_n'(\bar{z})} \neq 0 \text{ in } \mathbb{D}^*\right\}$	56
$\mathcal{MCVC}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MCVC}_p^n(h)\}$	56
$\mathcal{MCVS}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2\big(zf'\big)'(z)}{f_n'(z)+f_n'(-z)} \prec h(z),\right.$ $\left. f_n'(z) + f_n'(-z) \neq 0 \text{ in } \mathbb{D}^*\right\}$	52
$\mathcal{MCVS}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MCVS}_p^n(h)\}$	52
$\mathcal{MCVSC}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2\big(zf'\big)'(z)}{f_n'(z)+\overline{f_n'(-\bar{z})}} \prec h(z),\right.$ $\left. f_n'(z) + \overline{f_n'(-\bar{z})} \neq 0 \text{ in } \mathbb{D}^*\right\}$	59
$\mathcal{MCVSC}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MCVSC}_p^n(h)\}$	59
$\mathcal{MOC}_\alpha$	Class of α-convex functions in $\mathcal{A}$	6
$\mathcal{MQC}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{(zf'(z))'}{\varphi_n'(z)} \prec h(z),\right.$ $\left. \phi \in \mathcal{MCV}_p^n(h),\ \varphi_n'(z) \neq 0 \text{ in } \mathbb{D}^*\right\}$	49
$\mathcal{MQC}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MQC}_p^n(h)\}$	49
$\mathcal{MST}_p(\alpha)$	$\left\{f \in \mathcal{M}_p : -\operatorname{Re}\frac{1}{p}\frac{zf'(z)}{f(z)} > \alpha \quad (0 \leq \alpha < 1)\right\}$	42
$\mathcal{MST}_p(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{zf'(z)}{f(z)} \prec h(z)\right\}$	43
$\mathcal{MST}_p(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MST}_p(h)\}$	42
$\mathcal{MST}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{zf'(z)}{f_n(z)} \prec h(z),\ f_n(z) \neq 0 \text{ in } \mathbb{D}^*\right\}$	43
$\mathcal{MST}_p^n(g,h)$	$\{f \in \mathcal{M}_p : f * g \in \mathcal{MST}_p^n(h)\}$	43
$\mathcal{MSTC}_p^n(h)$	$\left\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2zf'(z)}{f_n(z)+\overline{f_n(\bar{z})}} \prec h(z), f_n(z) + \overline{f_n(\bar{z})} \neq 0 \text{ in } \mathbb{D}^*\right\}$	55

$\mathcal{MSTC}_p^n(g,h)$	$\big\{f \in \mathcal{M}_p : f * g \in \mathcal{MSTC}_p^n(h)\big\}$	55
$\mathcal{MSTS}_p^n(h)$	$\Big\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2zf'(z)}{f_n(z)-f_n(-z)} \prec h(z),$ $f_n(z) - f_n(-z) \neq 0 \text{ in } \mathbb{D}^*\}$	52
$\mathcal{MSTS}_p^n(g,h)$	$\big\{f \in \mathcal{M}_p : f * g \in \mathcal{MSTS}_p^n(h)\big\}$	52
$\mathcal{MSTSC}_p^n(h)$	$\Bigg\{f \in \mathcal{M}_p : -\frac{1}{p}\frac{2zf'(z)}{f_n(z)-\overline{f_n(-\bar{z})}} \prec h(z),$ $f_n(z) - \overline{f_n(-\bar{z})} \neq 0 \text{ in } \mathbb{D}^*\Bigg\}$	59
$\mathcal{MSTSC}_p^n(g,h)$	$\big\{f \in \mathcal{M}_p : f * g \in \mathcal{MSTSC}_p^n(h)\big\}$	59
$\mathbb{N}$	$\{1, 2, 3, \dots\}$	23
$\mathbb{N}_0$	$\mathbb{N} \bigcup \{0\}$	23
$\mathcal{P}$	$\{p \in \mathcal{H} : \text{with} \operatorname{Re} p(z) > 0,\ z \in \mathbb{D}\}$	11
$\mathcal{P}(\alpha)$	$\{p \in \mathcal{H} : \text{with} \operatorname{Re} p(z) > \alpha,\ z \in \mathbb{D}\}$	11
$\mathcal{P}[A,B]$	$\Big\{p \in \mathcal{H} : p(z) \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1)\Big\}$	12
$\mathcal{Q}$	Set of all functions $q(z)$ that analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$ such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$	31
$\mathcal{Q}_0$	$\{q \in \mathcal{Q} : q(0) = 0\}$	31
$\mathcal{Q}_1$	$\{q \in \mathcal{Q} : q(0) = 1\}$	31
$\mathcal{QC}$	Class of quasi-convex functions in $\mathcal{A}$	7
$\mathcal{QC}^n(h)$	$\Big\{f \in \mathcal{A} : \frac{(zf')'(z)}{\phi_n'(z)} \prec h(z), \phi \in \mathcal{CV}^n(h), \phi_n'(z) \neq 0 \text{ in } \mathbb{D}\Big\}$	38
$\mathcal{QC}_g^n(h)$	$\{f \in \mathcal{A} : f * g \in \mathcal{QC}^n(h)\}$	38
$\mathbb{R}$	Set of all real numbers	2
$\mathcal{R}_\alpha$	Class of prestarlike functions of order α in $\mathcal{A}$	34
$\mathcal{R}[A,B]$	$\Big\{f \in \mathcal{A} : f'(z) \prec \frac{1+Az}{1+Bz}, \quad (-1 \leq B < A \leq 1)\Big\}$	108

$\mathcal{R}[\alpha]$	$\mathcal{R}[\alpha,-\alpha]$	108
Re	Real part of a complex number	5
$\mathcal{S}$	Class of all normalized univalent functions f in $\mathcal{A}$	2
$\mathcal{S}_H$	Class of all sense-preserving harmonic functions f of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\overline{\sum_{k=1}^{\infty}b_kz^k}\quad(z\in\mathbb{D})$	26
$\mathcal{S}_H^0$	Class of all sense-preserving harmonic fucntions f of the form $f(z)=z+\sum_{k=2}^{\infty}a_kz^k+\overline{\sum_{k=2}^{\infty}b_kz^k}\quad(z\in\mathbb{D})$	27
$\mathcal{S}_p$	Class of parabolic starlike functions in $\mathcal{A}$	9
$\mathcal{S}_p(\alpha,\beta)$	Class of parabolic β-starlike functions of order α in $\mathcal{A}$	10
$\mathcal{S}^*_{\mathcal{C}}(h)$	$\left\{f\in\mathcal{A}:\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\prec h(z)\right\}$	36
$\mathcal{S}^*_{\mathcal{S}}(\alpha)$	$\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>\alpha\quad(0\leq\alpha<1/2)\right\}$	44
$\mathcal{S}^*_{\mathcal{S}}(h)$	$\left\{f\in\mathcal{A}:\frac{2zf'(z)}{f(z)-f(-z)}\prec h(z)\right\}$	36
$\mathcal{S}^*_{\mathcal{SC}}(h)$	$\left\{f\in\mathcal{A}:\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\prec h(z)\right\}$	37
$\mathcal{SCC}_\alpha$	Class of strongly close-to-convex functions of order α in $\mathcal{A}$	8
$\mathcal{SCV}_\alpha$	Class of strongly convex functions of order α in $\mathcal{A}$	8
$\mathcal{SST}_\alpha$	Class of strongly starlike functions of order α in $\mathcal{A}$	8
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	5
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}[A,B]$	$\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\in\mathcal{P}[A,B]\right\}$	12
$\mathcal{ST}(\varphi)$	$\left\{f\in\mathcal{A}:\frac{zf'(z)}{f(z)}\prec\varphi(z)\right\}$	14
$\mathcal{ST}_H$, $\mathcal{ST}_H^0$	Class of harmonic starlike functions	27,28
$\mathcal{ST}_s$	Class of starlike functions with respect to	7

SUBORDINASI DAN KONVOLUSI FUNGSI ANALISIS, FUNGSI MEROMORFI DAN FUNGSI HARMONIK

ABSTRAK

Tesis ini membincangkan fungsi analisis, fungsi meromorfi dan fungsi harmonik. Andaikan $\mathcal{A}_p$ sebagai kelas yang terdiri daripada fungsi-fungsi analisis ternormal dalam bentuk $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(p \in \mathbb{N})$, yang tertakrif pada cakera unit terbuka $\mathbb{D}$. Apabila $p = 1$, $\mathcal{A} := \mathcal{A}_1$. Tambahan pula, kelas yang terdiri daripada fungsi-fungsi meromorfi adalah berbentuk $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ $(p \in \mathbb{N})$, tertakrif pada cakera unit terbuka berliang $\mathbb{D}^*$ dan ditandakan sebagai $\mathcal{M}_p$ dengan $\mathcal{M} := \mathcal{M}_1$. Andaikan juga $\mathcal{S}_H$ sebagai kelas yang terdiri daripada fungsi-fungsi harmonik yang tertakrif pada cakera unit terbuka $\mathbb{D}$. Fungsi $f \in \mathcal{S}_H$ boleh ditulis sebagai $f = h + \bar{g}$, dengan $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$, $|b_1| < 1$. Tesis ini mengkaji empat masalah penyelidikan.

Bagi suatu fungsi $f \in \mathcal{M}_p$, f_n ditakrifkan sebagai $f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n+pk} f\left(\epsilon^k z\right)$, dengan $n \geq 1$ suatu integer, $\epsilon^n = 1$ and $\epsilon \neq 1$. Dengan menggunakan fungsi ini, kelas teritlak bagi fungsi meromorfi p-valen bak-bintang, cembung, hampir cembung dan kuasicembung terhadap titik n-lipat, serta juga kelas teritlak bagi fungsi meromorfi p-valen bak-bintang dan cembung, terhadap titik simetri n-lipat, titik konjugat dan titik konjugat simetri diperkenalkan. Sifat tutupan terhadap konvolusi bagi kelas-kelas tersebut dikaji.

Keputusan berkaitan subordinasi pembeza juga dikaji bagi fungsi-fungsi $f \in \mathcal{A}_p$. Dengan menggunakan teori subordinasi pembeza, suatu kelas fungsi diperoleh agar dapat memenuhi implikasi pembeza tertentu. Keputusan subordinasi pembeza yang melibatkan nisbah antara fungsi yang tertakrif melalui pengoperasi linear Dziok-Srivastava juga diperoleh. Konsep kedualan superordinasi pembeza juga dibincangkan untuk mendapat beberapa keputusan 'tersepit'. Tambahan

pula, masalah yang serupa bagi fungsi-fungsi meromorfi $f \in \mathcal{M}_p$, yang tertakrif melalui pengoperasi linear Liu-Srivastava dikaji.

Fungsi analisis ternormal f disebut bak-bintang Janowski jika $zf'(z)/f(z)$ adalah subordinat terhadap $(1 + Az)/(1 + Bz)$, $(-1 \leq B \leq A \leq 1)$. Dengan menggunakan teori subordinasi pembeza peringkat pertama, beberapa syarat cukup diperoleh agar implikasi berikut dipenuhi:

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z) \prec \frac{1+Cz}{1+Dz} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \qquad (1)$$

Masalah serupa dikaji dengan ungkapan $p(z)+zp'(z)/p^2(z)$ dan $p^2(z)+zp'(z)/p(z)$. Dengan menggunakan implikasi (1), keputusan subordinasi pembeza yang teritlak diperolehi dan digunakan kemudian pada pengoperasi kamiran Bernardi $F_\mu : \mathcal{A} \to \mathcal{A}$ dengan

$$F_\mu(z) := (\mu+1)\int_0^1 t^{\mu-1} f(tz)dt \quad (\mu > -1).$$

Tesis ini diakhiri dengan mengkaji kaitan fungsi harmonik dengan fungsi hipergeometri. Dua subkelas fungsi harmonik pada $\mathcal{S}_H$ diperkenalkan. Kelas-kelas tersebut ditandakan dengan $HP(\alpha, \beta)$ dan $G_H([\alpha_1], \gamma)$. Syarat-syarat cukup diperoleh bagi fungsi hipergeometri, fungsi beta tak lengkap dan suatu pengoperasi linear untuk terletak di dalam kelas $HP(\alpha, \beta)$. Turut diperoleh adalah, batas pekali, titik ekstrim, keputusan rangkuman dan tutupan terhadap suatu pengoperasi kamiran bagi fungsi-fungsi harmonik memuaskan kelas $G_H([\alpha_1], \gamma)$. Fungsi-fungsi harmonik dengan pekali negatif juga dilibatkan dalam kajian ini.

SUBORDINATION AND CONVOLUTION OF ANALYTIC, MEROMORPHIC AND HARMONIC FUNCTIONS

ABSTRACT

This thesis deals with analytic, meromorphic and harmonic functions. Let $\mathcal{A}_p$ denote the class of normalized analytic functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ $(p \in \mathbb{N})$, defined on the open unit disk $\mathbb{D}$ where p is a fixed positive integer. When $p = 1$, $\mathcal{A} := \mathcal{A}_1$. Further, the class of all meromorphic functions are of the form $f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k$ $(p \in \mathbb{N})$, defined in the punctured open unit disk $\mathbb{D}^*$ and denoted by $\mathcal{M}_p$ with $\mathcal{M} := \mathcal{M}_1$. Also, let $\mathcal{S}_H$ denote the class of sense-preserving harmonic functions defined in the unit disk $\mathbb{D}$. A function $f \in \mathcal{S}_H$ can be written as $f = h + \bar{g}$ where $h(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$, $|b_1| < 1$. Four research problems are discussed in this work.

Given a function $f \in \mathcal{M}_p$, let f_n be defined by $f_n(z) := \frac{1}{n} \sum_{k=0}^{n-1} \epsilon^{n+pk} f\left(\epsilon^k z\right)$, where $n \geq 1$ is any integer, $\epsilon^n = 1$ and $\epsilon \neq 1$. Using this function, general classes of meromorphic functions that are starlike, convex, close-to-convex and quasi-convex with respect to n-ply points, as well as meromorphic functions that are starlike and convex with respect to n-ply symmetric points, conjugate points and symmetric conjugate points are introduced. Closure properties under convolution of these classes are obtained.

Results associated with differential subordination are also investigated for functions $f \in \mathcal{A}_p$. By using the theory of differential subordination, a class of functions is determined to satisfy certain differential implication. Differential subordination results involving the ratios of functions defined through the Dziok-Srivastava linear operator are also obtained. The dual concept of differential superordination is also considered to obtain sandwich type results. Further, similar problems for meromorphic functions $f \in \mathcal{M}_p$, defined through the Liu-Srivastava linear operator are

investigated.

A normalized analytic function f is Janowski starlike if $zf'(z)/f(z)$ is subordinated to $(1 + Az)/(1 + Bz)$, $(-1 \leq B \leq A \leq 1)$. By making use of the theory of first-order differential subordination, sufficient conditions are obtained for the following implication to hold:

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z) \prec \frac{1+Cz}{1+Dz} \;\Rightarrow\; p(z) \prec \frac{1+Az}{1+Bz}. \tag{2}$$

Similar problem is also investigated with the expressions $p(z) + zp'(z)/p^2(z)$ and $p^2(z) + zp'(z)/p(z)$. Using the implication (2), a more general differential subordination result is obtained which is next applied to the Bernardi's integral operator $F_\mu : \mathcal{A} \to \mathcal{A}$ given by

$$F_\mu(z) := (\mu+1) \int_0^1 t^{\mu-1} f(tz) dt \quad (\mu > -1).$$

Finally, connections between harmonic functions and hypergeometric functions are investigated. Two subclasses of $\mathcal{S}_H$ are introduced. These classes are denoted by $HP(\alpha, \beta)$ and $G_H([\alpha_1], \gamma)$. Sufficient conditions are obtained for a hypergeometric function, an incomplete beta functions and an integral operator to be in the class $HP(\alpha, \beta)$. Further, coefficient bounds, extreme points, inclusion results and closure under an integral operator are also investigated for harmonic functions satisfying the class $G_H([\alpha_1], \gamma)$. Harmonic functions with negative coefficients are also considered in this investigation.

CHAPTER 1

INTRODUCTION

1.1 Univalent Functions

Geometric function theory is a branch of complex analysis which deals with the geometric properties of analytic functions. The theory of univalent functions deals with one-to-one analytic, meromorphic and harmonic functions.

Let $\mathbb{C}$ be the complex plane and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$. Let Ω be an open subset of $\mathbb{C}$. Let $f : \Omega \to \mathbb{C}$. The function f is *analytic* at $z_0 \in \Omega$ if it is differentiable at every point in some neighborhood of z_0 and it is analytic on Ω if it is analytic at all points in Ω. The analytic function f is *univalent* (Schlicht) in Ω, if it takes different points in Ω to different values, that is for any two distinct points z_1 and z_2 in Ω, $f(z_1) \neq f(z_2)$. The function f is *locally univalent* at a point $z_0 \in \Omega$, if it is univalent in some neighborhood of z_0. For an analytic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0. The restriction of one-to-oneness (injectivity) on functions defined on Ω provides a lot of information on the geometric and analytic properties of such functions.

We now recall the famous Riemann mapping theorem which states that, in any simply connected domain (a domain without any holes) Ω which is not the whole complex plane $\mathbb{C}$, there exists an analytic univalent function φ that maps Ω onto the unit disk.

Theorem 1.1 (Riemann Mapping Theorem) [35, p. 11] *Let Ω be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in Ω. Then there is a unique univalent analytic function f which maps Ω onto the unit disk $\mathbb{D}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

In view of this, the study of analytic univalent functions on a simply connected domain can be restricted to the study of these functions in the open unit disk $\mathbb{D}$.

Let $\mathcal{H}(\mathbb{D})$ be the class of functions analytic in $\mathbb{D}$. Let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

and let $\mathcal{H}_0 \equiv \mathcal{H}[0,1]$ and $\mathcal{H} \equiv \mathcal{H}[1,1]$. The open mapping theorem for a non-constant analytic function shows that the derivative of a univalent function can never vanish. Also translations and dilatations do not affect univalency. This enables us to consider functions f that are normalized by the conditions $f(0) = 0, f'(0) = 1$. Let $\mathcal{A}$ denote the class of such analytic functions f in $\mathbb{D}$. A function $f \in \mathcal{A}$ has the power series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{D}). \tag{1.1}$$

More generally, let $\mathcal{A}_p$ denote the class of all analytic functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (z \in \mathbb{D},\ p \in \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{1.2}$$

The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. Thus $\mathcal{S}$ is the class of all normalized univalent functions in $\mathbb{D}$. The function k defined by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n \quad (z \in \mathbb{D})$$

called the Koebe function which maps $\mathbb{D}$ onto the complex plane except for a slit along the half-line $(-\infty, -1/4]$, is an example of a function in $\mathcal{S}$. It plays a very important role in the study of the class $\mathcal{S}$. In fact, the Koebe function and its rotations $e^{-i\alpha} k\left(e^{i\alpha} z\right)$, $\alpha \in \mathbb{R}$ are the only extremal functions for various problem in the study of geometric function theory.

In 1916, Bieberbach [28] proved that $|a_2| \leq 2$ for $f \in \mathcal{S}$ and that equality holds if and only if f is a rotation of the Koebe function k. This result is known as Bieberbach Theorem. He also conjectured that $|a_n| \leq n\,(n \geq 2)$ for $f \in \mathcal{S}$, is generally valid. This conjecture is known as the Bieberbach's (or coefficient) conjecture. For the cases $n = 3$, and $n = 4$ the conjecture was proved by Lowner [67], and Garabedian and Schiffer [41], respectively. Pederson and Schiffer [96] proved the conjecture for $n = 5$. For $n = 6$, the conjecture was proved by Pederson [95] and Ozawa [90], independently. Finaly, in 1985, Louis de Branges [34], proved the Bieberbach's conjecture (now known as de Branges Theorem) for all coefficients n as in the following theorem:

Theorem 1.2 (de Branges Theorem) [34] *If $f \in \mathcal{S}$ is of the form (1.1), then*

$$|a_n| \leq n \quad (n \geq 2)$$

with equality if and only if, f is the Koebe function k or one of its rotations.

The simple coefficient inequality $|a_2| \leq 2$ yields many important properties of univalent functions. As a first application, we will state the covering theorem: if $f \in \mathcal{S}$, then the image of $\mathbb{D}$ under f contains a disk of radius $1/4$.

Theorem 1.3 (Koebe One-Quarter Theorem) [35, p. 31] *The range of every function $f \in \mathcal{S}$ contains the disk $\left\{w \in \mathbb{C} : |w| < \frac{1}{4}\right\}$.*

Another important consequence of the Bieberbach's theorem is the distortion theorem that gives sharp upper and lower bounds for $|f'(z)|$.

Theorem 1.4 (Distortion Theorem) [35, p. 32] *For each $f \in \mathcal{S}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \quad (|z| = r < 1).$$

The result is sharp.

The distortion theorem can be applied to obtain sharp upper and lower bounds for $|f(z)|$. This result is known as growth theorem.

Theorem 1.5 (Growth Theorem) [35, p. 33] *For each* $f \in \mathcal{S}$*,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \quad (|z| = r < 1).$$

The result is sharp.

One last implication of the Bieberbach's theorem is the rotation theorem where the sharp bound for $|\arg f'(z)|$ is obtained by using Loewner's method in [43].

Theorem 1.6 (Rotation Theorem) [35, p. 99] *For each* $f \in \mathcal{S}$*,*

$$|\arg f'(z)| \leq \begin{cases} 4sin^{-1}r & \left(r \leq \frac{1}{\sqrt{2}}\right), \\ \\ \pi + \log \frac{r^2}{1-r^2} & \left(r \geq \frac{1}{\sqrt{2}}\right), \end{cases}$$

where $|z| = r < 1$*. The bound is sharp.*

The difficulty in proving the Bieberbach conjecture led to various developments including new techniques for proving the conjecture as well as new classes of functions for which the conjecture was easier to handle. These classes of functions were defined by geometric considerations. Among the classes introduced are the class of convex functions, the class of starlike functions, the class of close-to-convex functions and the class of quasi-convex functions.

A domain Ω in the complex plane $\mathbb{C}$ is called *convex* if for every pair of points, the line segment joining them lies completely in the interior of Ω. That is, $w_1, w_2 \in \Omega$ implies that $tw_1 + (1-t)w_0 \in \Omega$ for $0 \leq t \leq 1$. A function $f \in \mathcal{S}$ is a convex if the image $f(\mathbb{D})$ is convex. The class of functions $f \in \mathcal{S}$ which are convex is denoted by $\mathcal{CV}$. A domain Ω in the complex plane $\mathbb{C}$ is called *starlike* with respect to a point $w_0 \in \Omega$, if the line segment joining w_0 to every other point $w \in \Omega$ lies in the interior of Ω. In other words, for any $w \in \Omega$ and $0 \le t \le 1$, $tw_0 + (1-t)w \in \Omega$. A function $f \in \mathcal{S}$ is starlike if the image $f(\mathbb{D})$ is starlike with respect to the origin. The class of functions $f \in \mathcal{S}$ which are starlike is denoted by $\mathcal{ST}$.

Analytically, a function $f \in \mathcal{CV}$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.3}$$

The condition (1.3) for convexity was first stated by Study [134]. Also, a function $f \in \mathcal{ST}$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.4}$$

The condition (1.4) for starlikeness is due to Nevanlinna [79]. In 1936, Robertson [107] introduced the concepts of convex functions of order α and starlike functions of order α, $0 \le \alpha < 1$. A function $f \in \mathcal{S}$ is said to be *convex of order* α if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}),$$

and *starlike of order* α if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

These classes are respectively denoted by $\mathcal{CV}(\alpha)$ and $\mathcal{ST}(\alpha)$. Note that $\mathcal{CV}(0) = \mathcal{CV}$ and $\mathcal{ST}(0) = \mathcal{ST}$.

Every convex function is evidently starlike. Thus $\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{S}$. However, there is a close analytic connection between convex and starlike functions. Alexander [7] first observed this in 1915 and proved the following theorem.

Theorem 1.7 (Alexander's Theorem) [35, p. 43] *Let $f \in \mathcal{S}$. Then $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$.*

Let $\alpha \in \mathbb{R}$. A function $f \in \mathcal{S}$ with $(f(z)/z)f'(z) \neq 0$ is said to be *α-convex* in $\mathbb{D}$ if and only if

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0 \quad (z \in \mathbb{D}).$$

We denote the class of α-convex functions by $\mathcal{MOC}_\alpha$. This class $\mathcal{MOC}_\alpha$ is a subclass of $\mathcal{S}$, and was introduced and studied by Mocanu [75]. If $\alpha = 1$, then an α-convex function is convex; and if $\alpha = 0$, then an α-convex function is starlike. In 1973, Miller et al. [74] proved that all α-convex functions are convex if $\alpha \geq 1$ and starlike if $\alpha < 1$.

Another subclass of $\mathcal{S}$ is the class of close-to-convex functions. Kaplan [57] introduced close-to-convex functions in 1952. A function $f \in \mathcal{A}$ is said to be *close-to-convex* if there is a real number $\theta, -\pi/2 < \theta < \pi/2$, and a starlike function g (not necessarily normalized) such that

$$\operatorname{Re}\left(e^{i\theta}\frac{zf'(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{D}).$$

Geometrically, close-to-convex functions maps each circle $z = re^{i\theta}$ with fixed $r < 1$ onto a simple closed curve with the property that the tangent vector cannot turn back on itself more than $-\pi$ radians. The class of functions $f \in \mathcal{A}$ that are close-to-convex is denoted by $\mathcal{CC}$. The subclasses of $\mathcal{S}$, namely convex, starlike and close-to-convex functions are related as follows:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CC} \subset \mathcal{S}.$$

Note that, it is not necessary to assume that f is univalent in the definition of a close-to-convex function since it follows from the following Noshiro-Warschawski theorem [81, 144] about functions defined in convex domain.

Theorem 1.8 *If a function f is analytic in a convex domain Ω and*

$$\operatorname{Re}\left(e^{i\theta} f'(z)\right) > 0 \quad (-\pi/2 \leq \theta \leq \pi/2)$$

in Ω, then f is univalent in Ω.

Kaplan [57] applied Noshiro-Warschawski theorem to prove that every close-to-convex function is univalent.

In 1980, Noor and Thomas [82] introduced the class of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be *quasi-convex* if and only if

$$\operatorname{Re}\left(\frac{(zf'(z))'}{g'(z)}\right) > 0 \quad (z \in \mathbb{D})$$

for some $g \in \mathcal{CV}$. The class of functions $f \in \mathcal{A}$ that are quasi-convex is denoted by $\mathcal{QC}$.

A function $f \in \mathcal{A}$ is said to be *starlike with respect to symmetric points* in $\mathbb{D}$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad (z \in \mathbb{D}). \tag{1.5}$$

This class was introduced and studied in 1959 by Sakaguchi [118]. Let the class of these functions be denoted by $\mathcal{ST}_s$. Geometrically, if $f \in \mathcal{ST}_s$, then for every $r < 1$, the angular velocity of $f(z)$ about the point $f(-z)$ is positive as z traverses the circle $|z| = r$ in the positive direction. Further investigations into the class of starlike functions with respect to symmetric points can be found in [31, 80, 92, 120, 138, 140–143]. El-Ashwah and Thomas [39] introduced and studied the classes consisting of *starlike functions with respect to conjugate points*, and *starlike functions with respect to symmetric conjugate points* defined respectively by the conditions

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)+\overline{f(\overline{z})}}\right)>0 \quad (z\in\mathbb{D}) \text{ and } \operatorname{Re}\left(\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}}\right)>0 \quad (z\in\mathbb{D}).$$

The classes of these functions be denoted respectively by $\mathcal{ST}_c$ and $\mathcal{ST}_{sc}$. Geometrically, if $f\in\mathcal{ST}_c$, then for every $r<1$, the angular velocity of $f(z)$ about the point $f(\bar{z})$ is positive as z traverses the circle $|z|=r$ in the positive direction and $f\in\mathcal{ST}_{sc}$ can be described similarly.

Brannan and Kirwan [29] introduced the classes of strongly convex and strongly starlike functions of order α. A function $f\in\mathcal{A}$ is said to be *strongly convex of order* α, $0<\alpha\leq 1$ if it satisfies

$$\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\pi\alpha}{2} \quad (z\in\mathbb{D}).$$

The class of all such functions is denoted by $\mathcal{SCV}_\alpha$. Similarly, a function $f\in\mathcal{A}$ is said to be *strongly starlike of order* α, $0<\alpha\leq 1$ if it satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi\alpha}{2} \quad (z\in\mathbb{D}).$$

The class of all such functions is denoted by $\mathcal{SST}_\alpha$. Since the condition $\operatorname{Re} w(z)>0$ is equivalent to $|\arg w(z)|<\pi/2$, we have $\mathcal{SCV}_1=\mathcal{CV}$ and $\mathcal{SST}_1=\mathcal{ST}$

Reade [106] introduced the class of strongly close-to-convex functions of order α. A function $f\in\mathcal{A}$ is said to be *strongly close-to-convex of order* α, $0<\alpha\leq 1$ if there is a real β and a convex function ϕ that satisfies

$$\left|\arg\left(e^{-i\beta}\frac{f'(z)}{\phi'(z)}\right)\right|\leq\alpha\frac{\pi}{2} \quad (z\in\mathbb{D}).$$

The class of all such functions is denoted by $\mathcal{SCC}_\alpha$. When $\beta=0$, $\mathcal{SCC}_1=\mathcal{CC}$.

In 1991, Goodman [45,46] introduced and investigated the classes of uniformly convex and uniformly starlike functions. A function $f \in \mathcal{S}$ is *uniformly convex (starlike)* if for every circular arc γ contained in $\mathbb{D}$ with center $\zeta \in \mathbb{D}$ the image arc $f(\gamma)$ is convex (starlike with respect to $f(\zeta)$). The class of all uniformly convex and uniformly starlike functions is denoted by $\mathcal{UCV}$ and $\mathcal{UST}$ respectively. Analytically, $f \in \mathcal{UCV}$ if and only if

$$\operatorname{Re}\left(1+(z-\zeta)\frac{f''(z)}{f'(z)}\right) \geq 0, \quad (z \neq \zeta, (z,\zeta) \in \mathbb{D}\times\mathbb{D}).$$

and $f \in \mathcal{UST}$ if and only if

$$\operatorname{Re}\left(\frac{f(z)-f(\zeta)}{(z-\zeta)f'(z)}\right) \geq 0 \quad (z \neq \zeta, (z,\zeta) \in \mathbb{D}\times\mathbb{D}).$$

In 1992, Ronning [109] and Ma and Minda [70] were able to give independently a one variable characterization for the class $\mathcal{UCV}$. The function $f \in \mathcal{S}$ belongs to $\mathcal{UCV}$ if and only if it satisfies

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{D}). \tag{1.6}$$

Using this one variable characterization of the class $\mathcal{UCV}$, it is easy to obtain transformations which preserve the class $\mathcal{UCV}$. However, a one variable characterization of the class $\mathcal{UST}$ is still unavailable. In exploring the possibility of an analogous result of Alexander's theorem between the classes $\mathcal{UCV}$ and $\mathcal{UST}$, Ronning [110], introduced a subclass $\mathcal{S}_p$ of *parabolic starlike* functions. The class $\mathcal{S}_p$ consist functions $F(z) = zf'(z)$, where f belongs to the class $\mathcal{UCV}$. Clearly $f \in \mathcal{S}_p$ if and only if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf'(z)}{f(z)}-1\right| \quad (z \in \mathbb{D}). \tag{1.7}$$

In order to interpret (1.6) and (1.7) geometrically, let

$$\Omega_{PAR} := \{w \in \mathbb{C} : \operatorname{Re} w > |w-1|\}.$$

The set Ω_{PAR} is the interior of the parabola

$$(\operatorname{Im} w)^2 < 2 \operatorname{Re} w - 1$$

and it is therefore symmetric with respect to the real axis and has $(1/2, 0)$ as its vertex. Then $f \in \mathcal{UCV}$ if and only if $1 + zf''(z)/f'(z) \in \Omega_{PAR}$ and $f \in \mathcal{S}_p$ if and only if $zf'(z)/f(z) \in \Omega_{PAR}$.

In [108,109], Ronning introduced the classes $\mathcal{UCV}(\alpha,\beta)$ consist of *uniformly β-convex functions of order α* and $\mathcal{S}_p(\alpha,\beta)$ consist of *parabolic β-starlike functions of order α*, $-1 < \alpha \leq 1,\ \beta \geq 0$, which generalize the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ respectively. The function $f \in \mathcal{A}$ belongs to $\mathcal{UCV}(\alpha,\beta)$ if and only if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right| \quad (z \in \mathbb{D}). \tag{1.8}$$

Similarly, the function $f \in \mathcal{A}$ belongs to $\mathcal{S}_p(\alpha,\beta)$ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in \mathbb{D}). \tag{1.9}$$

Indeed it follows from (1.9) and (1.8) that $f \in \mathcal{UCV}(\alpha,\beta)$ if and only if $zf' \in \mathcal{S}_p(\alpha,\beta)$. For $-1 < \alpha \leq 1,\ \beta \geq 0$, let

$$\Omega_{PAR}^{\alpha,\beta} := \{w \in \mathbb{C} : \operatorname{Re} w - \alpha > \beta|w-1|\}.$$

The set $\Omega_{PAR}^{\alpha,\beta}$ is the interior of the parabola

$$(\beta \operatorname{Im} w)^2 < (\operatorname{Re} w - \alpha)^2 - \beta^2 (\operatorname{Re} w - 1)^2.$$

Geometrically, $f \in \mathcal{UCV}(\alpha, \beta)$ if and only if $1 + zf''(z)/f'(z) \in \Omega_{PAR}^{\alpha,\beta}$ and $f \in \mathcal{S}_p(\alpha, \beta)$ if and only if $zf'(z)/f(z) \in \Omega_{PAR}^{\alpha,\beta}$. Clearly, $\mathcal{UCV}(0, 1) = \mathcal{UCV}$ and $\mathcal{S}_p(0, 1) = \mathcal{S}_p$.

Several subclasses of starlike and convex functions can be unified by using subordination between analytic functions. An analytic function $f \in \mathcal{A}$ is *subordinate* to an analytic function $g \in \mathcal{A}$, denoted by $f \prec g$, if there exists an analytic function ω, with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = g(\omega(z))$. In particular, if the function $g \in \mathcal{S}$, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. If $f \prec g$, we say g is superordinate to f.

Let $\mathcal{P}$ be the class of all analytic functions p of the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathbb{D}). \tag{1.10}$$

Any function in $\mathcal{P}$ is called a Caratheodory function. More generally, for $0 \leq \alpha < 1$, we denote by $\mathcal{P}(\alpha)$ the class of analytic functions $p \in \mathcal{P}$ with

$$\operatorname{Re} p(z) > \alpha \quad (z \in \mathbb{D}).$$

Most of the subclasses of functions in $\mathcal{S}$ are in one way or other related to the class $\mathcal{P}$. For example, the function $f \in \mathcal{ST}$ if and only if $zf(z)/f(z) \in \mathcal{P}$ and the function $f \in \mathcal{CV}$ if and only if $1 + zf''(z)/f'(z) \in \mathcal{P}$.

In terms of subordination, the analytic condition (1.10) can be written as

$$p(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}).$$

This follows since the mapping $q(z) = (1+z)/(1-z)$ maps $\mathbb{D}$ onto the right-half plane. In this light, the conditions (1.3) and (1.4) are respectively equivalent to

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}) \tag{1.11}$$

and

$$\frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}). \tag{1.12}$$

For $|B| \leq 1$ and $A \neq B$, a function p, analytic in $\mathbb{D}$ with $p(0) = 1$, belongs to the class $\mathcal{P}[A, B]$ if

$$p(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{D}).$$

In 1973, Janowski [55] introduced the class $\mathcal{P}[A, B]$ for $-1 \leq B < A \leq 1$. He introduced and investigated the classes of *Janowski convex* and *Janowski starlike* functions by replacing the function $(1+z)/(1-z)$ in (1.11) and (1.12) with $(1+Az)/(1+Bz)$. The classes of Janowski convex and Janowski starlike functions are denoted by $\mathcal{CV}[A, B]$ and $\mathcal{ST}[A, B]$ respectively. In particular $\mathcal{P}[A, B] \subset \mathcal{P}[1, -1] = \mathcal{P}$. It is also evident that, under the condition $p(0) = 1$ and $0 \leq \alpha < 1$, $\mathcal{P}[1-2\alpha, -1] = \mathcal{P}(\alpha)$. Thus, the classes $\mathcal{CV}[A, B]$ and $\mathcal{ST}[A, B]$ include the classes $\mathcal{CV}$ and $\mathcal{ST}$ respectively.

Rønning [110] and Ma and Minda [70] showed that the function $\phi_{PAR}(z)$ defined by

$$\phi_{PAR}(z) = 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2 \quad (z \in \mathbb{D})$$

maps $\mathbb{D}$ onto the parabolic region Ω_{PAR}. Therefore the classes $\mathcal{UCV}$ and $\mathcal{S}_p$ can be expressed in the form

$$\mathcal{UCV} = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi_{PAR}(z) \right\} \tag{1.13}$$

and

$$\mathcal{S}_p = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi_{PAR}(z) \right\}. \tag{1.14}$$

Analogous to (1.13) and (1.14), the classes $\mathcal{UCV}(\alpha, \beta)$ and $\mathcal{S}_p(\alpha, \beta)$ were also defined in terms of subordination in [2, 3, 93] for $0 \leq \alpha < 1$ and $\beta \geq 0$. Let $p(z) = 1 + zf''(z)/f'(z)$ or $p(z) = zf'(z)/f(z)$. Then the conditions (1.8) or (1.9) are rewritten in the form

$$p(z) \prec q_{\alpha,\beta}(z),$$

where the function $q_{\alpha,\beta}$ for $\beta = 0$ and $\beta = 1$ are

$$q_{\alpha,0}(z) = \frac{1 + (1 - 2\alpha)z}{1 - z}, \text{ and } q_{\alpha,1}(z) = 1 + \frac{2(1 - \alpha)}{\pi^2} \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2.$$

For $0 < \beta < 1$, the function $q_{\alpha,\beta}$ is

$$q_{\alpha,\beta}(z) = \frac{1 - \alpha}{1 - \beta^2} \cos \left(\frac{2}{\pi} \left(\cos^{-1} \beta \right) i \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) - \frac{\beta^2 - \alpha}{1 - \beta^2},$$

and if $\beta > 1$, then $q_{\alpha,\beta}$ has the form

$$q_{\alpha,\beta}(z) = \frac{1 - \alpha}{\beta^2 - 1} \sin \left(\frac{\pi}{2K(\beta)} \int_0^{u(z)/\sqrt{\beta}} \frac{dt}{\sqrt{1 - t^2 \sqrt{1 - \beta^2 t^2}}} \right) + \frac{\beta^2 - \alpha}{\beta^2 - 1},$$

where $u(z) = (z - \sqrt{\beta})/(1 - \sqrt{\beta} z)$ and K is such that $\beta = \cosh(\pi K'(z))/4K(z))$.

In an effort to unify the subclasses of univalent functions, Ma and Minda [68] gave a unified presentation of various subclasses of starlike and convex functions. For this purpose, let φ be an analytic function with positive real part in $\mathbb{D}$, normalized by the conditions $\varphi(0) = 1$ and $\varphi(0) > 0$, such that φ maps the unit disk $\mathbb{D}$ onto a region starlike with respect to 1 that is symmetric with respect to the real axis. They introduce the following classes

$$\mathcal{CV}(\varphi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \right\}$$

and

$$\mathcal{ST}(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}.$$

These functions are called Ma-Minda convex and starlike functions respectively. It is clear that for special choices of φ, these classes envelop several well-known subclasses of univalent function as special cases. For example, when $\varphi_{A,B}(z) = (1 + Az)/(1 + Bz)$ where $-1 \leq B < A \leq 1$, the class $\mathcal{ST}(\varphi_{A,B}) = \mathcal{ST}[A, B]$, and the class $\mathcal{CV}(\phi_{PAR}) = \mathcal{UCV}$.

Define the functions $k_\varphi, h_\varphi : \mathbb{D} \to \mathbb{C}$ respectively by

$$1 + \frac{zk''_\varphi(z)}{k'\varphi(z)} = \varphi(z) \quad (z \in \mathbb{D},\ k_\varphi \in \mathcal{A})$$

and

$$\frac{zh'_\varphi(z)}{h_\varphi(z)} = \varphi(z) \quad (z \in \mathbb{D},\ h_\varphi \in \mathcal{A}).$$

In [68], Ma and Minda introduced the functions k_φ and h_φ in $\mathcal{CV}(\varphi)$ and $\mathcal{ST}(\varphi)$ respectively, which turned out to be extremal for certain problems. For the class of Ma-Minda convex functions, the following theorem is holds.

Theorem 1.9 [68] *If $f \in \mathcal{CV}(\varphi)$, then, for $|z| = r$,*

$$k'_\varphi(-r) \leq |f'(z)| \leq k'_\varphi(r),$$

$$-k_\varphi(-r) \leq |f(z)| \leq k_\varphi(r).$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of k_φ. Also either f is a rotation of k_φ or $f(\mathbb{D})$ contains the disk $|w| \leq -k_\varphi(-1)$, where

$$-k_\varphi(-1) = \lim_{r \to 1^-}(-k_\varphi(-r)).$$

Further, for $|z_0| = r < 1$,

$$|\arg(f'(z_0))| \leq \max_{|z|=r} |\arg k'_\varphi(z)|.$$

Theorem 1.9 gives distortion, growth, covering and rotation results for the class $\mathcal{CV}(\varphi)$. Corresponding results for functions in $\mathcal{ST}(\varphi)$ were also obtained by Ma and Minda [68]. The additional assumptions $\min_{|z|=r}|\varphi(z)| = |\varphi(-r)|$ and $\min_{|z|=r}|\varphi(z)| = |\varphi(r)|$ are needed in the distortion theorem for $f \in \mathcal{ST}(\varphi)$.

More information on univalent functions can be found in the books by Goodman [44], Duren [35], Pommerenke [97], Graham and Kohr [47] and Rosenblum and Rovnyak [111]. Certain aspects of the subject have also been covered in the books by Nehari [78], Goluzin [42] and Hayman [50].

1.2 Differential Subordination and Differential Superordination

The theory of differential subordination and the dual theory of differential superordination were developed by Miller and Mocanu [72, 73]

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be univalent in $\mathbb{D}$. If p is analytic in $\mathbb{D}$ and satisfies the second order differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \tag{1.15}$$

then p is called a *solution of the differential subordination.* The univalent function q is called a *dominant* of the solution of the differential subordination or more simply dominant, if $p \prec q$ for all p satisfying (1.15). A dominant q_1 satisfying $q_1 \prec q$ for all dominants q of (1.15) is said to be the *best dominant* of (1.15). The best dominant is unique up to a rotation of $\mathbb{D}$. If $p(z) \in \mathcal{H}[a,n]$, then p will be called an (a,n)-*solution*, q an (a,n)-*dominant*, and q_1 the *best* (a,n)-*dominant.* Let $\Omega \subset \mathbb{C}$ and let (1.15) be replaced by

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \in \Omega, \text{ for all } z \in \mathbb{D}. \tag{1.16}$$

Even though this is a "differential inclusion" and $\psi\left(p(z), zp'(z), z^2p''(z); z\right)$ may not be analytic in $\mathbb{D}$, the condition in (1.16) will also be referred as a *second order differential subordination,* and the same definition of solution, dominant and best dominant as given above can be extended to this generalization. See [72] for more information on differential subordination.

Let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ and let h be analytic in $\mathbb{D}$. If p and

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big)$$

are univalent in $\mathbb{D}$ and satisfies the second order differential superordination

$$h(z) \prec \psi\big(p(z), zp'(z), z^2p''(z); z\big), \tag{1.17}$$

then p is called a *solution of the differential superordination.* An analytic function q is called a *subordinant* of the solution of the differential superordination or more simply subordinant, if $q \prec p$ for all p satisfying (1.17). A univalent subordinant q_1 satisfying $q \prec q_1$ for all subordinants q of (1.17) is said to be the *best subordinant* of (1.17). The best subordinant is unique up to a rotation of $\mathbb{D}$. Let $\Omega \subset \mathbb{C}$ and let (1.17) be replaced by

$$\Omega \subset \left\{\psi\big(p(z), zp'(z), z^2p''(z); z\big) | z \in \mathbb{D}\right\}. \tag{1.18}$$

Even though this more general situation is a "differential containment", the condition in (1.18) will also be referred as a *second order differential superordination* and the definition of solution, subordinant and best subordinant can be extended to this generalization. See [73] for more information on the differential superordination.

1.3 Convolution and Hypergeometric Functions

1.3.1 Convolution

For two analytic functions f and g given by their Taylor series $f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = b_1 z + b_2 z^2 + b_3 z^3 + \cdots = \sum_{k=1}^{\infty} b_k z^k$, the *convolution* or *Hadamard product,* denoted by $f * g$ is defined by the analytic function

$$(f * g)(z) = a_1 b_1 z + a_2 b_2 z^2 + a_3 b_3 z^3 + \cdots = \sum_{k=1}^{\infty} a_k b_k z^k.$$

The product is named after Jacques Hadamard [48] who published the first in-depth analysis of this product in 1899. The Hadamard product is commutative and associative. The first derivative of the product $f * g$ is related to the first derivatives of f and g by

$$z(f * g)'(z) = \big((zf)' * g\big)(z) = \big(f * (zg)'\big)(z).$$

The second derivative of the product is related to the first derivative of $f * g, f$ and g by

$$z^2(f*g)''(z) + z(f*g)'(z) = z\big(z(f*g)'\big)'(z) = z\big((zf')*g\big)'(z) = \big((zf')*(zg')\big)(z).$$

For any analytic function g with $g(0) = 0$, the function $f(z) = \sum_{k=1}^{\infty} z^n = z/(1-z)$ is an identity. Hence, $(f * g)(z) = g(z)$.

The importance of convolution lies in the fact that many linear operators in geometric function theory are special cases of the convolution operator $f \longmapsto f * g$ for an appropriate fixed g. Polya and Schoenberg [101] conjectured that the classes of convex functions $\mathcal{CV}$ and starlike fucntions $\mathcal{ST}$ are preserved under convolution with convex functions. In other words, if $f \in \mathcal{CV}$ and $g \in \mathcal{CV}(g \in \mathcal{ST})$, then $f * g \in \mathcal{CV}(f * g \in \mathcal{ST})$.

In 1973, Ruscheweyh and Sheil-Small [116] proved the Polya-Schoenberg conjecture. In fact, Ruscheweyh and Sheil-Small [116] prove that the class $\mathcal{CC}$ of close-to-convex functions is also closed under convolution with convex functions. They proved the following more general theorem that plays an important role in the theory of convolutions:

Theorem 1.10 [116, p. 128] *If $f \in \mathcal{CV}$, $g \in \mathcal{ST}$ and $\phi \in \mathcal{P}$, then $\frac{f*(\phi g)}{f*g} \in \mathcal{P}$.*

1.3.2 Hypergeometric Functions

The exploitation of hypergeometric functions in the proof of the Bieberbach conjecture by Louis de Branges has given function theorists a renewed impulse to study this special function. As a result there are a numbers of works on this topic [20, 30, 69, 71, 99, 117].

The *Pochhammer symbol* or ascending factorial is defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & (n=0); \\ a(a+1)(a+2)\ldots(a+n-1), & (n \in \mathbb{N}) \end{cases} \tag{1.19}$$

where $\Gamma(a)$, denotes the Gamma function.

It is well known that the functions $f(z) = (1-z)^{-1}$ and $g(z) = (1-z)^{-a}$ can be represented as in the following geometric series :

$$(1-z)^{-1}=\sum_{k=0}^{\infty} z^k. \tag{1.20}$$

$$(1-z)^{-a}=\sum_{k=0}^{\infty}\left(\begin{array}{c}-a\\ k\end{array}\right)(-z)^k=\sum_{k=0}^{\infty}\frac{(a)_k}{k!}z^k. \tag{1.21}$$

The geometric series representations (1.20) and (1.21) lead to the consideration of the function defined by

$$\Phi(a,c;z)=1+\frac{a}{c}\frac{z}{1!}+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\cdots. \tag{1.22}$$

where $a,c\in\mathbb{C}$ with $c\neq 0,-1,-2\cdots$. This function, called a *confluent* (or *Kummer*) *hypergeometric function*, satisfies Kummer's differential equation

$$zw''(z)+(c-z)w'(z)-aw(z)=0.$$

By using (1.19), (1.22) can be written in the form

$$\Phi(a,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{z^k}{k\,!}=\frac{\Gamma(c)}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}. \tag{1.23}$$

The function in (1.22) can have a generalized form as defined by the following function:

$$F(a,b,c;z)=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\cdots. \tag{1.24}$$

This function, called a *Gaussian hypergeometric function,* satisfies the hypergeometric differential equation

$$z(1-z)w''(z)+[c-(a+b+1)z]w'(z)-abw(z)=0.$$

Using the notation (1.19) in (1.24), $F(a,b,c;z)$ can be written as

$$F(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k\,!}=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)}\frac{z^k}{k\,!}. \tag{1.25}$$

More generally, for $\alpha_j\in\mathbb{C}\quad(j=1,2,\ldots,l)$ and $\beta_j\in\mathbb{C}\setminus\{0,-1,-2,\ldots\}$ $(j=1,2,\ldots m)$, the *generalized hypergeometric function* [100]

$${}_lF_m(z):=\ {}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z)$$

is defined by the infinite series

$${}_lF_m(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m;z):=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_l)_k}{(\beta_1)_k\ldots(\beta_m)_k}\frac{z^k}{k!} \tag{1.26}$$

$$(l\le m+1;l,m\in\mathbb{N}_0:=\mathbb{N}\cup\{0\})$$

where $(a)_n$ is the Pochhammer symbol defined by (1.19). The absence of parameters is emphasized by a dash. For example,

$${}_0F_1(-;b;z)=\sum_{k=0}^{\infty}\frac{z^k}{(b)_k k\,!},$$

is the *Bessel's function.* Also

$${}_0F_0(-;-;z)=\sum_{k=0}^{\infty}\frac{z^k}{k\,!}=\exp(z)$$

and

$${}_1F_0(a;-;z)=\sum_{k=0}^{\infty}\frac{(a)_k z^k}{k\,!}=\frac{1}{(1-z)^a}.$$

Similarly,

$${}_2F_1(a,b;b;z)=\frac{1}{(1-z)^a},\qquad {}_2F_1(1,1;1;z)=\frac{1}{1-z},$$

$$ {}_2F_1(1,1;2;z) = \frac{-\ln(1-z)}{z}, \text{ and } {}_2F_1(1,2;1;z) = \frac{1}{(1-z)^2}. $$

Recall that, for two functions $f(z)$ given by (1.2) and $g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$, the convolution (or Hadamard product) of f and g is defined by

$$ (f * g)(z) := z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k =: (g * f)(z). \tag{1.27} $$

Corresponding to the function

$$ h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^p \; {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z), $$

the *Dziok-Srivastava operator* [37, 38, 129]

$$ H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{A}_p \to \mathcal{A}_p $$

is defined in terms of convolution as follows:

$$ \begin{aligned} & H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) \\ & := h_p(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ & = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p} \cdots (\alpha_l)_{k-p}}{(\beta_1)_{k-p} \cdots (\beta_m)_{k-p}} \frac{a_k z^k}{(k-p)!}. \end{aligned} \tag{1.28} $$

For brevity,

$$ H_p^{l,m}[\alpha_1] f(z) := H_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z). $$

The linear (convolution) operator $H_p^{l,m}[\alpha_1] f(z)$ includes, as its special cases, various other linear operators in many earlier works in geometric function theory. Some of these special cases are described below.

The linear operator $\mathcal{F}:\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{F}(\alpha,\beta,\gamma)f(z)=H_1^{(2,1)}(\alpha,\beta;\gamma)f(z)$$

is *Hohlov linear operator* [51]. The linear operator $\mathcal{L}:\mathcal{A}\to\mathcal{A}$ defined by

$$\mathcal{L}(\alpha,\gamma)f(z)=H_1^{(2,1)}(\alpha,1;\gamma)f(z)=\mathcal{F}(\alpha,1,\gamma)f(z)$$

is the *Carlson and Shaffer linear operator* [30]. The differential operator $\mathcal{D}^\lambda$: $\mathcal{A}\to\mathcal{A}$ defined by the convolution:

$$\mathcal{D}^\lambda f(z):=\frac{z}{(1-z)^{\lambda+1}}*f(z)=H_1^{(2,1)}(\lambda+1,1;1)f(z),\quad (\lambda\geq -1, f\in\mathcal{A})$$

is the *Ruscheweyh derivative operator* [115]. This operator also implies,

$$\mathcal{D}^n f(z):=\frac{z\big(z^{n-1}f(z)\big)^{(n)}}{n\,!},\quad (n\in\mathbb{N}_0, f\in\mathcal{A}).$$

In 1969, Bernardi [27] considered the linear integral operator $F:\mathcal{A}\to\mathcal{A}$ defined by

$$F(z):=(\mu+1)\int_0^1 t^{\mu-1}f(tz)dt\quad (\mu>-1). \tag{1.29}$$

This operator is called the *generalized Bernardi-Libera-Livingston linear operator* [27, 62, 66]. The operator in (1.29) was investigated by Alexander [7] for $\mu=0$ and Libera [62] for $\mu=1$. This operator can be written as a special case of Dziok-Srivastava operator in the follwing form:

$$F(z)=H_1^{(2,1)}(\mu+1,1;\mu+2)f(z),\quad (\mu>-1, f\in\mathcal{A}).$$

It is well-known [27] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi-Libera-Livingston integral operator.

Definition 1.1 [88, 89] *The fractional derivative of order λ is defined by*

$$D_z^\lambda f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \quad (0 \leq \lambda < 1)$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 1.2 [88, 89] *Under the hypothesis of Definition 1.1, the fractional derivative of order $n+\lambda$ is defined, by*

$$D_z^{n+\lambda} f(z) := \frac{d^n}{dz^n} D_z^\lambda f(z) \quad (0 \leq \lambda < 1,\ n \in \mathbb{N}_0).$$

In 1987, Srivastava and Owa [130] studied a fractional derivative operator $\Omega^\lambda : \mathcal{A} \to \mathcal{A}$ defined by $\Omega^\lambda f(z) := \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z)$. The fractional derivative operator is a special case of the Dziok-Srivastava linear operator since

$$\begin{aligned} \Omega^\lambda f(z) &= H_1^{2,1}(2, 1; 2-\lambda) f(z) \\ &= \mathcal{L}(2, 2-\lambda) f(z), \quad (\lambda \notin \mathbb{N} \setminus \{1\}, f \in \mathcal{A}). \end{aligned}$$

1.4 Meromorphic Functions

A function f which is *meromorphic multivalent* in $\mathbb{D}$ can be redefined, by a simple fractional transformation to have it's pole of order p at the origin.

Let $\mathcal{M}_p$ denote the class of all meromorphic multivalent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \geq 1), \tag{1.30}$$

that are analytic in the open punctured unit disk

$$\mathbb{D}^* = \{z \in \mathbb{C} : 0 < |z| < 1\},$$

and let $\mathcal{M}_1 := \mathcal{M}$ (the class of all meromorphic univalent functions). For two functions $f, g \in \mathcal{M}_p$, where f given by (1.30) and

$$g(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of f and g is defined by the series

$$(f * g)(z) := \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k b_k z^k =: (g * f)(z).$$

Corresponding to the function

$$h_p^*(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := z^{-p} \; {}_lF_m(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z),$$

where ${}_lF_m$ is defined by (1.26), the *Liu-Srivastava operator* [64, 65, 102, 131] $\widetilde{H^*}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) : \mathcal{M}_p \to \mathcal{M}_p$ is defined by the Hadamard product

$$\begin{aligned} \widetilde{H^*}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) &:= h_p^*(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z) \\ &= \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p} \cdots (\alpha_l)_{k+p}}{(\beta_1)_{k+p} \cdots (\beta_m)_{k+p}} \frac{a_k z^k}{(k+p)!} \end{aligned} \tag{1.31}$$

where $\alpha_j \in \mathbb{C}\,(j = 1, 2, \ldots, l)$, $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots m)$, $l \leq m+1; l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $(a)_n$ is the Pochhammer symbol defined by (1.19).

For convenience, (1.31) is written as

$$\widetilde{H^*}_p^{l,m}[\beta_1] f(z) := \widetilde{H^*}_p^{(l,m)}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z).$$

1.5 Harmonic Functions

A real function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic* if all its second partial derivatives are continuous in Ω and it satisfies the Laplace equation:

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

A one-to-one mapping $u = u(x, y), v = v(x, y)$ from a region Ω in the xy-plane is a harmonic mapping if both u, v are harmonic. It is convenient to use the complex notation $z = x + iy$ and $w = u + iv$ and to write $w = f(z) = u(z) + iv(z)$.

In 1984, Clunie and Sheil-Small [32] found significant results for complex valued harmonic functions defined on a domain $\Omega \subset \mathbb{C}$ and given by $f = u + iv$ where u and v are both real harmonic in Ω. In this dissertation, the domain Ω will be the unit disk $\mathbb{D}$, which is simply connected. Since u and v are real harmonic functions, when Ω is simply connected there exist analytic functions F and G such that

$$u = \operatorname{Re} F = \frac{F + \overline{F}}{2} \text{ and } v = \operatorname{Im} G = \frac{G - \overline{G}}{2i}.$$

This observation gives the representation

$$f = \frac{F + \overline{F}}{2} + \frac{G - \overline{G}}{2} = \left(\frac{F + G}{2}\right) + \overline{\left(\frac{F - G}{2}\right)}.$$

Letting $h = (F + G)/2$ and $g = (F - G)/2$, the harmonic function f can be expressed as $f = h + \overline{g}$. The analytic functions h and g are called the analytic and co-analytic parts of f respectively.

A mapping is said to be a *sense preserving mapping* if it preserves the orientation, or sense, of the angle between two curves in the domain of the mapping. A sense preserving mapping does not necessarily preserve the magnitude of the angle between the intersecting curves.

The *Jacobian* of the function $f = u + iv$ is

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

If f is analytic function, then its Jacobian has the following form:

$$J_f(z) = (u_x)^2 + (v_x)^2 = |f'(z)|^2.$$

For analytic functions f, $J_f(z) \neq 0$ if and only if f is locally univalent in $\mathbb{D}$. Hans Lewy [60] showed in 1936 that this statement remains true for harmonic functions. If $f(z) = h(z) + \overline{g(z)}$, is harmonic function, then its Jacobian has the following form:

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$

A necessary and sufficient condition [32] for a harmonic function $f = h + \overline{g}$ to be locally univalent and sense preserving in a simply connected domain Ω is for $|h'(z)| > |g'(z)|$ in Ω. If $|h'(z)| < |g'(z)|$ throughout Ω, then f is sense reversing. Throughout this dissertation it will be assumed that the harmonic functions are sense preserving.

Let a harmonic function f of the unit disk $\mathbb{D}$ be given by $f = h + \overline{g}$ with $g(0) = 0$. Then the representation $f = h + \overline{g}$ is unique and is called the *canonical representation* of f. The power series expansions of h and g are given by

$$h(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k.$$

Denote by $\mathcal{S}_H$ the class of all sense-preserving harmonic functions of the unit disk $\mathbb{D}$ with $a_0 = 0$ and $a_1 = 1$. Hence, the power series expansions of h and g for the subclass $\mathcal{S}_H$ are given by

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.32}$$

This class was introduced and investigated by Clunie and Sheil-Small [32]. The class $\mathcal{S}_H$, contains the standard class $\mathcal{S}$ of analytic univalent functions. The subclasses of $\mathcal{S}_H$ consisting of functions for which $b_1 = 0$ is denoted by $\mathcal{S}_H^0$. Therefore, the series expansions of h and g for the subclass $\mathcal{S}_H^0$ are given by

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=2}^{\infty} b_k z^k. \tag{1.33}$$

Note that $\mathcal{S} \subset \mathcal{S}_H^0 \subset \mathcal{S}_H$. Clunie and Sheil-Small [32] proved that the class $\mathcal{S}_H$ is a normal family of functions and the class $\mathcal{S}_H^0$ is a compact normal family.

Let $\mathcal{CV}_H$ and $\mathcal{CV}_H^0$ denote the respective subclasses of $\mathcal{S}_H$ and $\mathcal{S}_H^0$ that consist of functions that map the unit disk $\mathbb{D}$ to a convex domain. Geometrically the classes of *harmonic convex* functions can be defined similar to analytic convex functions. Clunie and Sheil-Small gave an analytic description for harmonic convex functions [32, Theorem 5.7]. The theorem states that a harmonic function $f = h + \overline{g}$ is in the class $\mathcal{CV}_H$ of harmonic convex functions if and only if, the analytic functions $h - e^{i\varphi} g$ with $0 \leq \varphi < 2\pi$ are convex in the direction $\varphi/2$ and f is suitably normalized. Clunie and Sheil-Small [32] also proved that the image of the unit disk under any harmonic convex functions $f \in \mathcal{CV}_H^0$ contains the disk $\{w : |w| < 1/2\}$.

The class of functions $f \in \mathcal{S}_H$ is *harmonic starlike* if its range is starlike with respect to the origin. The class of harmonic starlike is denoted by $\mathcal{ST}_H$. Suppose the function $f = h + \overline{g}$ is in $\mathcal{S}_H$. Analogue to Alexander theorem, if $zh'(z) - \overline{zg'(z)}$ is starlike, then f is convex.

Two important examples of harmonic functions that Clunie and Sheil-Small

[32] examined are

$$l_0(z) = h(z) + \overline{g(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} + \frac{-\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2}$$

and the *harmonic Koebe function*

$$k_0(z) = H(z) + \overline{G(z)} = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \frac{\frac{1}{2}\overline{z}^2 + \frac{1}{6}\overline{z}^3}{(1-\overline{z})^3}.$$

The function l_0 maps $\mathbb{D}$ onto the half plane $\{w : \mathrm{Re}(w) > -1/2\}$, and the function k_0 maps $\mathbb{D}$ onto the complex plane $\mathbb{C}$ less a slit from $-\infty$ to $-1/6$ along the real axis.

From the harmonic Koebe function it is clear that

$$|a_n| = \frac{1}{6}(2n+1)(n+1), \quad |b_n| = \frac{1}{6}(2n-1)(n-1) \quad (n \geq 2)$$

for all functions f in $\mathcal{S}_H^0$. It has also been proved that for all functions $f \in \mathcal{S}_H^0$, the sharp inequality $|b_2| \leq \frac{1}{2}$ holds.

The class of functions $f \in \mathcal{S}_H^0$ that are harmonic starlike is denoted $\mathcal{ST}_H^0$. Sheill-Small [121] proved that for f of the form (1.33)

$$|a_n| \leq \frac{(n+1)(2n+1)}{6} \text{ and } |b_n| \leq \frac{(n-1)(2n-1)}{6}$$

hold true for $\mathcal{ST}_H^0$. It was shown in [32] by Clunie and Sheill-Small that

$$|a_n| \leq \frac{(n+1)}{2} \text{ and } |b_n| \leq \frac{(n-1)}{2}$$

for $f \in \mathcal{CV}_H^0$. In 1998, Silverman [123] showed that if f of the form (1.33), satisfies

$$\sum_{k=2}^{\infty} k\big(|a_k| + |b_k|\big) \leq 1, \tag{1.34}$$

then $f \in \mathcal{ST}_H^0$. Silverman [123] also proved that for f in the form (1.33), is in $\mathcal{CV}_H^0$, if

$$\sum_{k=2}^{\infty} k^2\big(|a_k| + |b_k|\big) \leq 1. \tag{1.35}$$

Silverman [123, 125] introduced the subclasses of $\mathcal{ST}_H^0$ and $\mathcal{CV}_H^0$, denoted by $\mathcal{TST}_H^0$ and $\mathcal{TCV}_H^0$, respectively, where $f = h + \overline{g}$,

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \geq 0 \text{ and } g(z) = -\sum_{k=2}^{\infty} b_k z^k, \quad b_k \geq 0. \tag{1.36}$$

Silverman [123] showed that (1.34) and (1.35) are necessary and sufficient conditions for f of the form (1.36) to be in the subclasses $\mathcal{TST}_H^0$ and $\mathcal{TCV}_H^0$ respectively.

1.6 Scope of the Thesis

In this thesis, certain properties of analytic, meromorphic and harmonic functions are investigated by using the techniques of subordination and convolution.

In Chapter 3, closure properties under convolution of general classes of meromorphic multivalent functions that are starlike, convex, close-to-convex and quasi-convex with respect to n-ply points, as well as meromorphic multivalent functions that are starlike and convex with respect to n-ply symmetric points, conjugate points and symmetric conjugate points are investigated. These classes are indeed extensions of the classes of meromorphic convex, starlike, close-to-convex and quasi-convex functions. Previous results are special cases of the results obtained here.

In Chapter 4, differential subordination and superordination results for multivalent analytic functions associated with Dziok-Srivastava linear operator are obtained. In addition, differential subordination and superordination results are also obtained for meromorphic multivalent functions in the punctured open unit disk $\mathbb{D}^*$ that are associated with the Liu-Srivastava linear operator. These results are obtained by investigating appropriate classes of admissible functions. Certain related sandwich-type results are also investigated.

Chapter 5 deals with the applications of the theory of first-order differential subordination to obtain sufficient conditions for a normalize analytic function f to be Janowski starlike. These conditions are expected to emerge from the investigation of the implication

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z) \prec \frac{1+Cz}{1+Dz} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz},$$

and other similar implications involving $p(z)+zp'(z)/p^2(z)$ and $p^2(z)+zp'(z)/p(z)$ where $p(z) \in \mathcal{P}$. The results obtained will also applied to Bernardi's integral operator to derive related results.

In Chapter 6 is to obtained results by investigating the connection between harmonic functions and hypergeometric functions. In this chapter sufficient conditions are obtained for a hypergeometric function and an integral operator to be in a subclass of harmonic univalent functions. In addition, coefficient bounds, extreme points, inclusion results and closure under an integral operator are also investigated for a new subclass of complex-valued harmonic univalent functions, using the Dziok-Srivastava operator for harmonic mappings. Results related to harmonic functions with negative coefficients are also obtained.

CHAPTER 2

PRELIMINARY RESULTS

In chapter 1, differential subordination, differential superordination, convolution and hypergeometic functions were defined. In this chapter, some important results related to differential subordination, differential superordination, convolution and hypergeometic functions are highlighted. These results will be applied in obtaining the main results.

2.1 Differential Subordination and Differential Superordination

Denote by $\mathcal{Q}$ the set of all functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$ where

$$E(q) = \{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$. Further let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 2.1 [72, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega \tag{2.1}$$

whenever $r = q(\zeta)$, $s = k\zeta q'(\zeta)$, and

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq n$. We write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, then the admissible condition (2.1) reduces to

$$\psi\big(q(\zeta), k\zeta q'(\zeta); z\big) \notin \Omega, \tag{2.2}$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq n$.

In particular when $q(z) = M(Mz + a)/(M + \overline{a}z)$, with $M > 0$ and $|a| < M$, then $q(\mathbb{D}) = \mathbb{D}_M := \{w : |w| < M\}$, $q(0) = a$, $E(q) = \emptyset$ and $q \in \mathcal{Q}$. In this case, we set $\Psi_n[\Omega, M, a] := \Psi_n[\Omega, q]$, and in the special case when the set $\Omega = \mathbb{D}_M$, the class is simply denoted by $\Psi_n[M, a]$.

The following theorem is a main result in the theory of differential subordinations:

Theorem 2.1 [72, Theorem 2.3b, p. 28] *Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ satisfies*

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega,$$

then $p(z) \prec q(z)$.

Definition 2.2 [73, Definition 3, p. 817] *Let Ω be a set in $\mathbb{C}$, $q(z) \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t; \zeta) \in \Omega \tag{2.3}$$

whenever $r = q(z), s = zq'(z)/m$, and

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq n \geq 1$. When $n = 1$, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

If $\psi : \mathbb{C}^2 \times \overline{\mathbb{D}} \to \mathbb{C}$ *If* $\psi : \mathbb{C}^2 \times \overline{\mathbb{D}} \to \mathbb{C}$, *then the admissible condition (2.3) reduces to*

$$\psi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \notin \Omega, \tag{2.4}$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ *and* $m \geq n$.

The following theorem is a key result in the theory of differential superordinations:

Theorem 2.2 [73, Theorem 1, p. 818] *Let* $\psi \in \Psi'_n[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{Q}(a)$ *and* $\psi(p(z), zp'(z), z^2p''(z); z)$ *is univalent in* $\mathbb{D}$, *then*

$$\Omega \subset \{\psi\big(p(z), zp'(z), z^2p''(z); z\big) : z \in \mathbb{D}\}$$

implies $q(z) \prec p(z)$.

The following lemma will be an important tool in obtaining the main results.

Lemma 2.1 [72] *Let* q *be univalent in the unit disk* $\mathbb{D}$ *and* ϑ *and* φ *be analytic in a domain* D *containing* $q(\mathbb{D})$ *with* $\varphi(w) \neq 0$ *when* $w \in q(\mathbb{D})$. *Set* $Q(z) := zq'(z)\varphi(q(z))$ *and* $h(z) := \vartheta(q(z)) + Q(z)$. *Suppose that either* h *is convex, or* Q *is starlike univalent in* $\mathbb{D}$. *In addition, assume that* $\operatorname{Re}\big(zh'(z)/Q(z)\big) > 0$ *for* $z \in \mathbb{D}$. *If* p *is analytic in* $\mathbb{D}$ *with* $p(0) = q(0)$, $p(\mathbb{D}) \subseteq D$ *and*

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)), \tag{2.5}$$

then $p(z) \prec q(z)$, *and* q *is the best dominant.*

2.2 Convolution

Ruscheweyh [114] introduced the class of prestarlike functions by using convolution.

Definition 2.3 *For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by*

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} : \ f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{ST}(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$.

Note that for $\alpha = 0$, $\mathcal{R}_0$ is the class of univalent convex functions $\mathcal{CV}$, and for $\alpha = 1/2$, $\mathcal{R}_{1/2}$ is the class of univalent starlike functions $\mathcal{ST}(1/2)$ of order 1/2.

The well-known result that the classes of starlike functions of order α and convex functions of order α are closed under convolution with prestarlike functions of order α is a consequence of the following:

Theorem 2.3 [113, Theorem 2.4] *Let $\alpha \leq 1$, $\phi \in \mathcal{R}_\alpha$ and $f \in \mathcal{ST}(\alpha)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathbb{D}) \subset \overline{co}\big(H(\mathbb{D})\big),$$

for any analytic function $H \in \mathcal{H}(\mathbb{D})$, where $\overline{co}\big(H(\mathbb{D})\big)$ denote the closed convex hull of $H(\mathbb{D})$.

2.3 Hypergeometric Functions

Lemma 2.2 [6, Lemma 10] *If $a, b, c > 0$, then*

(i)

$$F(a+k, b+k; c+k; 1) = \frac{(c)_k}{(c-a-b-k)_k} F(a, b; c; 1), \qquad (2.6)$$

for $k = 0, 1, 2, \ldots$ if $c > a + b + k$

(ii)

$$\sum_{n=1}^{\infty} n \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{ab}{c-a-b-1} F(a, b; c; 1) \qquad (2.7)$$

if $c > a + b + 1$

(iii)

$$\sum_{n=1}^{\infty} n^2 \frac{(a)_n (b)_n}{(c)_n (1)_n} = \left(\frac{(a)_2 (b)_2}{(c-a-b-2)_2} + \frac{ab}{c-a-b-1} \right) F(a,b;c;1) \qquad (2.8)$$

if $c > a + b + 2$.

CHAPTER 3

CONVOLUTIONS OF MEROMORPHIC MULTIVALENT FUNCTIONS WITH RESPECT TO N-PLY POINTS AND SYMMETRIC CONJUGATE POINTS

3.1 Introduction

The class of analytic function starlike with respect to symmetric points which was defined in Section 1.1, p. 7 can be defined more generally as in the following definition:

Definition 3.1 *Let h be a normalized function with $\operatorname{Re} h > 0$, and $h(0) = 1$. The class $\mathcal{S}^*_{\mathcal{S}}(h)$ consists of functions $f \in \mathcal{A}$ satisfying*

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec h(z).$$

Remark 3.1.1 *When $h(z) = (1+z)/(1-z)$, the class $\mathcal{S}^*_{\mathcal{S}}(h)$ reduces to the class of starlike functions with respect to symmetric points introduced by Sakaguchi* [118]. *Sakaguchi proved that the condition (1.5) is a necessary and sufficient condition for a function, $f \in \mathcal{A}$ to be univalent and starlike with respect to symmetric points. Later Das and Singh* [33] *extend the results of Sakaguchi for classes of convex and close-to-convex functions with respect to symmetric points.*

Similarly, the classes of analytic functions starlike with respect to conjugate and symmetric conjugate points which were defined in Section 1.1, p. 8 can be define in the following form respectively:

Definition 3.2 *Let h be a normalized function with $\operatorname{Re} h > 0$, and $h(0) = 1$. The class $\mathcal{S}^*_{\mathcal{C}}(h)$ consists of functions $f \in \mathcal{A}$ satisfying*

$$\frac{2zf'(z)}{f(z) + \overline{f(\bar{z})}} \prec h(z).$$

Definition 3.3 *Let h be a normalized function with* $\operatorname{Re} h > 0$, *and* $h(0) = 1$. *The class* $\mathcal{S}^*_{\mathcal{SC}}(h)$ *consists of functions* $f \in \mathcal{A}$ *satisfying*

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} \prec h(z).$$

Remark 3.1.2 *When* $h(z) = (1+z)/(1-z)$, *the classes* $\mathcal{S}^*_{\mathcal{C}}(h)$ *and* $\mathcal{S}^*_{\mathcal{SC}}(h)$ *reduces to the classes of starlike functions with respect to conjugate and symmetric conjugate points introduced by El-Ashwah and Thomas* [39].

Ravichandran [103] unified the classes of starlike, convex and close-to-convex functions with respect to $n-$ply symmetric points, conjugate points and symmetric conjugate points, and obtained several convolution properties. Ravichandran introduced these classes by using subordination as in the following definitions.

The classes of starlike and convex functions with respect to $n-$ply points are defined in the following definitions respectively:

Definition 3.4 [103] *Let* $\mathcal{ST}^n(h)$ *denote the class of all functions* $f \in \mathcal{A}$ *satisfying* $f_n(z)/z \neq 0$ *in* $\mathbb{D}$ *and*

$$\frac{zf'(z)}{f_n(z)} \prec h(z).$$

Denote by $\mathcal{ST}^n_g(h)$ *the class*

$$\mathcal{ST}^n_g(h) := \{f \in \mathcal{A} : f * g \in \mathcal{ST}^n(h)\}.$$

Definition 3.5 [103] *Let* $\mathcal{CV}^n(h)$ *denote the class of all functions* $f \in \mathcal{A}$ *satisfying* $f'_n(z) \neq 0$ *in* $\mathbb{D}$ *and*

$$\frac{(zf')'(z)}{f'_n(z)} \prec h(z).$$

Denote by $\mathcal{CV}^n_g(h)$ *the class*

$$\mathcal{CV}^n_g(h) := \{f \in \mathcal{A} : f * g \in \mathcal{CV}^n(h)\}.$$

Remark 3.1.3 *If* $n = 1$, *the classes introduced in the Definitions 3.4 and 3.5 coincide with the classes introduced by Shanmugam* [119].
If $g(z) = z/(1-z)$ *and* $h(z) = (1 + (1-2\beta)z)/(1-z)$, *then we have the classes introduced by Chand and Singh* [31]

The classes of close-to-convex and quasi-convex functions with respect to $n-$ply points are defined in the following definitions respectively:

Definition 3.6 [103] *Let* $\mathcal{CC}^n(h)$ *denote the class of all functions* $f \in \mathcal{A}$ *satisfying* $\phi_n(z)/z \neq 0$ *in* $\mathbb{D}$ *and*

$$\frac{zf'(z)}{\phi_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{ST}^n(h)$. *Denote by* $\mathcal{CC}_g^n(h)$ *the class*

$$\mathcal{CC}_g^n(h) := \{f \in \mathcal{A} : f * g \in \mathcal{CC}^n(h)\}.$$

Definition 3.7 [103] *Let* $\mathcal{QC}^n(h)$ *denote the class of all functions* $f \in \mathcal{A}$ *satisfying* $\phi_n'(z)/z \neq 0$ *in* $\mathbb{D}$ *and*

$$\frac{(zf')'(z)}{\phi_n'(z)} \prec h(z).$$

for some $\phi \in \mathcal{CV}^n(h)$. *Denote by* $\mathcal{QC}_g^n(h)$ *the class*

$$\mathcal{QC}_g^n(h) := \{f \in \mathcal{A} : f * g \in \mathcal{QC}^n(h)\}.$$

Remark 3.1.4 *If* $n = 1$, *the classes introduced in the Definitions 3.6 and 3.7 coincide with the classes introduced by Shanmugam* [119].

The classes of starlike and convex functions with respect to $n-$ply symmetric, conjugate and symmetric conjugate points are defined in the following definitions respectively:

Definition 3.8 [103] *Let* $\mathcal{STS}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $\big(f_n(z) - f_n(-z)\big)/z \neq 0$ *and satisfying*

$$\frac{2zf'(z)}{f_n(z) - f_n(-z)} \prec h(z).$$

Denote by $\mathcal{STS}_g^n(h)$ *the class*

$$\mathcal{STS}_g^n(h) := \left\{ f \in \mathcal{A} : f * g \in \mathcal{STS}^n(h) \right\}.$$

Definition 3.9 [103] *Let* $\mathcal{CVS}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $f_n'(z) + f_n'(-z) \neq 0$ *and satisfying*

$$\frac{2(zf')'(z)}{f_n'(z) + f_n'(-z)} \prec h(z).$$

Denote by $\mathcal{CVS}_g^n(h)$ *the class*

$$\mathcal{CVS}_g^n(h) := \left\{ f \in \mathcal{A} : f * g \in \mathcal{CVS}^n(h) \right\}.$$

Definition 3.10 [103] *Let* $\mathcal{STC}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $\big(f_n(z) + \overline{f_n}(\bar{z})\big)/z \neq 0$ *and satisfying*

$$\frac{2zf'(z)}{f_n(z) + \overline{f_n}(\bar{z})} \prec h(z).$$

Denote by $\mathcal{STC}_g^n(h)$ *the class*

$$\mathcal{STC}_g^n(h) := \left\{ f \in \mathcal{A} : f * g \in \mathcal{STC}^n(h) \right\}.$$

Definition 3.11 [103] *Let* $\mathcal{CVC}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $f_n'(z) + \overline{f_n'}(\bar{z}) \neq 0$ *and satisfying*

$$\frac{2(zf')'(z)}{f_n'(z) + \overline{f_n'}(\bar{z})} \prec h(z).$$

Denote by $\mathcal{CVC}_g^n(h)$ *the class*

$$\mathcal{CVC}_g^n(h) := \{f \in \mathcal{A} : f * g \in \mathcal{CVC}^n(h)\}.$$

Definition 3.12 [103] *Let* $\mathcal{STSC}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $\big(f_n(z) - \overline{f_n}(-\bar{z})\big)/z \neq 0$ *and satisfying*

$$\frac{2zf'(z)}{f_n(z) - \overline{f_n}(-\bar{z})} \prec h(z).$$

Denote by $\mathcal{STSC}_g^n(h)$ *the class*

$$\mathcal{STSC}_g^n(h) := \{f \in \mathcal{A} : f * g \in \mathcal{STSC}^n(h)\}.$$

Definition 3.13 [103] *Let* $\mathcal{CVSC}^n(h)$ *denote of all functions* $f \in \mathcal{A}$ *such that* $f_n'(z) + \overline{f_n'}(-\bar{z}) \neq 0$ *and satisfying*

$$\frac{2(zf')'(z)}{f_n'(z) + \overline{f_n'}(-\bar{z})} \prec h(z).$$

Denote by $\mathcal{CVSC}_g^n(h)$ *the class*

$$\mathcal{CVSC}_g^n(h) := \{f \in \mathcal{A} : f * g \in \mathcal{CVSC}^n(h)\}.$$

Remark 3.1.5 *If* $n = 1, g(z) = z/(1-z)$ *and* $h(z) = (1 + (1-2\alpha))/(1-z)$, *then, the classes introduced in Definitions 3.8 and 3.9 are the familiar classes*

$\mathcal{S}^*_{\mathcal{S}}(\alpha), \mathcal{C}_{\mathcal{S}}(\alpha)$ of starlike functions, convex functions with respect to symmetric points in $\mathbb{D}$ by Das and Singh [33]*. If $n = 1, g(z) = z/(1-z)$ and $h(z) = (1+z)/(1-z)$, then, the classes introduced in Definitions 3.10 through 3.13 are the classes of starlike and convex functions with respect to conjugate and symmetric conjugate points in $\mathbb{D}$ introduced by El-Ashwah and Thomas* [39]*.*

For the class defined in the Definition 3.4, Ravichandran [103] proved the following:

Theorem 3.1 *If $f \in \mathcal{ST}^n_g(h)$ and $\phi \in \mathcal{R}_\alpha$, then $\phi * f \in \mathcal{ST}^n_g(h)$.*

He also proved similar results for the classes defined in Definitions 3.4–3.13. These works were recently extended for multivalent functions by Ali *et al.* [18].

By using the convex hull method [113] and the method of differential subordination [72], Shanmugam [119] introduced and investigated convolution properties of various subclasses of analytic functions, whereas Ali *el al*. [11] and Supramaniam *et al.* [135] investigated these properties for subclasses of multivalent starlike and convex functions. Similar problems were also investigated for meromorphic functions in [19, 49, 146]. Motivated by the works in [18, 19, 49, 103, 119], in this chapter, certain subclasses of meromorphic multivalent functions in the punctured unit disk $\mathbb{D}^* := \{z \in \mathbb{C} : 0 < |z| < 1\}$ defined by means of convolution with a given fixed meromorphic multivalent function are introduced, and their closure properties under convolution are investigated.

3.2 Functions with Respect to n-ply Points

Let $\mathcal{M}_p$ denote the class of all meromorphic p-valent functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \geq 1), \tag{3.1}$$

that are analytic in the open punctured unit disk $\mathbb{D}^*$ with $\mathcal{M} := \mathcal{M}_1$. Analogous to classes of starlike and convex analytic functions, classes of meromorphic multivalent starlike and convex functions, and other related subclasses of meromorphic multivalent functions, are expressed in the form

$$\mathcal{MST}_p(g,h) := \left\{ f \in \mathcal{M}_p : \; -\frac{1}{p}\frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z) \right\},$$

where g is a fixed function in $\mathcal{M}_p$, and h a suitably normalized analytic function with positive real part. For instance, the class of meromorphic multivalent starlike functions of order α, $0 \leq \alpha < 1$, defined by

$$\mathcal{MST}_p(\alpha) := \left\{ f \in \mathcal{M}_p : \; -\operatorname{Re}\frac{1}{p}\frac{zf'(z)}{f(z)} > \alpha \right\}$$

is a particular case of $\mathcal{MST}_p(g,h)$ with $g(z) = 1/(z^p(1-z))$ and $h(z) = (1+(1-2\alpha)z)/(1-z)$.

Remark 3.2.1 *When $p = 1$ and $\alpha = 0$, the class $\mathcal{MST}_p(\alpha) = \mathcal{MST}_1(0)$ will reduced to the well known class of meromorphic univalent starlike function.*

In this section, four classes $\mathcal{MST}_p^n(g,h)$, $\mathcal{MCV}_p^n(g,h)$, $\mathcal{MCC}_p^n(g,h)$ and $\mathcal{MQC}_p^n(g,h)$ of meromorphic multivalent functions with respect to n-ply points are introduced and the convolution properties of these new subclasses are investigated. These new subclasses extend the classical classes of meromorphic multivalent starlike, convex, close-to-convex and quasi-convex functions respectively.

Let the function $g \in \mathcal{M}_p$ be fixed, and h be a convex univalent function with positive real part satisfying $h(0) = 1$. On certain occasions, we would additionally require that $\operatorname{Re} h(z) < 1 + (1-\alpha)/p$, where $0 \leq \alpha < 1$.

Definition 3.14 *Let $n \geq 1$ be any integer, $\epsilon^n = 1$ and $\epsilon \neq 1$. For $f \in \mathcal{M}_p$ of the form (3.1), let the function with n-ply points $f_n \in \mathcal{M}_p$ be defined by*

$$f_n(z) := \frac{1}{n}\sum_{k=0}^{n-1} \epsilon^{n+pk} f\left(\epsilon^k z\right) = z^{-p} + a_{n-p}z^{n-p} + a_{2n-p}z^{2n-p} + \cdots . \qquad (3.2)$$

The classes of meromorphic multivalent starlike and convex functions with respect to n-ply points are defined in the following definitions:

Definition 3.15 *The class* $\mathcal{MST}_p^n(h)$ *consists of functions* $f \in \mathcal{M}_p$ *satisfying* $f_n(z) \neq 0$ *in* $\mathbb{D}^*$ *and the subordination*

$$-\frac{1}{p}\frac{zf'(z)}{f_n(z)} \prec h(z).$$

The general class $\mathcal{MST}_p^n(g,h)$ *consists of functions* $f \in \mathcal{M}_p$ *for which* $f * g$, *belongs to* $\mathcal{MST}_p^n(h)$.

Definition 3.16 *The class* $\mathcal{MCV}_p^n(h)$ *consists of functions* $f \in \mathcal{M}_p$ *satisfying* $f_n'(z) \neq 0$ *in* $\mathbb{D}^*$ *and the subordination*

$$-\frac{1}{p}\frac{\left(zf'(z)\right)'}{f_n'(z)} \prec h(z).$$

The general class $\mathcal{MCV}_p^n(g,h)$ *consists of functions* $f \in \mathcal{M}_p$ *for which* $f*g$, *belongs to* $\mathcal{MCV}_p^n(h)$.

Remark 3.2.2 *If* $g(z) := 1/z^p(1-z)$, *then the classes* $\mathcal{MST}_p^n(g,h)$ *and* $\mathcal{MCV}_p^n(g,h)$ *coincide with* $\mathcal{MST}_p^n(h)$ *and* $\mathcal{MCV}_p^n(h)$ *respectively. If* $p = 1$ *and* $n = 1$, *then the classes* $\mathcal{MST}_p^n(g,h)$ *and* $\mathcal{MCV}_p^n(g,h)$ *reduced respectively to* $\mathcal{MST}(g,h)$ *and* $\mathcal{MCV}(g,h)$ *which were introduced and investigated in* [49]. *The notation* $\mathcal{MST}_p(h)$ *will be used for the class* $\mathcal{MST}_p^1(h)$.

Theorem 3.2 *Let* h *be a convex univalent function satisfying*

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1), \tag{3.3}$$

and $\phi \in \mathcal{M}_p$ *with* $z^{p+1}\phi \in \mathcal{R}_\alpha$.

(i) *If* $f \in \mathcal{MST}_p^n(g,h)$, *then* $\phi * f \in \mathcal{MST}_p^n(g,h)$.

(ii) If $f \in \mathcal{MCV}_p^n(g,h)$, then $\phi * f \in \mathcal{MCV}_p^n(g,h)$.

Proof. (i) The proof begins by showing that if $f \in \mathcal{MST}_p^n(h)$, then $\phi * f \in \mathcal{MST}_p^n(h)$. Let $f \in \mathcal{MST}_p^n(h)$, and define the functions H and ψ by

$$H(z) := -\frac{zf'(z)}{pf_n(z)} \quad \text{and} \quad \psi(z) := z^{p+1} f_n(z).$$

Thus for any fixed $z \in \mathbb{D}$,

$$-\frac{zf'(z)}{pf_n(z)} \in h(\mathbb{D}) \tag{3.4}$$

Replacing z by $\epsilon^k z$ in (3.2), it follows that

$$\begin{aligned}
f_n(\epsilon^k z) &= (\epsilon^k z)^{-p} + a_{n-p}(\epsilon^k z)^{n-p} + a_{2n-p}(\epsilon^k z)^{2n-p} + \dots \\
&= \epsilon^{-pk} z^{-p} + a_{n-p}\epsilon^{k(n-p)} z^{n-p} + a_{2n-p}\epsilon^{k(2n-p)} z^{2n-p} + \dots \\
&= \epsilon^{-pk} z^{-p} + a_{n-p}\epsilon^{kn-kp} z^{n-p} + a_{2n-p}\epsilon^{2kn-kp} z^{2n-p} + \dots \\
&= \epsilon^{-pk} z^{-p} + a_{n-p}(1)\epsilon^{-pk} z^{n-p} + a_{2n-p}(1)\epsilon^{-kp} z^{2n-p} + \dots \quad (\epsilon^n = 1) \\
&= \epsilon^{-pk} \left(z^{-p} + a_{n-p} z^{n-p} + a_{2n-p} z^{2n-p} + \dots \right) \\
&= \epsilon^{-pk} f_n(z)
\end{aligned}$$

Therefore

$$f_n(\epsilon^k z) = \epsilon^{-pk} f_n(z). \tag{3.5}$$

By replacing z by $\epsilon^k z$ in (3.4), it follows that

$$-\frac{(\epsilon^k z) f'(\epsilon^k z)}{pf_n(\epsilon^k z)} \in h(\mathbb{D}).$$

In view of (3.5), it is clear that

$$-\frac{(\epsilon^k z) f'(\epsilon^k z)}{pf_n(\epsilon^k z)} = -\frac{(\epsilon^k z) f'(\epsilon^k z)}{p\epsilon^{-pk} f_n(z)}$$

$$= -\frac{z\epsilon^{k+pk} f'(\epsilon^k z)}{p f_n(z)}.$$

Thus, the containment (3.4) becomes

$$-\frac{z\epsilon^{k(1+p)} f'(\epsilon^k z)}{p f_n(z)} \in h(\mathbb{D}).$$

Since $h(\mathbb{D})$ is a convex domain, this yields

$$-\frac{1}{n}\sum_{k=0}^{n-1} \frac{\epsilon^{k(1+p)} z f'(\epsilon^k z)}{p f_n(z)} \in h(\mathbb{D}). \tag{3.6}$$

From (3.2), it is evident that

$$\begin{aligned} f_n'(z) &= \frac{1}{n}\sum_{k=0}^{n-1} \epsilon^{(n+pk)} f'(\epsilon^k z)\epsilon^k \\ &= \frac{1}{n}\sum_{k=0}^{n-1} \epsilon^{k(1+p)} f'(\epsilon^k z). \end{aligned} \tag{3.7}$$

Now, by using (3.7) in (3.6), it follows that

$$-\frac{z f_n'(z)}{p f_n(z)} \in h(\mathbb{D})$$

and thus in terms of subordination becomes

$$-\frac{z f_n'(z)}{p f_n(z)} \prec h(z). \tag{3.8}$$

Hence $f_n \in \mathcal{MST}_p(h)$. Logarithmic differentiation of $\psi(z) = z^{p+1} f_n(z)$ yields

$$\frac{z\psi'(z)}{\psi(z)} = \frac{z f_n'(z)}{f_n(z)} + p + 1. \tag{3.9}$$

By applying (3.3) in (3.8), it follows that

$$-\operatorname{Re}\left(\frac{zf_n'(z)}{pf_n(z)}\right) < 1 + \frac{1-\alpha}{p}$$

which in turn implies

$$\operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) > -p - 1 + \alpha. \tag{3.10}$$

From (3.9), it follows that

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) = \operatorname{Re}\left(\frac{zf_n'(z)}{f_n(z)}\right) + p + 1$$

and by using (3.3)

$$\operatorname{Re}\left(\frac{z\psi'(z)}{\psi(z)}\right) > -p - 1 + \alpha + p + 1 = \alpha. \tag{3.11}$$

Inequality (3.11) shows that the function ψ belongs to $\mathcal{ST}(\alpha)$. Computation shows that

$$\begin{aligned}
-\frac{z(\phi * f)'(z)}{p(\phi * f)_n(z)} &= \frac{-zp^{-1}(\phi * f)'(z)}{(\phi * f)_n(z)} \\
&= \frac{\big(\phi * (-p^{-1}zf')\big)(z)}{(\phi * f_n)(z)} \\
&= \frac{\big(\phi * (Hf_n)\big)(z)}{(\phi * f_n)(z)} \\
&= \frac{\big(z^{p+1}\phi(z)\big) * \big(H(z)\psi(z)\big)}{\big(z^{p+1}\phi(z)\big) * (\psi(z))}.
\end{aligned}$$

Since $z^{p+1}\phi \in \mathcal{R}_\alpha$ and $\psi \in \mathcal{ST}(\alpha)$, Theorem 2.3 yields

$$\frac{\big(z^{p+1}\phi(z)\big) * \big(H(z)\psi(z)\big)}{\big(z^{p+1}\phi(z)\big) * (\psi(z))} \in \overline{co}(H(\mathbb{D})).$$

The subordination $H \prec h$ implies

$$-\frac{z(\phi * f)'(z)}{p(\phi * f)_n(z)} \prec h(z).$$

Thus $\phi * f \in \mathcal{MST}_p^n(h)$. The general case follows from the fact that

$$f \in \mathcal{MST}_p^n(g, h) \Leftrightarrow f * g \in \mathcal{MST}_p^n(h). \tag{3.12}$$

Indeed, if $f \in \mathcal{MST}_p^n(g, h)$, then $f * g \in \mathcal{MST}_p^n(h)$, and therefore $\phi * f * g \in \mathcal{MST}_p^n(h)$, or equivalently $\phi * f \in \mathcal{MST}_p^n(g, h)$.

(ii) The identity

$$-\frac{\big(z(g * f)'(z)\big)'}{p(g * f)_n'(z)} = -\frac{z\big(g * -p^{-1}zf'\big)'(z)}{p\big(g * -p^{-1}zf'\big)_n(z)}$$

shows that

$$f \in \mathcal{MCV}_p^n(g, h) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MST}_p^n(g, h), \tag{3.13}$$

and by the result of part (i), it is clear that

$$\phi * \left(-\frac{zf'}{p}\right) \in \mathcal{MST}_p^n(g, h)$$

which implies

$$-\frac{z(\phi * f)'}{p} \in \mathcal{MST}_p^n(g, h).$$

Hence, in view of (3.13), $\phi * f \in \mathcal{MCV}_p^n(g, h)$. ∎

Corollary 3.2.1 *The following inclusions hold:*

(i) $\mathcal{MST}_p^n(g, h) \subset \mathcal{MST}_p^n(\phi * g, h)$

(ii) $\mathcal{MCV}_p^n(g, h) \subset \mathcal{MCV}_p^n(\phi * g, h)$

Proof. (i) In view of the fact (3.12) and Theorem 3.2(i), it is evident that $\phi * f * g \in \mathcal{MST}_p^n(h)$, or equivalently $f * (\phi * g) \in \mathcal{MST}_p^n(h)$. Hence $f \in \mathcal{MST}_p^n(\phi * g, h)$.

(ii) In view of the fact that $f \in \mathcal{MCV}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MCV}_p^n(h)$, and Theorem 3.2(ii), it follows that $\phi * f * g \in \mathcal{MCV}_p^n(h)$ or $f * (\phi * g) \in \mathcal{MCV}_p^n(h)$. Therefore $f \in \mathcal{MCV}_p^n(\phi * g, h)$. ∎

When $n = 1$ and $p = 1$, the following corollaries are easily deduced from Theorem 3.2(i) and Theorem 3.2(ii) respectively:

Corollary 3.2.2 [49, Theorem 3.3] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$, *and $\phi \in \mathcal{M}$ with $z^2\phi \in \mathcal{R}_\alpha$. If $f \in \mathcal{MST}(g,h)$ then $\phi * f \in \mathcal{MST}(g,h)$.*

Corollary 3.2.3 [49, Theorem 3.6] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$, *and $\phi \in \mathcal{M}$ with $z^2\phi \in \mathcal{R}_\alpha$. If $f \in \mathcal{MCV}(g,h)$ then $\phi * f \in \mathcal{MCV}(g,h)$.*

If $n = 1, p = 1$ and $g(z) = 1/z(1-z)$, then the following result follows from Theorem 3.2(i):

Corollary 3.2.4 [49, Theorem 3.5] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$, *and $\phi \in \mathcal{M}$ with $z^2\phi \in \mathcal{R}_\alpha$. If $f \in \mathcal{MST}(h)$, then $f \in \mathcal{MST}(\phi, h)$*

The classes of meromorphic multivalent close-to-convex and quasi-convex functions with respect to n-ply points are defined in the following definitions:

Definition 3.17 *The class $\mathcal{MCC}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying the subordination*

$$-\frac{1}{p}\frac{zf'(z)}{\phi_n(z)} \prec h(z)$$

for some $\phi \in \mathcal{MST}_p^n(h)$ with $\phi_n(z) \neq 0$ in $\mathbb{D}^*$. The general class $\mathcal{MCC}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MCC}_p^n(h)$.

Definition 3.18 *The class $\mathcal{MQC}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying the subordination*

$$-\frac{1}{p}\frac{\left(zf'(z)\right)'}{\varphi_n'(z)} \prec h(z)$$

for some $\varphi \in \mathcal{MCV}_p^n$ with $\varphi_n'(z) \neq 0$ in $\mathbb{D}^*$. The general class $\mathcal{MQC}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MQC}_p^n(h)$.

Remark 3.2.3 *If $g(z) := 1/z^p(1-z)$, then the classes $\mathcal{MCC}_p^n(g,h)$ and $\mathcal{MQC}_p^n(g,h)$ coincide with $\mathcal{MCC}_p^n(h)$ and $\mathcal{MQC}_p^n(h)$ respectively. If $p = 1$ and $n = 1$, then the classes $\mathcal{MCC}_p^n(g,h)$ and $\mathcal{MQC}_p^n(g,h)$ reduced respectively to $\mathcal{MCC}(g,h)$ and $\mathcal{MQC}(g,h)$ which were introduced and investigated in* [49].

Theorem 3.3 *Let h be a convex univalent function satisfying*

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1),$$

and $\phi \in \mathcal{M}_p$ with $z^{p+1}\phi \in \mathcal{R}_\alpha$.

(i) If $f \in \mathcal{MCC}_p^n(g,h)$ with respect to a function $\varphi \in \mathcal{MST}_p^n(g,h)$, then $\phi * f \in \mathcal{MCC}_p^n(g,h)$ with respect to $\phi * \varphi \in \mathcal{MST}_p^n(g,h)$.

(ii) If $f \in \mathcal{MQC}_p^n(g,h)$ with respect to a function $\varphi \in \mathcal{MCV}_p^n(g,h)$, then $\phi * f \in \mathcal{MQC}_p^n(g,h)$ with respect to $\phi * \varphi \in \mathcal{MCV}_p^n(g,h)$.

Proof. (i) It is sufficient to prove that $\phi * f \in \mathcal{MCC}_p^n(h)$ whenever $f \in \mathcal{MCC}_p^n(h)$. Let $f \in \mathcal{MCC}_p^n(h)$ with respect to a function $\varphi \in \mathcal{MST}_p^n(h)$. Since

$$\varphi \in \mathcal{MST}_p^n(h),$$

Theorem 3.2 yields

$$\phi * \varphi \in \mathcal{MST}_p^n(h).$$

Also it is easy to see from (3.8) and (3.3) that

$$\varphi_n \in \mathcal{MST}_p(h) \quad \text{and} \quad z^{p+1}\varphi_n(z) \in \mathcal{ST}(\alpha).$$

Now, define the functions $H(z)$ and $\psi(z)$ by

$$H(z) = -\frac{zf'(z)}{p\varphi_n(z)} \quad \text{and} \quad \psi(z) = z^{p+1}\varphi_n(z).$$

Since $z^{p+1}\phi(z) \in R_\alpha$ and $H(z) \prec h(z)$, an application of Theorem 2.3 shows that

$$\begin{aligned}
&-\frac{z(\phi * f)'(z)}{p(\phi * \varphi)_n(z)} \\
&= \frac{\big(\phi * (-zp^{-1}f')\big)(z)}{(\phi * \varphi_n)(z)} \\
&= \frac{\big(\phi * (H(z)\varphi_n)\big)(z)}{(\phi * \varphi_n)(z)} \\
&= \frac{\big(z^{p+1}\phi(z)\big) * (H(z)\psi(z))}{\big(z^{p+1}\phi(z)\big) * (\psi(z))} \prec h(z).
\end{aligned}$$

Thus $\phi * f \in \mathcal{MCC}_p^n(h)$. The general case follows from the fact that

$$f \in \mathcal{MCC}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MCC}_p^n(h). \tag{3.14}$$

Indeed, if $f \in \mathcal{MCC}_p^n(g,h)$, then $f * g \in \mathcal{MCC}_p^n(h)$, and therefore $\phi * f * g \in \mathcal{MCC}_p^n(h)$, or equivalently $\phi * f \in \mathcal{MCC}_p^n(g,h)$.

(ii) The proof begins with the identity

$$-\frac{\big(z(g * f)'(z)\big)'}{p(g * \varphi)_n'(z)} = -\frac{z\left(g * -\frac{zf'}{p}\right)'}{\left(g * -\frac{z\varphi'}{p}\right)_n},$$

which shows that the function

$$f \in \mathcal{MQC}_p^n(g,h) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MCC}_p^n(g,h). \tag{3.15}$$

From the result of part (i), it follows that

$$\phi * \left(-\frac{zf'}{p}\right) \in \mathcal{MCC}_p^n(g,h)$$

which implies

$$-\frac{z}{p}(\phi * f)'(z) \in \mathcal{MCC}_p^n(g,h).$$

Hence, in view of (3.15) $\phi * f \in \mathcal{MQC}_p^n(g,h)$. This complete the proof. ∎

Corollary 3.2.5 *The following inclusions hold:*

(i) $\mathcal{MCC}_p^n(g,h) \subset \mathcal{MCC}_p^n(\phi * g,h)$

(ii) $\mathcal{MQC}_p^n(g,h) \subset \mathcal{MQC}_p^n(\phi * g,h)$

Proof. (i) From the fact (3.14), it is clear that if $f \in \mathcal{MCC}_p^n(g,h)$, then $f * g \in \mathcal{MCC}_p^n(h)$. Theorem 3.3(i), yields $\phi * f * g \in \mathcal{MCC}_p^n(h)$, or $f * (\phi * g) \in \mathcal{MCC}_p^n(h)$. Therefore $f \in \mathcal{MCC}_p^n(\phi * g,h)$.

(ii) From the fact that $f \in \mathcal{MQC}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MQC}_p^n(h)$, and Theorem 3.3(ii), it follows that $\phi * f * g \in \mathcal{MQC}_p^n(h)$ or $f * (\phi * g) \in \mathcal{MQC}_p^n(h)$ Hence $f \in \mathcal{MQC}_p^n(\phi * g,h)$. This complete the proof. ∎

When $n = 1$ and $p = 1$, the following corollaries are special cases of Theorem 3.3(i) and Theorem 3.3(ii) respectively:

Corollary 3.2.6 [49, Theorem 3.7] *Under the conditions of Corollary 3.2.2, if $f \in \mathcal{MCC}(g,h)$ with respect to $\psi \in \mathcal{MST}(g,h)$, then $\phi * f \in \mathcal{MCC}(g,h)$ with respect to $\phi * \psi \in \mathcal{MST}(g,h)$.*

Corollary 3.2.7 [49, Theorem 3.12] *Let h and ϕ satisfy the conditions of Corollary 3.2.2. If $f \in \mathcal{MQC}(g,h)$, then $\phi * f \in \mathcal{MQC}(g,h)$.*

3.3 Functions with Respect to n-ply Symmetric Points

In this section and the following two sections, it is assumed that p is an odd number. As before, it is assumed that the function $g \in \mathcal{M}_p$ is a fixed function and the function h is convex univalent with positive real part satisfying $h(0) = 1$.

The classes of meromorphic multivalent starlike and convex functions with respect to n-ply symmetric points are defined in the following definitions:

Definition 3.19 *For odd integer p, the class $\mathcal{MSTS}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying $f_n(z) - f_n(-z) \neq 0$ in $\mathbb{D}^*$ and the subordination*

$$-\frac{1}{p}\frac{2zf'(z)}{f_n(z) - f_n(-z)} \prec h(z). \tag{3.16}$$

The general class $\mathcal{MSTS}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MSTS}_p^n(h)$.

Definition 3.20 *For odd integer p, the class $\mathcal{MCVS}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying $f_n'(z) + f_n'(-z) \neq 0$ in $\mathbb{D}^*$ and the subordination*

$$-\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + f_n'(-z)} \prec h(z).$$

The general class $\mathcal{MCVS}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MCVS}_p^n(h)$.

Theorem 3.4 *Let h be a convex univalent function satisfying*

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1),$$

and $\phi \in \mathcal{M}_p$ *with* $z^{p+1}\phi \in \mathcal{R}_\alpha$.

(i) If $f \in \mathcal{MSTS}_p^n(g,h)$, *then* $\phi * f \in \mathcal{MSTS}_p^n(g,h)$.

(ii) If $f \in \mathcal{MCVS}_p^n(g,h)$, *then* $\phi * f \in \mathcal{MCVS}_p^n(g,h)$.

Proof. (i) It is enough to show that if f is in $\mathcal{MSTS}_p^n(h)$, then so is $\phi * f$. Define the functions F_n, H and Ψ by

$$F_n(z) = \frac{f_n(z) - f_n(-z)}{2}, \quad H(z) := -\frac{zf'(z)}{pF_n(z)} \quad \text{and} \quad \Psi(z) := z^{p+1}F_n(z).$$

When $n = 1,\ F_{(1)}(z) = F(z)$.

Let $f \in \mathcal{MSTS}_p^n(h)$. Thus for any fixed $z \in \mathbb{D}$,

$$-\frac{zf'(z)}{pF_n(z)} \in h(\mathbb{D}). \tag{3.17}$$

Replacing z by $-z$ in (3.17) and taking the convex combinations, it follows that

$$\begin{aligned} -\frac{zf'(z)}{2pF_n(z)} - \frac{(-z)f'(-z)}{2pF_n(-z)} &= -\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \frac{(-z)f'(-z)}{F_n(-z)}\right) \\ &= -\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \frac{(-z)f'(-z)}{-F_n(z)}\right) \\ &= -\frac{z\left(\frac{f'(z)+f'(-z)}{2}\right)}{pF_n(z)} \in h(\mathbb{D}) \end{aligned}$$

which implies

$$-\frac{zF'(z)}{pF_n(z)} \in h(\mathbb{D}).$$

This shows that the function $F \in \mathcal{MST}_p^n(h)$, and the proof of Theorem 3.2 now shows that $F_n \in \mathcal{MST}_p(h)$, or

$$-\frac{zF_n'(z)}{pF_n(z)} \prec h(z). \tag{3.18}$$

A computation from

$$\Psi(z) = z^{p+1}F_n(z)$$

shows that

$$\frac{z\Psi'(z)}{\Psi(z)} = \frac{zF_n'(z)}{F_n(z)} + p + 1. \tag{3.19}$$

By applying (3.3) in (3.18), it follows that

$$-\operatorname{Re}\left(\frac{zF_n'(z)}{pF_n(z)}\right) < 1 + \frac{1-\alpha}{p}$$

which implies

$$\operatorname{Re}\left(\frac{zF_n'(z)}{F_n(z)}\right) > -p - 1 + \alpha. \tag{3.20}$$

From (3.19), it is clear that

$$\operatorname{Re}\left(\frac{z\Psi'(z)}{\Psi(z)}\right) = \operatorname{Re}\left(\frac{zF_n'(z)}{F_n(z)}\right) + p + 1$$

and by using (3.20)

$$\operatorname{Re}\left(\frac{z\Psi'(z)}{\Psi(z)}\right) > -p - 1 + \alpha + p + 1 = \alpha. \tag{3.21}$$

Inequality (3.21) shows that the function Ψ belongs to $\mathcal{ST}(\alpha)$. Another computation shows that

$$-\frac{2}{p}\frac{z(\phi * f)'(z)}{(\phi * f)_n(z) - (\phi * f)_n(-z)} = \frac{\phi(z) * -zp^{-1}f'(z)}{\phi(z) * \frac{f_n(z)-f_n(-z)}{2}} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)}.$$

Since $z^{p+1}\phi \in \mathcal{R}_\alpha$ and $\Psi \in \mathcal{ST}(\alpha)$, Theorem 2.3 yields

$$\frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} = \frac{\big(z^{p+1}\phi(z)\big) * \big(H(z)\psi(z)\big)}{\big(z^{p+1}\phi(z)\big) * \big(\psi(z)\big)} \in \overline{co}\big(H(\mathbb{D})\big),$$

and because $H(z) \prec h(z)$, it follows that

$$-\frac{2}{p}\frac{z(\phi * f)'(z)}{(\phi * f)_n(z) - (\phi * f)_n(-z)} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z).$$

Hence $\phi * f \in \mathcal{MSTS}_p^n(h)$.The general result is obtained from the fact that

$$f \in \mathcal{MSTS}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MSTS}_p^n(h)$$

which gives $\phi * f * g \in \mathcal{MSTS}_p^n(h)$ or equivalently $\phi * f \in \mathcal{MSTS}_p^n(g,h)$.

(ii) The identity

$$-\frac{2(z(g*f)'(z))'}{p\big((g*f)_n'(z) + (g*f)_n'(-z)\big)} = -\frac{2z\left(g * -\frac{zf'}{p}\right)'(z)}{p\left(\left(g * -\frac{zf'}{p}\right)_n(z) - \left(g * -\frac{zf'}{p}\right)_n(-z)\right)}$$

shows that

$$f \in \mathcal{MCVS}_p^n(g,h) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MSTS}_p^n(g,h), \tag{3.22}$$

and, by the result of part (i), it is clear that

$$\phi * \left(-\frac{zf'}{p}\right) \in \mathcal{MSTS}_p^n(g,h)$$

which implies

$$-\frac{z(\phi * f)'}{p} \in \mathcal{MSTS}_p^n(g,h).$$

Hence, in view of (3.22), $\phi * f \in \mathcal{MCVS}_p^n(g,h)$. ∎

3.4 Functions with Respect to n-ply Conjugate Points

The classes of meromorphic multivalent starlike and convex functions with respect to n-ply conjugate points are defined in the following definitions:

Definition 3.21 *For odd integer p, the class $\mathcal{MSTC}_p^n(h)$ consists of functions*

$f \in \mathcal{M}_p$ *satisfying* $f_n(z) + \overline{f_n(\bar{z})} \neq 0$ *in* $\mathbb{D}^*$ *and the subordination*

$$-\frac{1}{p}\frac{2zf'(z)}{f_n(z) + \overline{f_n(\bar{z})}} \prec h(z). \tag{3.23}$$

The general class $\mathcal{MSTC}_p^n(g,h)$ *consists of functions* $f \in \mathcal{M}_p$ *for which* $f * g$*, belongs to* $\mathcal{MSTC}_p^n(h)$.

Definition 3.22 *For odd integer* p*, the class* $\mathcal{MCVC}_p^n(h)$ *consists of functions* $f \in \mathcal{M}_p$ *satisfying* $f_n'(z) + \overline{f_n'(\bar{z})} \neq 0$ *in* $\mathbb{D}^*$ *and the subordination*

$$-\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(\bar{z})}} \prec h(z).$$

The general class $\mathcal{MCVC}_p^n(g,h)$ *consists of functions* $f \in \mathcal{M}_p$ *for which* $f * g$*, belongs to* $\mathcal{MCVC}_p^n(h)$.

Theorem 3.5 *Let* h *be a convex univalent function satisfying*

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1),$$

and $\phi \in \mathcal{M}_p$ *with* $z^{p+1}\phi \in \mathcal{R}_\alpha$ *where* ϕ *has real coefficient.*

(i) If $f \in \mathcal{MSTC}_p^n(g,h)$*, then* $\phi * f \in \mathcal{MSTC}_p^n(g,h)$.

(ii) If $f \in \mathcal{MCVC}_p^n(g,h)$*, then* $\phi * f \in \mathcal{MCVC}_p^n(g,h)$.

Proof. (i) The proof is similar to the proof of Theorem 3.4(i) except for the definition of the function F_n. Here the function F_n is defined by

$$F_n(z) = \frac{f_n(z) + \overline{f_n(\bar{z})}}{2}.$$

Let the functions H and Ψ be defined by

$$H(z) := -\frac{zf'(z)}{pF_n(z)} \quad \text{and} \quad \Psi(z) := z^{p+1}F_n(z).$$

Let $f \in \mathcal{MSTC}_p^n(h)$. Thus for any fixed $z \in \mathbb{D}$,

$$-\frac{zf'(z)}{pF_n(z)} \in h(\mathbb{D}). \tag{3.24}$$

Replacing z by $\overline{z}$ in (3.24) and taking the convex combinations, it follows that

$$\begin{aligned}
-\frac{zf'(z)}{2pF_n(z)} - \overline{\left(\frac{(\bar{z})f'(\bar{z})}{2pF_n(\bar{z})}\right)} &= -\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \overline{\left(\frac{(\bar{z})f'(\bar{z})}{F_n(\bar{z})}\right)}\right) \\
&= -\frac{1}{2p}\left(\frac{zf'(z)}{F_n(z)} + \frac{z\overline{f'(\bar{z})}}{F_n(z)}\right) \\
&= -\frac{z\left(\frac{f'(z)+\overline{f'(\bar{z})}}{2}\right)}{pF_n(z)} \in h(\mathbb{D})
\end{aligned}$$

which implies

$$-\frac{zF'(z)}{pF_n(z)} \in h(\mathbb{D}).$$

This shows that the function $F \in \mathcal{MST}_p^n(h)$, and the proof of Theorem 3.3 now shows that $F_n \in \mathcal{MST}_p(h)$. Since h is a convex function with

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p},$$

it follows that

$$\operatorname{Re}\frac{z\Psi'(z)}{\Psi(z)} = \operatorname{Re}\frac{zF_n'(z)}{F_n(z)} + p + 1 > \alpha,$$

and hence $\Psi \in \mathcal{ST}(\alpha)$. A computation shows that

$$-\frac{2z(\phi * f)'(z)}{p\left((\phi * f)_n(z) + \overline{(\phi * f)_n(\bar{z})}\right)} = \frac{\phi(z) * -zp^{-1}f'(z)}{\phi(z) * \frac{f_n(z)+\overline{f_n(\bar{z})}}{2}} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)}.$$

Since $z^{p+1}\phi \in \mathcal{R}_\alpha$ and $z^{p+1}F_n \in \mathcal{ST}(\alpha)$, Theorem 2.3 yields

$$\frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} = \frac{\big(z^{p+1}\phi(z)\big) * \big(H(z)z^{p+1}F_n(z)\big)}{\big(z^{p+1}\phi(z)\big) * \big(z^{p+1}F_n(z)\big)} \in \overline{co}\big(H(\mathbb{D})\big),$$

and because $H(z) \prec h(z)$, it follows that

$$-\frac{2}{p}\frac{z(\phi * f)'(z)}{(\phi * f)_n(z) + \overline{(\phi * f)_n(\bar{z})}} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z).$$

Hence $\phi * f \in \mathcal{MSTC}_p^n(h)$. The general result is obtained from the fact that

$$f \in \mathcal{MSTC}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MSTC}_p^n(h)$$

which gives $\phi * f * g \in \mathcal{MSTC}_p^n(h)$ or equivalently $\phi * f \in \mathcal{MSTC}_p^n(g,h)$.

(ii) The identity

$$-\frac{2\big(z(g * f)'(z)\big)'}{p\left((g * f)'_n(z) + \overline{(g * f)'_n(\bar{z})}\right)} = -\frac{2z\left(g * -\frac{zf'}{p}\right)'(z)}{p\left(\left(g * -\frac{zf'}{p}\right)_n(z) + \overline{\left(g * -\frac{zf'}{p}\right)_n(\bar{z})}\right)}$$

shows that

$$f \in \mathcal{MCVC}_p^n(g,h) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MSTC}_p^n(g,h), \tag{3.25}$$

and by the result of part (i), it is clear that

$$\phi * \left(-\frac{zf'}{p}\right) \in \mathcal{MSTC}_p^n(g,h)$$

which implies

$$-\frac{z(\phi * f)'}{p} \in \mathcal{MSTC}_p^n(g,h).$$

Hence, in view of (3.25) $\phi * f \in \mathcal{MCVC}_p^n(g,h)$. ∎

3.5 Functions with Respect to n-ply Symmetric Conjugate Points

The classes of meromorphic multivalent starlike and convex functions with respect to n-ply symmetric conjugate points are defined in the following definitions:

Definition 3.23 *For odd integer p, the class $\mathcal{MSTSC}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying $f_n(z) - \overline{f_n(-\bar{z})} \neq 0$ in $\mathbb{D}^*$ and the subordination*

$$-\frac{1}{p}\frac{2zf'(z)}{f_n(z) - \overline{f_n(-\bar{z})}} \prec h(z). \tag{3.26}$$

The general class $\mathcal{MSTSC}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MSTSC}_p^n(h)$.

Definition 3.24 *For odd integer p, the class $\mathcal{MCVSC}_p^n(h)$ consists of functions $f \in \mathcal{M}_p$ satisfying $f_n'(z) + \overline{f_n'(-\bar{z})} \neq 0$ in $\mathbb{D}^*$ and the subordination*

$$-\frac{1}{p}\frac{2\left(zf'\right)'(z)}{f_n'(z) + \overline{f_n'(-\bar{z})}} \prec h(z).$$

The general class $\mathcal{MCVSC}_p^n(g,h)$ consists of functions $f \in \mathcal{M}_p$ for which $f * g$, belongs to $\mathcal{MCVSC}_p^n(h)$.

Theorem 3.6 *Let h be a convex univalent function satisfying*

$$\operatorname{Re} h(z) < 1 + \frac{1-\alpha}{p} \quad (0 \leq \alpha < 1),$$

and $\phi \in \mathcal{M}_p$ with $z^{p+1}\phi \in \mathcal{R}_\alpha$ where ϕ has real coefficient.

(i) If $f \in \mathcal{MSTSC}_p^n(g,h)$, then $\phi * f \in \mathcal{MSTSC}_p^n(g,h)$.

(ii) If $f \in \mathcal{MCVSC}_p^n(g,h)$, then $\phi * f \in \mathcal{MCVSC}_p^n(g,h)$.

Proof. (i) Again the proof is analogous to the proof of Theorem 3.4(i) except for the definition of the function F_n. Here the function F_n is defined by

$$F_n(z) = \frac{f_n(z) - \overline{f_n(-\overline{z})}}{2}$$

and let the functions H and Ψ be defined by

$$H(z) := -\frac{zf'(z)}{pF_n(z)} \quad \text{and} \quad \Psi(z) := z^{p+1}F_n(z).$$

Let $f \in \mathcal{MSTSC}_p^n(h)$. Thus for any fixed $z \in \mathbb{D}$,

$$-\frac{zf'(z)}{pF_n(z)} \in h(\mathbb{D}). \tag{3.27}$$

Replacing z by $-\overline{z}$ in (3.27) and taking the convex combinations, it follows that

$$-\frac{zf'(z)}{2pF_n(z)} - \overline{\left(\frac{(-\overline{z})f'(-\overline{z})}{2pF_n(-\overline{z})}\right)} = -\frac{z\left(\frac{f'(z)+\overline{f'(-\overline{z})}}{2}\right)}{pF_n(z)} \in h(\mathbb{D})$$

which implies

$$-\frac{zF'(z)}{pF_n(z)} \prec h(\mathbb{D}).$$

This shows that the function $F \in \mathcal{MST}_p^n(h)$, and the proof of Theorem 3.3 now shows that $F_n \in \mathcal{MST}_p(h)$. Since h is a convex function with
$\operatorname{Re} h(z) < 1 + (1-\alpha)/p$, it follows that

$$\operatorname{Re}\frac{z\Psi'(z)}{\Psi(z)} = \operatorname{Re}\frac{zF_n'(z)}{F_n(z)} + p + 1 > \alpha,$$

and hence $\Psi \in \mathcal{ST}(\alpha)$. A computation shows that

$$-\frac{2z(\phi * f)'(z)}{p\left((\phi * f)_n(z) - \overline{(\phi * f)_n(-\overline{z})}\right)} = \frac{\phi(z) * -zp^{-1}f'(z)}{\phi(z) * \frac{f_n(z)-\overline{f_n(-\overline{z})}}{2}} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)}.$$

Since $z^{p+1}\phi \in \mathcal{R}_\alpha$ and $z^{p+1}F_n \in \mathcal{ST}(\alpha)$, Theorem 2.3 yields

$$\frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} = \frac{\big(z^{p+1}\phi(z)\big) * \big(H(z)z^{p+1}F_n(z)\big)}{\big(z^{p+1}\phi(z)\big) * \big(z^{p+1}F_n(z)\big)} \in \overline{co}\big(H(\mathbb{D})\big),$$

and because $H(z) \prec h(z)$, it follows that

$$-\frac{2}{p}\frac{z(\phi * f)'(z)}{(\phi * f)_n(z) - \overline{(\phi * f)_n(-\overline{z})}} = \frac{(\phi * HF_n)(z)}{(\phi * F_n)(z)} \prec h(z).$$

Hence $\phi * f \in \mathcal{MSTSC}_p^n(h)$. The general result is obtained from the fact that

$$f \in \mathcal{MSTSC}_p^n(g,h) \Leftrightarrow f * g \in \mathcal{MSTSC}_p^n(h)$$

which gives $\phi * f * g \in \mathcal{MSTSC}_p^n(h)$ or equivalently $\phi * f \in \mathcal{MSTSC}_p^n(g,h)$.

(ii) The identity

$$-\frac{2(z(g * f)'(z))'}{p\left((g * f)_n'(z) + \overline{(g * f)_n'(-\overline{z})}\right)} = -\frac{2z\left(g * -\frac{zf'}{p}\right)'(z)}{p\left(\left(g * -\frac{zf'}{p}\right)_n(z) - \overline{\left(g * -\frac{zf'}{p}\right)_n(-\overline{z})}\right)}$$

shows that

$$f \in \mathcal{MCVSC}_p^n(g,h) \Leftrightarrow -\frac{zf'}{p} \in \mathcal{MSTSC}_p^n(g,h), \tag{3.28}$$

and by the result of part (i), it is clear that

$$\phi * \left(-\frac{zf'}{p}\right) \in \mathcal{MSTSC}_p^n(g,h)$$

which implies

$$-\frac{z(\phi * f)'}{p} \in \mathcal{MSTSC}_p^n(g,h).$$

Hence, in view of (3.28) $\phi * f \in \mathcal{MCVSC}_p^n(g,h)$. ∎

CHAPTER 4

DIFFERENTIAL SANDWICH RESULTS FOR MULTIVALENT FUNCTIONS

4.1 Introduction

Various authors have used the Dziok-Srivastava operator

$$H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) = z^p + \sum_{k=p+1}^{\infty} \frac{(\alpha_1)_{k-p}\cdots(\alpha_l)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_m)_{k-p}} \frac{a_k z^k}{(k-p)!}.$$

The operator $H_p^{l,m}[\alpha_1] := H_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)$ satisfy the relation

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z\big(H_p^{l,m}[\alpha_1]f(z)\big)' + (\alpha_1 - p)H_p^{l,m}[\alpha_1]f(z).$$

In this chapter, the Dziok-Srivastava operator will be defined in terms of the notation $\widetilde{H}_p^{l,m}[\beta_1]$ as follows:

$$\widetilde{H}_p^{l,m}[\beta_1]f(z) := \widetilde{H}_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z)$$

where $\widetilde{H}_p^{l,m}[\beta_1]$ and the parameter β_1 in the denominator satisfy the recurrence relation

$$\beta_1 \widetilde{H}_p^{l,m}[\beta_1]f(z) = z\big(\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\big)' + (\beta_1 - p)\widetilde{H}_p^{l,m}[\beta_1+1]f(z). \tag{4.1}$$

The Dziok-Srivastava operator with respect to the parameter β_1 which satisfies the recurrence relation (4.1) was first investigated by Srivastava *et al.* [132]

Various authors also have investigated the Liu-Srivastava operator

$$\widetilde{H^*}_p^{(l,m)}(\alpha_1,\ldots,\alpha_l;\beta_1,\ldots,\beta_m)f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} \frac{(\alpha_1)_{k+p}\cdots(\alpha_l)_{k+p}}{(\beta_1)_{k+p}\cdots(\beta_m)_{k+p}} \frac{a_k z^k}{(k+p)!}$$

where is framed using the notation $H_p^{l,m}[\alpha_1]f(z)$. Their works exploited the recurrence relation involving the parameter α_1 in the numerator satisfying

$$\alpha_1 H_p^{l,m}[\alpha_1+1]f(z) = z\big(H_p^{l,m}[\alpha_1]f(z)\big)' + (\alpha_1+p)H_p^{l,m}[\alpha_1]f(z).$$

In this chapter, the parameter β_1 in the denominator of Liu-Srivastava operator which also satisfies a recurrence relation will be investigated. Denoting the Liu-Srivastava operator by $\widetilde{H^*}_p^{l,m}[\beta_1]f(z)$, it can be shown that

$$\beta_1 \widetilde{H^*}_p^{l,m}[\beta_1]f(z) = z\big(\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)\big)' + (\beta_1+p)\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z). \quad (4.2)$$

4.2 Subordination of the Dziok-Srivastava Operator

In this section, the differential subordination result of Miller and Mocanu in Theorem 2.1 is extended for functions associated with the Dziok-Srivastava linear operator $\widetilde{H}_p^{l,m}[\beta_1]$. A similar problem was studied by Aghalary *et al.* [1]. Related results may also found in the works of [12–15, 23, 24, 58]. The following class of admissible functions will be required to obtain the main results.

Definition 4.1 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}[0,p]$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever*

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\beta_1+1-p)q(\zeta)}{\beta_1+1} \quad (\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(\beta_1+1)w + (p-\beta_1)(\beta_1-p+1)u}{(\beta_1+1)v + (p-\beta_1-1)u} - (2(\beta_1-p)+1)\right) \geq k\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\setminus E(q)$ and $k \geq p$.

In the particular case $q(z) = Mz, \quad M > 0$, and in view of Definition 4.1, the following definition is immediate.

Definition 4.2 *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+\beta_1+1-p}{\beta_1+1}Me^{i\theta}, \frac{L+(\beta_1-p+1)(2k+\beta_1-p)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \notin \Omega \tag{4.3}$$

whenever $z \in \mathbb{D}$, $\theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ,
$\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq p$.

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$.

Theorem 4.1 *Let $\phi \in \Phi_H[\Omega, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{4.4}$$

then

$$\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z), \qquad (z \in \mathbb{D}).$$

Proof. Define the analytic function p in $\mathbb{D}$ by

$$p(z) := \widetilde{H}_p^{l,m}[\beta_1+2]f(z). \tag{4.5}$$

In view of the relation (4.1), it follows that

$$(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z) = z\big(\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\big)' + (\beta_1+1-p)\widetilde{H}_p^{l,m}[\beta_1+2]f(z). \tag{4.6}$$

Using (4.5), (4.6) can be written as

$$\widetilde{H}_p^{l,m}[\beta_1+1]f(z) = \frac{zp'(z) + (\beta_1+1-p)p(z)}{\beta_1+1}. \tag{4.7}$$

From (4.1), further computations show that

$$\widetilde{H}_p^{l,m}[\beta_1]f(z) = \frac{z^2p''(z) + 2(\beta_1 - p + 1)zp'(z) + (\beta_1 - p)(\beta_1 - p + 1)p(z)}{\beta_1(\beta_1 + 1)}. \quad (4.8)$$

Define the transformation ψ from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{s + (\beta_1 + 1 - p)r}{\beta_1 + 1},\ w = \frac{t + 2(\beta_1 - p + 1)s + (\beta_1 - p)(\beta_1 - p + 1)r}{\beta_1(\beta_1 + 1)}, \quad (4.9)$$

where

$$r = p(z),\ s = zp'(z) \quad \text{and} \quad t = z^2p''(z).$$

Let

$$\begin{aligned}\psi(r, s, t; z) &= \phi(u, v, w; z)\\ &= \phi\left(r, \frac{s + (\beta_1 + 1 - p)r}{\beta_1 + 1}, \frac{t + 2(\beta_1 - p + 1)s + (\beta_1 - p)(\beta_1 - p + 1)r}{\beta_1(\beta_1 + 1)}; z\right).\end{aligned} \quad (4.10)$$

From (4.5), (4.7) and (4.8), the equation (4.10), yields

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) = \phi\left(\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right). \quad (4.11)$$

Hence (4.4) becomes

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega.$$

To complete the proof, it is left to show that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.1. Note that

$$v = \frac{s + (\beta_1 + 1 - p)r}{\beta_1 + 1}$$

implies

$$s = v(\beta_1 + 1) - (\beta_1 + 1 - p)r.$$

Computation from

$$w = \frac{t + 2(\beta_1 - p + 1)s + (\beta_1 - p)(\beta_1 - p + 1)r}{\beta_1(\beta_1 + 1)}$$

shows that

$$\frac{t}{s} + 1 = \frac{w\beta_1(\beta_1 + 1) - (\beta_1 - p)(\beta_1 - p + 1)u}{v(\beta_1 + 1) - (\beta_1 + 1 - p)u} - \big(2(\beta_1 - p) + 1\big) \quad (u = r).$$

Hence $\psi \in \Psi_p[\Omega, q]$. By Theorem 2.1, $p(z) \prec q(z)$ or

$$\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z). \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_H[h(\mathbb{D}), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 4.1.

Theorem 4.2 *Let $\phi \in \Phi_H[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \prec h(z), \tag{4.12}$$

then

$$\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z).$$

The next result is an extension of Theorem 4.1 to the case where the behavior of q on $\partial\mathbb{D}$ is not known.

Corollary 4.2.1 *Let $\Omega \subset \mathbb{C}$, q be univalent in $\mathbb{D}$ and $q(0) = 0$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$*

and

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z).$$

Proof. Theorem 4.1 yields $\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q_\rho(z)$. The result follows easily from the subordination $q_\rho(z) \prec q(z)$. ∎

Theorem 4.3 *Let h and q be univalent in $\mathbb{D}$, with $q(0) = 0$, set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Let $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy one of the following conditions:*

1. *$\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
2. *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}_p$ satisfies (4.12), then

$$\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z).$$

Proof. (1) The function q_ρ is univalent on $\overline{\mathbb{D}}$, and therefore $E(q_\rho)$ is empty and $q_\rho \in \mathcal{Q}$. The class $\Phi_H[h, q_\rho]$ is an admissible class and from Theorem 4.1 we obtain

$$\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q_\rho(z).$$

Since $q_\rho \prec q$ we deduce $\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z)$.

(2) From (4.11), it is evident that

$$\phi \in \Phi_H[h_\rho, q_\rho]$$

implies

$$\psi \in \Psi_H[h_p, q_p].$$

If we let $p_\rho(z) = p(\rho z)$, then

$$\psi\big(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z\big) = \psi\big(p(\rho z), \rho zp'(\rho z), \rho^2z^2p''(\rho z); \rho z\big) \in h_\rho(\mathbb{D}).$$

By using Thorem 2.1, with ρz, a function mapping $\mathbb{D}$ into $\mathbb{D}$ we obtain $p_\rho(z) \prec q_\rho(z)$, for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$ we obtain $p(z) \prec q(z)$ or $\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z)$. ∎

The next theorem yields the best dominant of the differential subordination (4.12).

Theorem 4.4 *Let h be univalent in $\mathbb{D}$, $\phi : \mathbb{C}^3\times\mathbb{D} \to \mathbb{C}$ and $\beta_1 \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$. Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2q''(z); z\right) = h(z) \tag{4.13}$$

has a solution q with $q(0) = 0$ and satisfy one of the following conditions:

1. *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$,*
2. *q is univalent in $\mathbb{D}$ and $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
3. *q is univalent in $\mathbb{D}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}_p$ satisfies (4.12), then

$$\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q(z),$$

and $q(z)$ is the best dominant.

Proof. In view of Theorem 4.2 and Theorem 4.3 we deduce that q is a dominant of (4.12) . Since q satisfies (4.13), it is also a solution of (4.12) and therefore q will be dominated by all dominants. Hence q is the best dominant. ∎

Corollary 4.2.2 *Let $\phi \in \Phi_H[\Omega, M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$\left|\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < M.$$

Proof. The result follows by taking $q(z) = Mz$ in Theorem 4.1. ∎

Corollary 4.2.3 *Let $\phi \in \Phi_H[M]$ and $\beta_1 \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)\right| < M,$$

then

$$\left|\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < M.$$

Proof. By taking $\Omega = q(\mathbb{D})$ in Corollary 4.2.2, the result is obtained. ∎

Taking $\phi(u, v, w; z) = v$ in Corollary 4.2.3 leads to the following example.

Example 4.1 *If $\operatorname{Re}\beta_1 \geq (p-3)/2$ and $f \in \mathcal{A}_p$ satisfies*

$$\left|\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\right| < M,$$

then

$$\left|\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < M.$$

Proof. By taking $\phi(u, v, w; z) = v$ in Corollary 4.2.3, we have to find the condition so that $\phi \in \Phi_H[M]$, that is, the admissibility condition (4.3) is satisfied. This follows from

$$\left|\phi\left(Me^{i\theta}, \frac{k+\beta_1+1-p}{\beta_1+1}Me^{i\theta}, \frac{L+(\beta_1-p+1)(2k+\beta_1-p)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right)\right| \geq M$$

which implies

$$\left|\frac{k+\beta_1+1-p}{\beta_1+1}Me^{i\theta}\right| \geq M$$

or

$$|k+\beta_1+1-p| \geq |\beta_1+1|. \tag{4.14}$$

Preceding inequality (4.14), shows that

$$\operatorname{Re}\beta_1 \geq \frac{p-k}{2}-1.$$

Since $k \geq 1$, then it is sufficient to write

$$\operatorname{Re}\beta_1 \geq \frac{p-3}{2}$$

for (4.14) holds true.

Hence, from Corollary 4.2.3, if $\operatorname{Re}\beta_1 \geq (p-3)/2$ and $\left|\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\right| < M$ then $\left|\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < M$. ∎

Corollary 4.2.4 *Let $M>0$ and $\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\widetilde{H}_p^{l,m}[\beta_1+1]f(z)+\left(\frac{p}{\beta_1+1}-1\right)\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\right| < \frac{Mp}{|\beta_1+1|},$$

then

$$|\widetilde{H}_p^{l,m}[\beta_1+2]f(z)| < M.$$

Proof. Let $\phi(u,v,w;z) = v+\big(p/(\beta_1+1)-1\big)u$ and $\Omega = h(\mathbb{D})$ where $h(z) = Mz/|\beta_1+1|$, $M>0$. It is enough to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (4.3) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta}, \frac{k+\beta_1+1-p}{\beta_1+1}Me^{i\theta}, \frac{L+(\beta_1-p+1)(2k+\beta_1-p)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right)\right|$$

$$\begin{aligned}
&= \left| \left(\frac{k+\beta_1+1-p}{\beta_1+1} \right) Me^{i\theta} + \left(\frac{p}{\beta_1+1} - 1 \right) Me^{i\theta} \right| \\
&= \frac{kM}{|\beta_1+1|} \\
&\geq \frac{Mp}{|\beta_1+1|}
\end{aligned}$$

for $z \in \mathbb{D},\ \theta \in \mathbb{R},\ \beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq p$. From Corollary 4.2.2, the required result is obtained. ∎

Definition 4.3 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever*

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + \beta_1 q(\zeta)}{\beta_1 + 1} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1\big((\beta_1+1)w + (1-\beta_1)u\big)}{(\beta_1+1)v - \beta_1 u} + 1 - 2\beta_1 \right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ and $k \geq 1$.

In the particular case $q(z) = Mz, \quad M > 0$, and in view of Definition 4.3, the following definition is immediate.

Definition 4.4 *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+\beta_1}{\beta_1+1} Me^{i\theta}, \frac{L + \beta_1(2k+\beta_1-1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z \right) \notin \Omega \tag{4.15}$$

whenever $z \in \mathbb{D},\ \theta \in \mathbb{R}$, $\operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ,
$\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$.

Theorem 4.5 *Let $\phi \in \Phi_{H,1}[\Omega, q]$ and $\beta_1 \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{4.16}$$

then

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \prec q(z).$$

Proof. Define the analytic function p in $\mathbb{D}$ by

$$p(z) := \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}. \tag{4.17}$$

A computation shows that

$$z\big(\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\big)' = (p-1)z^{p-1}p(z) + z^p p'(z). \tag{4.18}$$

By replacing (4.18) in (4.6), it follows that

$$\begin{aligned}&(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\\ &= (p-1)z^{p-1}p(z) + z^p p'(z) + (\beta_1+1-p)\widetilde{H}_p^{l,m}[\beta_1+2]f(z).\end{aligned}$$

Hence,

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}} = \frac{zp'(z)+\beta_1 p(z)}{\beta_1+1}. \tag{4.19}$$

From (4.19), it follows that

$$\widetilde{H}_p^{l,m}[\beta_1+1]f(z) = \frac{z^p p'(z) + \beta_1 z^{p-1}p(z)}{\beta_1+1}$$

and

$$\begin{aligned}&(\widetilde{H}_p^{l,m}[\beta_1+1]f(z))'\\ &= \frac{1}{\beta_1+1}\Big(pz^{p-1}p'(z) + z^p p''(z) + \beta_1(p-1)z^{p-2}p(z) + \beta_1 z^{p-1}p'(z)\Big)\end{aligned}$$

or equivalently

$$\begin{aligned}
&\frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\big)'}{z^{p-1}} \\
&= \frac{1}{\beta_1+1}\Big(pzp'(z)+z^2p''(z)+\beta_1(p-1)p(z)+\beta_1 zp'(z)\Big). \qquad (4.20)
\end{aligned}$$

In view of (4.1), it yields

$$\frac{\beta_1\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}} = \frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\big)'}{z^{p-1}} + \frac{(\beta_1-p)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}. \qquad (4.21)$$

Using (4.19) and (4.20), (4.21) rewritten as

$$\begin{aligned}
&\frac{\beta_1\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}} \\
&= \left(\frac{pzp'(z)+z^2p''(z)+\beta_1(p-1)p(z)+\beta_1 zp'(z)}{\beta_1+1}\right) \\
&\qquad\qquad + \left(\frac{(\beta_1-p)\big(zp'(z)+\beta_1 p(z)\big)}{\beta_1+1}\right) \\
&= \frac{z^2p''(z)+2\beta_1 zp'(z)+\beta_1(\beta_1-1)p(z)}{(\beta_1+1)}.
\end{aligned}$$

Hence

$$\frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}} = \frac{z^2p''(z)+2\beta_1 zp'(z)+\beta_1(\beta_1-1)p(z)}{\beta_1(\beta_1+1)}. \qquad (4.22)$$

Define the transformation ψ from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ v=\frac{s+\beta_1 r}{\beta_1+1},\ w=\frac{t+2\beta_1 s+\beta_1(\beta_1-1)r}{\beta_1(\beta_1+1)} \qquad (4.23)$$

where

$$r=p(z),\ s=zp'(z) \quad \text{and} \quad t=z^2p''(z).$$

Let

$$\psi(r,s,t;z) = \phi(u,v,w;z) = \phi\left(r, \frac{s+\beta_1 r}{\beta_1+1}, \frac{t+2\beta_1 s+\beta_1(\beta_1-1)r}{\beta_1(\beta_1+1)}; z\right). \quad (4.24)$$

From (4.17), (4.19) and (4.22), equation (4.24), leads to

$$\begin{aligned}&\psi\big(p(z), zp'(z), z^2p''(z); z\big)\\ &= \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right).\end{aligned} \quad (4.25)$$

Hence (4.16) becomes

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega.$$

To complete the proof the admissibility condition for $\phi \in \Phi_{H,1}[\Omega, q]$ is shown to be equivalent to the admissibility condition for ψ as given in Definition 2.1. Note that

$$v = \frac{s+\beta_1 r}{\beta_1+1}$$

implies

$$s = v(\beta_1+1) - \beta_1 r.$$

A computation from

$$w = \frac{t+2\beta_1 s+\beta_1(\beta_1-1)r}{\beta_1(\beta_1+1)}$$

shows that

$$\frac{t}{s}+1 = \frac{w\beta_1(\beta_1+1)-\beta_1(\beta_1-1)u}{v(\beta_1+1)-\beta_1 u} + 1 - 2\beta_1 \quad (u=r).$$

Hence $\psi \in \Psi[\Omega, q]$. By Theorem 2.1, $p(z) \prec q(z)$ or

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \prec q(z). \qquad \blacksquare$$

As in the previous case, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case, the class $\Phi_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 4.5.

Theorem 4.6 *Let $\phi \in \Phi_{H,1}[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right) \prec h(z),$$

then

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \prec q(z).$$

Corollary 4.2.5 *Let $\phi \in \Phi_{H,1}[\Omega, M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right) \in \Omega,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}\right| < M.$$

Proof. The result follows by taking $q(z) = Mz$ in Theorem 4.5. $\blacksquare$

Corollary 4.2.6 *Let $\phi \in \Phi_{H,1}[M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right)\right| < M,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}\right| < M.$$

Proof. By taking $\Omega = q(\mathbb{D})$ in Corollary 4.2.5, the result is obtained. ∎

Taking $\phi(u, v, w; z) = v$ in Corollary 4.2.6 leads to the following example.

Example 4.2 *If* $\operatorname{Re}\beta_1 \geq -1$ *and* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}\right| < M,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}\right| < M.$$

Proof. By taking $\phi(u, v, w; z)$ in Corollary 4.2.6, we have to find the condition so that $\phi \in \Phi_{H,1}[M]$, that is, the admissibility condition (4.15) is satisfied. This follows

$$\left|\phi\left(Me^{i\theta}, \frac{k+\beta_1}{\beta_1+1}Me^{i\theta}, \frac{L+\beta_1(2k+\beta_1-1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right)\right| \geq M$$

which implies

$$\left|\frac{k+\beta_1}{\beta_1+1}Me^{i\theta}\right| \geq M$$

or

$$|k+\beta_1| \geq |\beta_1+1|. \tag{4.26}$$

Following inequality (4.26), a computation shows that

$$\operatorname{Re}\beta_1 \geq \frac{-(1+k)}{2}.$$

Since $k \geq 1$, then it is sufficient to write

$$\operatorname{Re}\beta_1 \geq -1$$

for (4.26) holds true. Hence, from Corollary 4.2.6, if $\operatorname{Re}\beta_1 \geq -1$ and

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}\right| < M,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}\right| < M.$$

This completes the proof. ∎

Remark 4.2.1 *The analogue of Corollary 4.2.3 and Corollary 4.2.6 can be obtained by choosing suitable admissible function.*

Definition 4.5 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,2}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever*

$$u = q(\zeta),\ v = \frac{(\beta_1+1)q(\zeta)}{(\beta_1+2) - k\zeta q'(\zeta) - q(\zeta)}, \quad (\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\dots\}, \quad q(\zeta) \neq 0),$$

$$\operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2)-(\beta+1)u-vu}\left(\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right) - \frac{\beta+1}{v} - 1\right) \geq k\operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}\setminus E(q)$ and $k \geq 1$.

In the particular case $q(z) = 1 + Mz$, $M > 0$, and in view of Definition 4.5, the following definition is immediate.

Definition 4.6 *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(1+Me^{i\theta}, \frac{(\beta_1+1)(1+Me^{i\theta})}{(\beta_1+1)-Me^{i\theta}(k+1)},\right.$$
$$\left.\frac{\beta_1(1+Me^{i\theta})\big(\beta_1+1-Me^{i\theta}(k+1)\big)}{\big(\beta_1+1-Me^{i\theta}(k+1)\big)\big(\beta_1-2Me^{i\theta}(k+1)\big)-(1+Me^{i\theta})(L+2Me^{i\theta})}; z\right) \notin \Omega$$

whenever $z \in \mathbb{D},\ \theta \in \mathbb{R}$,

$$\mathrm{Re}\left(\frac{\big((\beta_1+1)-Me^{i\theta}(k+1)\big)(1-kMe^{i\theta})-L(1+Me^{i\theta})}{kMe^{i\theta}(1+Me^{i\theta})}\right) \geq (k+4)$$

for all real θ, $\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$ *and* $k \geq 1$.

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,2}[\Omega, M]$ is simply denoted by $\Phi_{H,2}[M]$.

Theorem 4.7 *Let* $\phi \in \Phi_{H,2}[\Omega, q]$ *and* $\beta_1 \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{4.27}$$

then
$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

Proof. Define the analytic function p in $\mathbb{D}$ by

$$p(z) := \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}. \tag{4.28}$$

By using (4.28), a computation shows that get

$$\frac{zp'(z)}{p(z)} = \frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+3]f(z)\big)'}{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)} - \frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\big)'}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}. \tag{4.29}$$

In view of the relation (4.1), it follows that

$$(\beta_1+2)\widetilde{H}_p^{l,m}[\beta_1+2]f(z) = z\big(\widetilde{H}_p^{l,m}[\beta_1+3]f(z)\big)' + (\beta_1+2-p)\widetilde{H}_p^{l,m}[\beta_1+3]f(z). \tag{4.30}$$

Using (4.6) and (4.30), (4.29) can be written as

$$\begin{aligned}\frac{zp'(z)}{p(z)} &= \frac{(\beta_1+2)\widetilde{H}_p^{l,m}[\beta_1+2]f(z) - (\beta_1+2-p)\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)} \\ &\quad - \left(\frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z) - (\beta_1+1-p)\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}\right)\end{aligned}$$

$$= \frac{(\beta_1+2)\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)} - \frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - 1.$$

By (4.28), it follows that

$$\begin{aligned}\frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} &= \frac{(\beta_1+2)}{p(z)} - \frac{zp'(z)}{p(z)} - 1 \\ &= \frac{(\beta_1+2) - zp'(z) - p(z)}{p(z)}\end{aligned}$$

or

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} = \frac{(\beta_1+2) - zp'(z) - p(z)}{(\beta_1+1)p(z)}.$$

Hence,

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} = \frac{(\beta_1+1)p(z)}{(\beta_1+2) - zp'(z) - p(z)}. \tag{4.31}$$

Further computations show that

$$\frac{z\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'}{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)} = \frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+2]f(z)\big)'}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - \frac{z\big(\widetilde{H}_p^{l,m}[\beta_1+1]f(z)\big)'}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} \tag{4.32}$$

Using (4.1) and (4.6), (4.32) can be written as

$$\frac{z\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'}{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)} = \frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)-(\beta_1+1-p)\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}$$

$$-\left(\frac{\beta_1\widetilde{H}_n^{l,m}\beta_1 f(z)-(\beta_1-p)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)$$

$$= \frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - \frac{\beta_1\widetilde{H}_n^{l,m}\beta_1 f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} - 1,$$

or

$$\frac{\beta_1\widetilde{H}_n^{l,m}\beta_1 f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} = \frac{(\beta_1+1)\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - \frac{z\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'}{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)} - 1. \tag{4.33}$$

Further computation shows that

$$\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'$$
$$= \frac{(\beta_1+1)p'(z)\big((\beta_1+2)-zp'(z)-p(z)\big)-\big(-zp''(z)-2p'(z)\big)\big((\beta_1+1)p(z)\big)}{\big((\beta_1+2)-zp'(z)-p(z)\big)^2}$$

and

$$\frac{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'}{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)} = \frac{p'(z)}{p(z)} + \frac{zp''(z)+2p'(z)}{(\beta_1+2)-zp'(z)-p(z)}.$$

Hence

$$\frac{z\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)'}{\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}\right)} = \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)+2zp'(z)}{(\beta_1+2)-zp'(z)-p(z)} \tag{4.34}$$

Using (4.31) and (4.34), (4.33) can be written as

$$\frac{\beta_1\widetilde{H}_p^{l,m}[\beta_1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)} = \frac{(\beta_1+2)-zp'(z)-p(z)}{p(z)} - \frac{zp'(z)}{p(z)} - \left(\frac{z^2p''(z)+2zp'(z)}{(\beta_1+2)-zp'(z)-p(z)}\right) - 1$$

which implies

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)} = \frac{\beta_1}{\frac{\beta_1-2-zp'(z)-p(z)}{p(z)} - \frac{zp'(z)}{p(z)} - \frac{\left(z^2p''(z)+2zp'(z)\right)}{\beta_1+2-zp'(z)-p(z)} - 1}. \tag{4.35}$$

Define the transformation ψ from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r,\ v = \frac{(\beta_1+1)r}{\beta_1+2-s-r},\ w = \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}, \tag{4.36}$$

where

$$r = p(z),\ s = zp'(z) \quad \text{and} \quad t = z^2p''(z).$$

Let

$$\begin{aligned}\psi(r,s,t;z) &:= \phi(u,v,w;z)\\ &= \phi\left(r, \frac{(\beta_1+1)r}{\beta_1+2-s-r}, \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1}; z\right).\end{aligned} \tag{4.37}$$

From (4.28), (4.31) and (4.35), the equation(4.37), yields

$$\begin{aligned} &\psi\big(p(z), zp'(z), z^2p''(z); z\big) \\ &= \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right). \end{aligned} \tag{4.38}$$

Hence (4.27) becomes

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) \in \Omega.$$

To complete the proof the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is shown to be equivalent to the admissibility condition for ψ as given in Definition 2.1. Note that

$$v = \frac{(\beta_1+1)r}{\beta_1+2-s-r}$$

implies

$$s = \frac{v(\beta_1+2)-(\beta_1+1)r-rv}{v}. \tag{4.39}$$

By using (4.39) in

$$w = \frac{\beta_1}{\frac{\beta_1+2-s-r}{r} - \frac{s}{r} - \frac{(t+2s)}{\beta_1+2-s-r} - 1},$$

a computation shows that

$$\frac{t}{s}+1 = \frac{(\beta_1+1)u}{v(\beta_1+2)-(\beta_1+1)u-vu}\left(\frac{\beta_1+1}{v} - \frac{\beta_1}{w} - 1\right) - \frac{\beta_1+1}{v} - 1 \quad (u=r) \tag{4.40}$$

and hence $\psi \in \Psi[\Omega, q]$. By Theorem 2.1, $p(z) \prec q(z)$ or

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z). \qquad \blacksquare$$

As in the previous cases, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_{H,2}[h(\mathbb{D}), q]$ is written as $\Phi_{H,2}[h, q]$. The following result is an immediate consequence of Theorem 4.7.

Theorem 4.8 *Let $\phi \in \Phi_{H,2}[h, q]$ and $\beta_1 \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h(z),$$

then

$$\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

Corollary 4.2.7 *Let $\phi \in \Phi_{H,2}[\Omega, M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

Proof. The result follows by taking $q(z) = 1 + Mz$ in Theorem 4.7. ∎

Corollary 4.2.8 *Let $\phi \in \Phi_{H,2}[M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left|\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

Proof. By taking $\Omega = q(\mathbb{D})$ in Corollary 4.2.7, the result is obtained. ∎

4.3 Superordination of the Dziok-Srivastava Operator

The dual problem of differential subordination, that is, differential superordination of the Dziok-Srivastava linear operator is investigated in this section. For this purpose, the following class of admissible functions will be required.

Definition 4.7 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[0,p]$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever*

$$u = q(z), \quad v = \frac{zq'(z) + m(\beta_1 + 1 - p)q(z)}{m(\beta_1 + 1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(\beta_1+1)w + (p-\beta_1)(\beta_1 - p + 1)u}{(\beta_1+1)v + (p - \beta_1 - 1)u} - (2(\beta_1 - p) + 1)\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq p$.

Theorem 4.9 *Let $\phi \in \Phi'_H[\Omega, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$, $\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$\Omega \subset \left\{\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) : z \in \mathbb{D}\right\} \quad (4.41)$$

implies

$$q(z) \prec \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z).$$

Proof. From (4.11) and (4.41), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\}.$$

From (4.9), it is clear that the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Theorem 2.2, $q(z) \prec p(z)$ or

$$q(z) \prec \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z). \quad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi'_H[h(\mathbb{D}), q]$ is written as $\Phi'_H[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 4.9.

Theorem 4.10 *Let h be analytic in $\mathbb{D}$, $\phi \in \Phi'_H[h, q]$ and $\beta_1 \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$, $\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(\widetilde{H}_p^{l,m}[\beta_1 + 2]f(z), \widetilde{H}_p^{l,m}[\beta_1 + 1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \tag{4.42}$$

implies

$$q(z) \prec \widetilde{H}_p^{l,m}[\beta_1 + 2]f(z).$$

Theorem 4.9 and 4.10 can only be used to obtain subordinants of differential superordination of the form (4.41) or (4.42). The following theorem proves the existence of the best subordinant of (4.42) for certain ϕ.

Theorem 4.11 *Let h be analytic in $\mathbb{D}$, $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ and $\beta_1 \in \mathbb{C}\setminus\{0, -1, -2, \ldots\}$. Suppose that the differential equation*

$$\phi\big(q(z), zq'(z), z^2q''(z); z\big) = h(z) \tag{4.43}$$

has a solution $q \in \mathcal{Q}_0$*. If* $\phi \in \Phi'_H[h,q]$*,* $f \in \mathcal{A}_p$*,* $\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{Q}_0$ *and*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h(z) \prec \phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

implies

$$q(z) \prec \widetilde{H}_p^{l,m}[\beta_1+2]f(z)$$

and q *is the best subordinant.*

Proof. In view of Theorem 4.9 and Theorem 4.10 we deduce that q is a subordinant of (4.42). Since q satisfies (4.43), it is also a solution of (4.42) and therefore q will be subordinated by all subordinants. Hence q is the best subordinant. ∎

Combining Theorems 4.2 and 4.10, we obtain the following sandwich-type theorem.

Corollary 4.3.1 *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{D}$, h_2 *be univalent function in* $\mathbb{D}$*,* $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$*,* $\phi \in \Phi_H[h_2,q_2] \cap \Phi'_H[h_1,q_1]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$*. If* $f \in \mathcal{A}_p$*,* $\widetilde{H}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{H}[0,p] \cap \mathcal{Q}_0$ *and*

$$\phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h_1(z) \prec \phi\left(\widetilde{H}_p^{l,m}[\beta_1+2]f(z), \widetilde{H}_p^{l,m}[\beta_1+1]f(z), \widetilde{H}_p^{l,m}[\beta_1]f(z); z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \widetilde{H}_p^{l,m}[\beta_1+2]f(z) \prec q_2(z).$$

Definition 4.8 *Let* Ω *be a set in* $\mathbb{C}$ *and* $q \in \mathcal{H}_0$ *with* $zq'(z) \neq 0$. *The class of admissible functions* $\Phi'_{H,1}[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ *that satisfy the admissibility condition* $\phi(u, v, w; \zeta) \in \Omega$ *whenever*

$$u = q(z), \quad v = \frac{zq'(z) + m\beta_1 q(z)}{m(\beta_1+1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1\big((\beta_1+1)w + (1-\beta_1)u\big)}{(\beta_1+1)v - \beta_1 u} + 1 - 2\beta_1\right) \leq \frac{1}{m} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ *and* $m \geq 1$.

Next, the dual result of Theorem 4.5 for differential superordination will be given.

Theorem 4.12 *Let* $\phi \in \Phi'_{H,1}[\Omega, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{D}$, *then*

$$\Omega \subset \left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right) : z \in \mathbb{D}\right\} \tag{4.44}$$

implies

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}.$$

Proof. From (4.25) and (4.44), it follows that

$$\Omega \subset \left\{\phi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\}$$

From (4.23), it follows that the admissibility condition for $\phi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 2.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}. \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi'_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi'_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 4.12.

Theorem 4.13 *Let* $q \in \mathcal{H}_0$, h *be analytic on* $\mathbb{D}$, $\phi \in \Phi'_{H,1}[h, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \in \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h(z) \prec \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right)$$

implies

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}.$$

Combining Theorems 4.6 and 4.13, the following sandwich-type theorem is obtained.

Corollary 4.3.2 *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{D}$, h_2 *be univalent function in* $\mathbb{D}$, $q_2 \in \mathcal{Q}_0$ *with* $q_1(0) = q_2(0) = 0$, $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \in \mathcal{H}_0 \cap \mathcal{Q}_0$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h_1(z) \prec \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{z^{p-1}}, \frac{\widetilde{H}_p^{l,m}[\beta_1]f(z)}{z^{p-1}}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{z^{p-1}} \prec q_2(z).$$

Definition 4.9 *Let* Ω *be a set in* $\mathbb{C}$, $q(z) \neq 0$, $zq'(z) \neq 0$ *and* $q \in \mathcal{H}$. *The class of admissible functions* $\Phi'_{H,2}[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ *that satisfy the admissibility condition* $\phi(u, v, w; \zeta) \in \Omega$ *whenever*

$$u = q(z),\ v = \frac{m(\beta_1+1)q(z)}{m(\beta_1+2) - zq'(z) - mq(z)}, \quad (\beta_1 \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}, \quad q(z) \neq 0),$$

$$\operatorname{Re}\left(\frac{\beta_1 u(\beta_1+1)(w-1)}{w\big((\beta_1+1)(v-u)+v(1-u)\big)} - \frac{(\beta+1)}{v} - 1\right) \leq \frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ *and* $m \geq 1$.

Now, the dual result of Theorem 4.7 for differential superordination will be given.

Theorem 4.14 *Let* $\phi \in \Phi'_{H,2}[\Omega, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right)$$

is univalent in $\mathbb{D}$, *then*

$$\Omega \subset \left\{\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) : z \in \mathbb{D}\right\} \tag{4.45}$$

implies

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}.$$

Proof. From (4.38) and (4.45), it follows that

$$\Omega \subset \left\{ \phi \left(p(z), zp'(z), z^2p''(z); z \right) : z \in \mathbb{D} \right\}.$$

In view of (4.36), the admissibility condition for $\phi \in \Phi'_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 2.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}. \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi'_{H,2}[h(\mathbb{D}), q]$ is written as $\Phi'_{H,2}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 4.14.

Theorem 4.15 *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{D}$, $\phi \in \Phi'_{H,2}[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ and*

$$\phi \left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z \right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi \left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z \right)$$

implies

$$q(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}.$$

Combining Theorems 4.8 and 4.15 yield the following sandwich-type theorem.

Corollary 4.3.3 *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be univalent function in $\mathbb{D}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, $\phi \in \Phi_{H,2}[h_2, q_2] \cap \Phi'_{H,2}[h_1, q_1]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{A}_p$, $\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right)$$

is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \phi\left(\frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h_2(z),$$

implies

$$q_1(z) \prec \frac{\widetilde{H}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H}_p^{l,m}[\beta_1+2]f(z)} \prec q_2(z).$$

4.4 Subordination of the Liu-Srivastava Operator

In this section, differential subordination results are obtained for multivalent meromorphic functions associated with the Liu-Srivastava linear operator in the punctured unit disk. A similar problem was studied by Aghalary *et al.* [1,4], and related results can be found in the works of [9, 10, 12–15, 25, 58, 63]. The following class of admissible functions will be required to obtain main results.

Definition 4.10 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever*

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (\beta_1+1)q(\zeta)}{\beta_1+1} \qquad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(w-u)}{(v-u)} - (2\beta_1+1)\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ *and* $k \geq 1$.

Choosing $q(z) = 1 + Mz,\ M > 0$, Definition 4.10 easily gives the following definition.

Definition 4.11 *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ such that*

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{k + \beta_1 + 1}{\beta_1 + 1}Me^{i\theta}, 1 + \frac{L + (\beta_1 + 1)(2k + \beta_1)Me^{i\theta}}{\beta_1(\beta_1 + 1)}; z\right) \notin \Omega \tag{4.46}$$

whenever $z \in \mathbb{D},\ \theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ,
$\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ and $k \geq 1$.

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$.

Theorem 4.16 *Let $\phi \in \Phi_H[\Omega, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ satisfies*

$$\left\{\phi\left(z^p \widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{4.47}$$

then

$$z^p \widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z) \qquad (z \in \mathbb{D}).$$

Proof. Define the analytic function p in $\mathbb{D}$ by

$$p(z) := z^p \widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z). \tag{4.48}$$

In view of the relation (4.2), it follows that

$$(\beta_1 + 1)\widetilde{H^*}_p^{l,m}[\beta_1 + 1]f(z) = z\big(\widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z)\big)' + (\beta_1 + 1 + p)\widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z). \tag{4.49}$$

Using (4.48), a computation from (4.49) shows that

$$(\beta_1+1)\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z) = \frac{zp'(z)-p.p(z)}{z^p} + \frac{(\beta+1+p)p(z)}{z^p}$$

or equivalently

$$z^p\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z) = \frac{1}{\beta_1+1}\big((\beta_1+1)p(z)+zp'(z)\big). \tag{4.50}$$

By using (4.50), further computations from (4.2) show that

$$\beta_1\widetilde{H^*}_p^{l,m}[\beta_1]f(z) = z\left(\frac{(\beta_1+1)p(z)+zp'(z)}{(\beta_1+1)z^p}\right)' + (\beta_1+p)\left(\frac{(\beta_1+1)p(z)+zp'(z)}{(\beta_1+1)z^p}\right)$$

which implies

$$\beta_1(\beta_1+1)\widetilde{H^*}_p^{l,m}[\beta_1]f(z) = \frac{z^2p''(z)+2(\beta_1+1)zp'(z)+\beta_1(\beta_1+1)p(z)}{z^p}.$$

Hence

$$z^p\widetilde{H^*}_p^{l,m}[\beta_1]f(z) = \frac{1}{\beta_1(\beta_1+1)}\big(z^2p''(z)+2(\beta_1+1)zp'(z)\big)+p(z). \tag{4.51}$$

Define the transformation ψ from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,\ v=\frac{s+(\beta_1+1)r}{\beta_1+1},\ w=\frac{t+2(\beta_1+1)s+(\beta_1)(\beta_1+1)r}{\beta_1(\beta_1+1)} \tag{4.52}$$

where

$$r=p(z),\ s=zp'(z)\quad\text{and}\quad t=z^2p''(z).$$

Let

$$\begin{aligned}\psi(r,s,t;z) &= \phi(u,v,w;z)\\ &= \phi\left(r,\frac{s+(\beta_1+1)r}{\beta_1+1},\frac{t+2(\beta_1+1)s+(\beta_1)(\beta_1+1)r}{\beta_1(\beta_1+1)};z\right).\end{aligned} \tag{4.53}$$

From (4.48), (4.50) and (4.51), equation (4.53) yields

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big)$$
$$= \phi\left(z^p\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z), z^p\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z), z^p\widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right). \quad (4.54)$$

Hence (4.47) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

To complete the proof, it is left to show that the admissibility condition for $\phi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.1. Note that

$$v = \frac{s + (\beta_1 + 1)r}{\beta_1 + 1}$$

implies

$$s = (\beta_1 + 1)(v - r).$$

A computation from

$$w = \frac{t + 2(\beta_1 + 1)s + (\beta_1)(\beta_1 + 1)r}{\beta_1(\beta_1 + 1)}$$

shows that

$$\frac{t}{s} + 1 = \frac{\beta_1(w - u)}{(v - u)} - (2\beta_1 + 1) \quad (u = r)$$

and hence $\psi \in \Psi_n[\Omega, q]$. By Theorem 2.1, $p(z) \prec q(z)$ or

$$z^p\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) \prec q(z). \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_H[h(\mathbb{D}), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 4.16.

Theorem 4.17 *Let $\phi \in \Phi_H[h, q]$ with $q(0) = 1$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ and satisfies*

$$\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right) \prec h(z), \tag{4.55}$$

then

$$z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z).$$

The next result is an extension of Theorem 4.16 to the case where the behavior of q on $\partial\mathbb{D}$ is not known.

Corollary 4.4.1 *Let $\Omega \subset \mathbb{C}$, q be univalent in $\mathbb{D}$ and $q(0) = 1$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ and*

$$\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z).$$

Proof. Theorem 4.16 yields $z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q_\rho(z)$. The result now follows from the fact that $q_\rho(z) \prec q(z)$. ∎

Theorem 4.18 *Let h and q be univalent in $\mathbb{D}$, with $q(0) = 1$, set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Let $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfy one of the following conditions:*

1. *$\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0, 1)$, or*

2. *there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{M}_p$ *satisfies* (4.55), then

$$z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z).$$

Proof. (1) The function q_ρ is univalent on $\overline{\mathbb{D}}$, and therefore $E(q_\rho)$ is empty and $q_\rho \in \mathcal{Q}$. The class $\Phi_H[h, q_\rho]$ is an admissible class and from Theorem 4.16 we obtain

$$z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q_\rho(z).$$

Since $q_\rho \prec q$ we deduce $z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z)$.
(2) From (4.54), it is evident that $\phi \in \Phi_H[h_\rho, q_\rho]$ implies $\psi \in \Psi_H[h_p, q_p]$. If we let $P_\rho(z) = P(\rho z)$, then

$$\psi\big(p_\rho(z), zp'_\rho(z), z^2 p''_\rho(z); \rho z\big) = \psi\big(p(\rho z), \rho z p'(\rho z), \rho^2 z^2 p''(\rho z); \rho z\big) \in h_\rho(\mathbb{D}).$$

By using Thorem 2.1, with ρz, a function mapping $\mathbb{D}$ into $\mathbb{D}$ we obtain
$p_\rho(z) \prec q_\rho(z)$, for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$ we obtain $p \prec q$ or
$z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \prec q(z)$. ∎

The next theorem yields the best dominant of the differential subordination (4.55).

Theorem 4.19 *Let* h *be univalent in* $\mathbb{D}$, $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *Suppose that the differential equation*

$$\phi\left(q(z), zq'(z), z^2 q''(z); z\right) = h(z) \tag{4.56}$$

has a solution q *with* $q(0) = 1$ *and satisfy one of the following conditions:*

1. $q \in \mathcal{Q}_1$ *and* $\phi \in \Phi_H[h, q]$,

2. q *is univalent in* $\mathbb{D}$ *and* $\phi \in \Phi_H[h, q_\rho]$ *for some* $\rho \in (0,1)$, *or*

3. q *is univalent in* $\mathbb{D}$ *and there exists* $\rho_0 \in (0,1)$ *such that* $\phi \in \Phi_H[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{M}_p$ *satisfies* (4.55), then

$$z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z) \prec q(z),$$

and q is the best dominant.

Proof. In view of Theorem 4.17 and Theorem 4.18 we deduce that q is a dominant of (4.55) . Since q satisfies (4.56), it is also a solution of (4.55) and therefore q will be dominated by all dominants. Hence q is the best dominant. ∎

Corollary 4.4.2 *Let* $\phi \in \Phi_H[\Omega, M]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$. *If* $f \in \mathcal{M}_p$ *satisfies*

$$\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right) \in \Omega,$$

then

$$\left|z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z) - 1\right| < M.$$

Proof. The result follows by taking $q(z) = 1 + Mz$ in Theorem 4.16. ∎

Corollary 4.4.3 *Let* $\phi \in \Phi_H[M]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$. *If* $f \in \mathcal{M}_p$ *satisfies*

$$\left|\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right) - 1\right| < M,$$

then

$$\left|z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z) - 1\right| < M.$$

Proof. By taking $\Omega = q(\mathbb{D})$ in Corollary 4.4.2, the result is obtained. ∎

The following example is easily obtained by taking $\phi(u,v,w;z) = v$ in Corollary 4.4.3.

Example 4.3 *If* $\operatorname{Re}\beta_1 \geq -3/2$ *and* $f \in \mathcal{M}_p$ *satisfies*

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z) - 1 \right| < M,$$

then

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) - 1 \right| < M.$$

Proof. By taking $\phi(u,v,w;z) = v$ in Corollary 4.4.3, we have to find the condition so that $\phi \in \Phi_H[M]$, that is, the admissibility condition (4.46) is satisfied. This follows from

$$\left| \phi\left(1 + Me^{i\theta}, 1 + \frac{k+\beta_1+1}{\beta_1+1}Me^{i\theta}, 1 + \frac{L + (\beta_1+1)(2k+\beta_1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \right| \geq M$$

which implies

$$\left| \frac{k+\beta_1+1}{\beta_1+1} Me^{i\theta} \right| \geq M$$

or

$$|k+\beta_1+1| \geq |\beta_1+1|. \tag{4.57}$$

Using inequality (4.57), further computations show that

$$\operatorname{Re}\beta_1 \geq \frac{-(k+2)}{2}.$$

Since $k \geq 1$, then

$$\operatorname{Re}\beta_1 \geq -\frac{3}{2}$$

is sufficient for (4.57) to hold true. Hence, from Corollary 4.4.3, if $\operatorname{Re}\beta_1 \geq -3/2$

and

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z) - 1 \right| < M$$

then

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) - 1 \right| < M.$$

This complete the proof. ∎

Corollary 4.4.4 *Let $M > 0$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. If $f \in \mathcal{M}_p$, and*

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z) - z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) \right| < \frac{M}{|\beta_1+1|},$$

then

$$\left| z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) - 1 \right| < M.$$

Proof. Let $\phi(u, v, w; z) = v - u$ and $\Omega = h(\mathbb{D})$ where $h(z) = \frac{M}{|\beta_1+1|}z$, $M > 0$. It is sufficient to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissible condition (4.46) is satisfied. This follows since

$$\begin{aligned}
&\left| \phi\left(1 + Me^{i\theta}, 1 + \frac{k+\beta_1+1}{\beta_1+1}Me^{i\theta}, 1 + \frac{L + (\beta_1+1)(2k+\beta_1)Me^{i\theta}}{\beta_1(\beta_1+1)}; z\right) \right| \\
&= \left| 1 + \frac{k+\beta_1+1}{\beta_1+1}Me^{i\theta} - 1 - Me^{i\theta} \right| \\
&= \left| \frac{kM}{\beta_1+1} \right| \\
&\geq \frac{Mn}{|\beta_1+1|}
\end{aligned}$$

$z \in \mathbb{D}$, $\theta \in \mathbb{R}$, $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$ and $k \geq 1$. From Corollary 4.4.2, the required result is obtained. ∎

Definition 4.12 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfying the* *admissibility condition* $\phi(u, v, w; z) \notin \Omega$ *whenever*

$$u = q(\zeta),\ v = \frac{(\beta_1 + 1)q(\zeta)}{(\beta_1 + 2) - k\zeta q'(\zeta) - q(\zeta)}, \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \quad q(\zeta) \neq 0),$$

$$\operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2) - (\beta+1)u - vu}\left(\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right) - \frac{\beta+1}{v} - 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D} \setminus E(q)$ *and* $k \geq 1$.

In the particular case $q(z) = 1 + Mz,\ M > 0$, Definition 4.12 yields the following definition.

Definition 4.13 *Let* Ω *be a set in* $\mathbb{C}$ *and* $M > 0$*. The class of admissible functions* $\Phi_{H,1}[\Omega, M]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ *satisfying*

$$\phi\left(1 + Me^{i\theta}, \frac{(\beta_1 + 1)(1 + Me^{i\theta})}{(\beta_1 + 1) - Me^{i\theta}(k+1)},\right.$$
$$\left.\frac{\beta_1(1 + Me^{i\theta})\big(\beta_1 + 1 - Me^{i\theta}(k+1)\big)}{\big(\beta_1 + 1 - Me^{i\theta}(k+1)\big)\big(\beta_1 - 2Me^{i\theta}(k+1)\big) - (1 + Me^{i\theta})(L + 2kMe^{i\theta})}; z\right) \notin \Omega$$

whenever $z \in \mathbb{D},\ \theta \in \mathbb{R}$, $\operatorname{Re}(Le^{-i\theta}) \geq (k-1)kM$ *for all real* θ,

$\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ *and* $k \geq 1$.

In the special case $\Omega = q(\mathbb{D}) = \{\omega : |\omega - 1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is simply denoted by $\Phi_{H,1}[M]$.

Theorem 4.20 *Let* $\phi \in \Phi_{H,1}[\Omega, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$*. If* $f \in \mathcal{M}_p$ *satisfies*

$$\left\{\phi\left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1 + 3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1 + 1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1 + 1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z\right) : z \in \mathbb{D}\right\} \subset \Omega, \tag{4.58}$$

then

$$\frac{\widetilde{H^*}_p^{l,m}[\beta_1 + 3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1 + 2]f(z)} \prec q(z).$$

Proof. The proof is similar to Theorem 4.7. ∎

As in the previous cases, if $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi_{H,1}[h, q]$. The following result is an immediate consequence of Theorem 4.20.

Theorem 4.21 *Let $\phi \in \Phi_{H,1}[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h(z),$$

then

$$\frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)} \prec q(z).$$

Corollary 4.4.5 *Let $\phi \in \Phi_{H,1}[\Omega, M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ satisfies*

$$\phi\left(\frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1]f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)} - 1\right| < M.$$

Proof. The result follows by taking $q(z) = 1 + Mz$ in Theorem 4.20. ∎

Corollary 4.4.6 *Let $\phi \in \Phi_{H,1}[M]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$ satisfies*

$$\left|\phi\left(\frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1]f(z)}; z\right) - 1\right| < M,$$

then

$$\left| \frac{\widetilde{H*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H*}_p^{l,m}[\beta_1+2]f(z)} - 1 \right| < M.$$

Proof. By taking $\Omega = q(\mathbb{D})$ in Corollary 4.4.5, the result is obtained. ∎

4.5 Superordination of the Liu-Srivastava Operator

The dual problem of differential subordination, that is, differential superordination of the Liu-Srivastava linear operator is investigated in this section. For this purpose, the following class of admissible functions will be required.

Definition 4.14 *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever*

$$u = q(z), \quad v = \frac{zq'(z) + m(\beta_1+1)q(z)}{m(\beta_1+1)} \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{\beta_1(w-u)}{(v-u)} - (2\beta_1+1)\right) \leq \frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ and $m \geq 1$.

Theorem 4.22 *Let $\phi \in \Phi'_H[\Omega, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$, $z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$\Omega \subset \left\{\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right) : z \in \mathbb{D}\right\} \tag{4.59}$$

implies

$$q(z) \prec z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z).$$

Proof. From (4.54) and (4.59), it follows that

$$\Omega \subset \left\{\psi\left(p(z), zp'(z), z^2p''(z); z\right) : z \in \mathbb{D}\right\}.$$

From (4.52), it is clear that the admissibility condition for $\phi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'_p[\Omega, q]$, and by Theorem 2.2, $q(z) \prec p(z)$ or

$$q(z) \prec z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z).$$

∎

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi'_H[h(\mathbb{D}), q]$ is written as $\Phi'_H[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 4.22.

Theorem 4.23 *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{D}$, $\phi \in \Phi'_H[h, q]$ and $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. If $f \in \mathcal{M}_p$, $z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right) \tag{4.60}$$

implies

$$q(z) \prec z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z).$$

Theorem 4.22 and 4.23 can only be used to obtain subordinants of differential superordination of the form (4.59) or (4.60). The following theorem proves the existence of the best subordinant of (4.60) for certain ϕ.

Theorem 4.24 *Let h be analytic in $\mathbb{D}$, $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ and $\beta_1 \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$. Suppose that the differential equation*

$$\phi\big(q(z), zq'(z), z^2 q''(z); z\big) = h(z) \tag{4.61}$$

has a solution $q \in \mathcal{Q}_1$. If $\phi \in \Phi'_H[h, q]$, $f \in \mathcal{M}_p$, $z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z) \in \mathcal{Q}_1$ and

$$\phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, then

$$h(z) \prec \phi\left(z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1 + 1]f(z), z^p \widetilde{H*}_p^{l,m}[\beta_1]f(z); z\right)$$

implies

$$q(z) \prec z^p \widetilde{H*}_p^{l,m}[\beta_1 + 2]f(z),$$

and q is the best subordinant.

Proof. In view of Theorem 4.22 and Theorem 4.23 we deduce that q is a subordinant of (4.60) . Since q satisfies (4.61), it is also a solution of (4.60) and therefore q will be subordinated by all subordinants. Hence q is the best subordinant. ∎

Combining Theorems 4.17 and 4.23, we obtain the following sandwich-type theorem.

Corollary 4.5.1 *Let h_1 and q_1 be analytic functions in $\mathbb{D}$, h_2 be a univalent function in $\mathbb{D}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$, $\phi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$ and*

$\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{M}_p$, $z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) \in \mathcal{H} \cap \mathcal{Q}_1$ *and*

$$\phi\left(z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h_1(z) \prec \phi\left(z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1+1]f(z), z^p \widetilde{H^*}_p^{l,m}[\beta_1]f(z); z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec z^p \widetilde{H^*}_p^{l,m}[\beta_1+2]f(z) \prec q_2(z).$$

Definition 4.15 *Let* Ω *be a set in* $\mathbb{C}$ *and* $q \in \mathcal{H}$ *with* $zq'(z) \neq 0$. *The class of admissible functions* $\Phi'_{H,1}[\Omega, q]$ *consists of those functions* $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ *satisfying the admissibility condition* $\phi(u, v, w; \zeta) \in \Omega$ *whenever*

$$u = q(z),\ v = \frac{m(\beta_1+1)q(z)}{m(\beta_1+2) - zq'(z) - mq(z)}, \quad (\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

$$\operatorname{Re}\left(\frac{(\beta+1)u}{v(\beta+2) - (\beta+1)u - vu}\left(\frac{\beta+1}{v} - \frac{\beta}{w} - 1\right) - \frac{\beta+1}{v} - 1\right) \leq \frac{1}{m}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathbb{D}$, $\zeta \in \partial\mathbb{D}$ *and* $m \geq 1$.

Next the dual result of Theorem 4.20 for differential superordination is given.

Theorem 4.25 *Let* $\phi \in \Phi'_{H,1}[\Omega, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{M}_p$, $\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z\right)$$

is univalent in $\mathbb{D}$*, then*

$$\Omega \subset \left\{ \phi \left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z \right) : z \in \mathbb{D} \right\} \tag{4.62}$$

implies

$$q(z) \prec \frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}.$$

Proof. From (4.38) and (4.62), it follows that

$$\Omega \subset \left\{ \phi \left(p(z), zp'(z), z^2p''(z); z \right) : z \in \mathbb{D} \right\}.$$

From (4.36), the admissibility condition for $\phi \in \Phi'_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 2.2. Hence $\psi \in \Psi'[\Omega, q]$, and by Theorem 2.2, $q(z) \prec p(z)$ or

$$q(z) \prec \frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}. \qquad \blacksquare$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping h of $\mathbb{D}$ onto Ω. In this case the class $\Phi'_{H,1}[h(\mathbb{D}), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 4.25.

Theorem 4.26 *Let* $q \in \mathcal{H}$*,* h *be analytic in* $\mathbb{D}$*,* $\phi \in \Phi'_{H,1}[h, q]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$*. If* $f \in \mathcal{M}_p$*,* $\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{Q}_1$ *and*

$$\phi \left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z \right)$$

is univalent in $\mathbb{D}$, *then*

$$h(z) \prec \phi\left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z\right)$$

implies

$$q(z) \prec \frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}.$$

Combining Theorems 4.21 and 4.26, the following sandwich-type theorem is obtained.

Corollary 4.5.2 *Let* h_1 *and* q_1 *be analytic functions in* $\mathbb{D}$, h_2 *be univalent function in* $\mathbb{D}$, $q_2 \in \mathcal{Q}_1$ *with* $q_1(0) = q_2(0) = 1$, $\phi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$ *and* $\beta_1 \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. *If* $f \in \mathcal{M}_p$, $\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ *and*

$$\phi\left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z\right)$$

is univalent in $\mathbb{D}$, *then*

$$h_1(z) \prec \phi\left(\frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}, \frac{\widetilde{H^*}_p^{l,m}[\beta_1+1]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1]f(z)}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\widetilde{H^*}_p^{l,m}[\beta_1+3]f(z)}{\widetilde{H^*}_p^{l,m}[\beta_1+2]f(z)} \prec q_2(z).$$

CHAPTER 5

HALF-PLANE DIFFERENTIAL SUBORDINATION CHAIN

5.1 Introduction

As defined in Section 1.3.2, p. 22, Bernardi [27] introduced the linear integral operator $F_\mu : \mathcal{A} \to \mathcal{A}$ defined by

$$F_\mu(z) := (\mu+1)\int_0^1 t^{\mu-1} f(tz)dt \quad (\mu > -1). \tag{5.1}$$

It is well-known [27] that the classes of starlike, convex and close-to-convex functions are closed under the Bernardi integral transform.

Parvatham considered the class $\mathcal{R}[\alpha]$ of functions $f \in \mathcal{A}$ satisfying

$$\left|f'(z) - 1\right| < \alpha \left|f'(z) + 1\right| \quad (z \in \mathbb{D},\ 0 < \alpha \leq 1),$$

or equivalently

$$f'(z) \prec \frac{1+\alpha z}{1-\alpha z} \quad (z \in \mathbb{D},\ 0 < \alpha \leq 1),$$

and obtained the following result:

Theorem 5.1 [94, Theorem 2, p. 440] *Let $\mu \geq 0$, $0 < \alpha \leq 1$ and δ be given by*

$$\delta := \alpha\left(\frac{2-\alpha+\mu(1-\alpha)}{1+\mu(1-\alpha)}\right).$$

If the functions $f \in \mathcal{R}[\delta]$, then the function F_μ given by Bernardi's integral (5.1) is in $\mathcal{R}[\alpha]$.

The class $\mathcal{R}[\alpha]$ can be extended to the general class $\mathcal{R}[A, B]$ consisting of all analytic functions $f \in \mathcal{A}$ satisfying

$$f'(z) \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathbb{D},\ -1 \leq B < A \leq 1),$$

or the equivalent inequality

$$\left|f'(z)-1\right| < \left|A-Bf'(z)\right| \quad (z\in\mathbb{D},\ -1\le B<A\le 1).$$

For $0\le\alpha<1$, the class $\mathcal{R}[1-2\alpha,-1]$ consists of functions $f\in\mathcal{A}$ for which

$$\operatorname{Re} f'(z)>\alpha \quad (z\in\mathbb{D},\ 0<\alpha\le 1),$$

and $\mathcal{R}[1-\alpha,0]$ is the class of functions $f\in\mathcal{A}$ satisfying the condition

$$\left|f'(z)-1\right| < 1-\alpha \quad (z\in\mathbb{D},\ 0\le\alpha<1).$$

When $0<\alpha\le 1$, the class $\mathcal{R}[\alpha,-\alpha]$ is the class $\mathcal{R}[\alpha]$ considered by Parvatham [94].

Silverman [122], Obradovič and Tuneski [87] and many others (see [84–86, 104, 105]) have studied properties of functions defined in terms of the expression

$$\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f(z)}{zf'(z)}\right).$$

In fact, Silverman [122] obtained the order of starlikeness for functions in the class $\mathcal{G}_b$ defined by

$$\mathcal{G}_b := \left\{ f\in\mathcal{A} : \left|\left(1+\frac{zf''(z)}{f'(z)}\right)\left(\frac{f(z)}{zf'(z)}\right)-1\right| < b,\ 0<b\le 1,\ z\in\mathbb{D}\right\}.$$

Obradovič and Tuneski [87] improved the result of Silverman [122] by showing

$$\mathcal{G}_b\subset\mathcal{ST}[0,-b]\subset\mathcal{ST}\big(2/(1+\sqrt{1+8b})\big).$$

Tuneski [139] took it further and obtained conditions for the inclusion

$\mathcal{G}_b \subset \mathcal{ST}[A,B]$ to hold. Letting $zf'(z)/f(z) =: p(z)$, the inclusion $\mathcal{G}_b \subset \mathcal{ST}[A,B]$ is evidently a special case of the differential chain

$$1+\beta\frac{zp'(z)}{p(z)^2} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{5.2}$$

For $f \in \mathcal{A}$ and $0 \leq \alpha < 1$ Frasin and Darus [40] showed that

$$\frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)} \prec \frac{(1-\alpha)z}{2-\alpha} \Rightarrow \left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1-\alpha.$$

Letting $z^2 f'(z)/f^2(z)$ as $p(z)$, the above implication is a special case of the differential chain

$$1+\beta\frac{zp'(z)}{p(z)} \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{5.3}$$

Nunokawa et al. [83] showed that when p is analytic in $\mathbb{D}$ with $p(0) = 1$, then

$$1+zp'(z) \prec 1+z \Rightarrow p(z) \prec 1+z. \tag{5.4}$$

They applied this differential implication to obtain a criterion for normalized analytic functions to be univalent. Clearly (5.4) is a special case of the differential chain

$$1+\beta zp'(z) \prec \frac{1+Dz}{1+Ez} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{5.5}$$

The implications (5.2), (5.3) and (5.5) have been investigated in [16]. Analogous results were also obtained in [8] by considering the expressions $1+\beta zp'(z)$, $1+\beta(zp'(z)/p(z))$, $1+\beta(zp'(z)/p^2(z))$ and $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)$ as subordinate functions to the function $\sqrt{1+z}$ that maps $\mathbb{D}$ onto the right-half of the lemniscate of Bernoulli . Each led to the deduction that p is subordinated to $\sqrt{1+z}$.

Singh and Gupta [126] showed that whenever p and q are analytic in $\mathbb{D}$ with

$p(0) = 1 = q(0)$ and

$$(1-\alpha)p(z) + \alpha p^2(z) + \alpha\gamma z p'(z) \prec (1-\alpha)q(z) + \alpha q^2(z) + \alpha\gamma z q'(z),$$

then $p(z) \prec q(z)$. They also investigated similar problems in [127] and [128].

The present work investigates the differential implication

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z) \prec \frac{1+Cz}{1+Dz} \Rightarrow p(z) \prec \frac{1+Az}{1+Bz}. \tag{5.6}$$

Using the implication (5.6), in this thesis, a more general result relating to the Briot-Bouquet differential subordination is obtained which is then applied to the Bernardi's integral operator on the class $\mathcal{R}[C, D]$. Analogous results are obtained by considering the expressions $p(z) + zp'(z)/p^2(z)$ and $p^2(z) + zp'(z)/p(z)$. These results are then used to obtain sufficient conditions for normalized analytic functions in $\mathbb{D}$ to be Janowski starlike.

5.2 Some Subordination Results

Lemma 5.1 *Let $a, b, c, d \in \mathbb{R}$.*

(i) If $a \neq 0$ and $b^2 < 3ac$, then

$$\min_{|t|\leq 1}\left(at^3 + bt^2 + ct + d\right) = b + d - |a + c|.$$

(ii) If $a \neq 0$ and $b^2 \geq 3ac$, then

$$\begin{aligned}&\min_{|t|\leq 1}\left(at^3 + bt^2 + ct + d\right)\\ &= \min\left(\frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}}\right), b + d - |a + c|\right).\end{aligned}$$

(iii) If $a = 0$, then

$$\min_{|t|\le 1}\left(bt^2+ct+d\right) = \begin{cases} \frac{4bd-c^2}{4b} & (b>0 \quad and \quad |c|<2b), \\ b+d-|c| & (b\le 0 \quad or \quad |c|\ge 2b). \end{cases}$$

Proof. (i) When $a \neq 0$, let $f(t) = at^3 + bt^2 + ct + d$. Then $f'(t) = 3at^2 + 2bt + c$ and $f''(t) = 6at + 2b$. Thus $f'(t) = 0$ if

$$t = \frac{-b \pm \sqrt{b^2-3ac}}{3a}.$$

Let

$$t_0 = \frac{-b+\sqrt{b^2-3ac}}{3a} \quad \text{and} \quad t_1 = \frac{-b-\sqrt{b^2-3ac}}{3a}. \tag{5.7}$$

If $b^2 < 3ac$, then

$$\begin{aligned} \min_{|t|\le 1}\left(at^3+bt^2+ct+d\right) &= \min\left(f(1), f(-1)\right) \\ &= \min\left\{a+b+c+d,\ b+d-(a+c)\right\} \\ &= b+d-|a+c|. \end{aligned}$$

(ii) Let $a \neq 0$ and $b^2 \ge 3ac$. Using (5.7),

$$\begin{aligned} f''(t_0) &= f''\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right) \\ &= 6a\left(\frac{-b+\sqrt{b^2-3ac}}{3a}\right)+2b \\ &= 2\sqrt{b^2-3ca} \\ &\ge 0 \end{aligned}$$

and so $t_0 = \frac{-b+\sqrt{b^2-3ac}}{3a}$ gives minimum value.

When $t_1 = \frac{-b-\sqrt{b^2-3ac}}{3a}$,

$$\begin{aligned} f''\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right) &= 6a\left(\frac{-b-\sqrt{b^2-3ac}}{3a}\right) + 2b \\ &= -2\sqrt{b^2-3ca} \\ &\leq 0 \end{aligned}$$

and so $t_1 = \frac{-b-\sqrt{b^2-3ac}}{3a}$ gives maximum value. It is clear that $t = t_0$ gives minimum value. Also

$$\begin{aligned} f(t_0) &= f\left(\frac{-b+\sqrt{(b^2-3ac)}}{3a}\right) \\ &= \frac{2ab^3 - 9a^2bc + 27a^3d - 2a(b^2-3ac)^{\frac{3}{2}}}{27a^3} \\ &= \frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2-3ac)^{\frac{3}{2}}\right). \end{aligned}$$

Thus,

$$\min_{|t|<1} f(t) = \min\left(f(t_0), f(1), f(-1)\right),$$

or equivalently

$$\begin{aligned} &\min_{|t|\leq 1}\left(at^3 + bt^2 + ct + d\right) \\ &= \min\left(\frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2-3ac)^{\frac{3}{2}}\right), b + d - |a+c|\right). \end{aligned}$$

This complete the proof for part (ii).

(iii) Let $a = 0$. The result is clear for $b = 0$. Let $f(t) = bt^2 + ct + d$. Then $f'(t) = 2bt + c$ and $f''(t) = 2b$. Thus $f'(t) = 0$ if $t = -\frac{c}{2b}$. For $b > 0$, we first note that

$$f(1) = b + d + c, \quad f(-1) = b + d - c, \quad f\left(-\frac{c}{2b}\right) = \frac{-c^2 + 4bd}{4b}$$

and

$$b + d - |c| \geq \frac{-c^2 + 4bd}{4b}$$

is equivalent to

$$4b^2 + 4bd - 4b|c| \geq -c^2 + 4bd$$

or

$$(2b - |c|)^2 \geq 0$$

which is trivially true. Thus, for $b > 0$,

$$f(1), f(-1) \geq f(-c/2b).$$

<u>Case(1) ($b > 0,\ |c| < 2b$)</u>

In this case, the function $f(t)$ attains its minimum at $t = -\frac{c}{2b}$. Thus the global minimum of $f(t)$ is given by

$$\min_{|t|\leq 1} f(t) = \min\left(f(1), f(-1), f\left(-\frac{c}{2b}\right)\right) = f\left(-\frac{c}{2b}\right) = \frac{-c^2 + 4bd}{4b}.$$

<u>Case(2) (Otherwise)</u>

In this case, either $b < 0$ or $|c| \geq 2b$. If $b < 0$, then the point $t = -c/2b$ gives maximum while for $|c| \geq 2b$, the point $t = -c/2b$ lies at the boundary or outside [-1,1]. Thus the minimum occurs only at boundary and thus the global minimum is given by

$$\begin{aligned}
\min_{|t|\leq 1} f(t) &= \min(f(1), f(-1)) \\
&= \min(b + d + c, b + d - c) \\
&= b + d - |c|
\end{aligned}$$

This complete the proof for part (iii). ∎

Now an application of Lemma 2.1 yields the following result.

Lemma 5.2 *Let α, β, A and B be real numbers satisfying $-1 \leq B \leq 0$, $A \neq B$, $\beta \neq 0$, and*

$$\frac{\alpha+\beta+1}{\beta} \geq \left|\frac{(1-\alpha-\beta)B+2\alpha A}{\beta}\right|. \tag{5.8}$$

Let $q(z) = (1+Az)/(1+Bz)$. If p is analytic in $\mathbb{D}$ with $p(0) = 1$, and

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta z p'(z) \prec (1-\alpha)q(z)+\alpha q^2(z)+\beta z q'(z), \tag{5.9}$$

then $p \in \mathcal{P}[A,B]$, and q is the best dominant.

Proof. Let the function $q : \mathbb{D} \to \mathbb{C}$ be defined by

$$q(z) = \frac{1+Az}{1+Bz}. \tag{5.10}$$

A computation from (5.10) gives

$$\beta z q'(z) = \frac{\beta z(A-B)}{(1+Bz)^2}.$$

Further computation shows that

$$\begin{aligned}
&(1-\alpha)q(z)+\alpha q^2(z)+\beta z q'(z)\\
&= (1-\alpha)\left(\frac{1+Az}{1+Bz}\right)+\alpha\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\beta z(A-B)}{(1+Bz)^2}\\
&= \frac{1+\big((1+\alpha+\beta)A+(1-\alpha-\beta)B\big)z+\left((1-\alpha)AB+\alpha A^2\right)z^2}{(1+Bz)^2}.
\end{aligned} \tag{5.11}$$

Therefore

$$
\begin{aligned}
&(1-\alpha)q(z)+\alpha q^2(z)+\beta zq'(z)\\
&=\frac{1+\big((1+\alpha+\beta)A+(1-\alpha-\beta)B\big)z+\big((1-\alpha)AB+\alpha A^2\big)\,z^2}{(1+Bz)^2}.
\end{aligned}
\tag{5.12}
$$

Define the function ϑ and φ by

$$
\vartheta(w)=(1-\alpha)w+\alpha w^2, \qquad \varphi(w)=\beta
$$

so that (5.9) becomes (2.5). Clearly the functions ϑ and φ are analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$. Also let $Q,h:\mathbb{D}\to\mathbb{C}$ be the functions defined by

$$
Q(z):=zq'(z)\varphi(q(z))=\beta zq'(z)
$$

and

$$
h(z):=\vartheta(q(z))+Q(z)=(1-\alpha)q(z)+\alpha q^2(z)+\beta zq'(z).
$$

Since q is convex, the function $zq'(z)$ is starlike, and therefore Q is starlike univalent in $\mathbb{D}$. A computation yield

$$
\begin{aligned}
\frac{zh'(z)}{Q(z)}&=\frac{z(1-\alpha)q'(z)}{Q(z)}+\frac{2\alpha zq(z)q'(z)}{Q(z)}+\frac{zQ'(z)}{Q(z)}\\
&=\frac{\frac{\alpha+\beta+1}{\beta}+\frac{1}{\beta}\big((1-\alpha-\beta)B+2\alpha A\big)z}{1+Bz}.
\end{aligned}
$$

Therefore

$$
\frac{zh'(z)}{Q(z)}=\frac{a+bz}{1+Bz}
$$

where $a=\frac{\alpha+\beta+1}{\beta}$ and $b=\frac{1}{\beta}\big((1-\alpha-\beta)B+2\alpha A\big)$.

In view of (5.8), it follows that

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) = \operatorname{Re} a\left(\frac{1+\frac{b}{a}z}{1+Bz}\right) \geq 0.$$

By Lemma 2.1, if p is analytic in $\mathbb{D}$ with $p(0)=1$, and

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z) \prec (1-\alpha)q(z)+\alpha q^2(z)+\beta zq'(z),$$

then $p \in \mathcal{P}[A,B]$, and q is the best dominant. This complete the proof. ∎

Setting $p(z) = \frac{zf'(z)}{f(z)}$, $\beta = \alpha\lambda$ and $B=-1$ where $\alpha,\lambda \in \mathbb{R}$, $\alpha,\lambda > 0$ in Lemma 5.2, we obtain the following result of Singh and Gupta [126]:

Corollary 5.2.1 [126] *Let α and λ be positive real numbers. Assume that $-1 < A \leq 1$ satisfies $A \leq 1/\alpha$ whenever $\alpha > 1$. If $f \in \mathcal{A}, f(z)/z \neq 0$ in $\mathbb{D}$ and*

$$\frac{zf'(z)}{f(z)}\left(1-\alpha+\alpha(1-\lambda)\frac{zf'(z)}{f(z)}+\alpha\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right) \prec h(z),$$

where

$$h(z) = \frac{1+\Big(\big(1+\alpha(1+\lambda)\big)A-\big(1-\alpha(1+\lambda)\big)\Big)z+\big(\alpha A^2-(1-\alpha)A\big)z^2}{(1-z)^2}, \tag{5.13}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1-z},$$

and $(1+Az)/(1-z)$ is the best dominant.

Remark 5.2.1 *By judicious choices for A, α and λ in Corollary 5.2.1, Singh and Gupta* [126] *obtained a number of known results in* [61, 91, 127].

The following result which is obtained by considering the expression

$p(z)+zp'(z)/p^2(z)$, gives sufficient condition for an analytic function to be in the class $\mathcal{P}[A,B]$.

Lemma 5.3 *Let $-1<B<A\leq 1$ and*

$$\frac{A-B}{1-AB}\leq\frac{1}{\sqrt{2}}. \tag{5.14}$$

Let $q(z)=(1+Az)/(1+Bz)$. If p is analytic and $p(z)\neq 0$ in $\mathbb{D}$, $p(0)=1$ and

$$p(z)+\frac{zp'(z)}{p^2(z)}\prec q(z)+\frac{zq'(z)}{q^2(z)}, \tag{5.15}$$

then $p\in\mathcal{P}[A,B]$ and q is the best dominant.

Proof. As in the proof of Lemma 5.2, let the function q be defined by

$$q(z)=\frac{1+Az}{1+Bz} \tag{5.16}$$

and by taking the logarithmic differentiation on (5.16) gives

$$q(z)+\frac{zq'(z)}{q^2(z)}=\frac{1+(4A-B)z+(3A^2-B^2+BA)z^2+A^3z^3}{(1+Bz)(1+Az)^2}. \tag{5.17}$$

Now to show that (5.15) implies $p(z)\prec q(z)$, let the functions ϑ and φ defined by

$$\vartheta(w)=w, \qquad \varphi(w)=\frac{1}{w^2}.$$

Since $w\neq 0$, the functions ϑ and φ are analytic in $\mathbb{C}$ and $\varphi(w)\neq 0$.
Let $Q,h:\mathbb{D}\to\mathbb{C}$ be the functions defined by

$$Q(z):=zq'(z)\varphi(q(z))=\frac{zq'(z)}{q^2(z)}=\frac{z(A-B)}{(1+Az)^2} \tag{5.18}$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = q(z) + \frac{zq'(z)}{q^2(z)} = \frac{1+Az}{1+Bz} + \frac{z(A-B)}{(1+Az)^2}. \tag{5.19}$$

A calculation from (5.18) yields

$$\frac{zQ'(z)}{Q(z)} = \frac{1-Az}{1+Az}.$$

Further computations show that

$$\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) = \mathrm{Re}\left(\frac{1-Az}{1+Az}\right) > \frac{|1-A|^2}{1-|A|^2} > 0 \tag{5.20}$$

for $-1 < A \leq 1$, which shows Q is starlike univalent in $\mathbb{D}$. Now (5.18) and (5.19) yield

$$\frac{zh'(z)}{Q(z)} = q^2(z) + \frac{zQ'(z)}{Q(z)}.$$

From (5.20) it is clear that $\mathrm{Re}\left(\frac{zQ'(z)}{Q(z)}\right) \geq 0$. Therefore, to show $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ it is enough to show $\mathrm{Re}(q^2(z)) > 0$.

$$q(z) = \frac{1+Az}{1+Bz}$$

maps the disk $\mathbb{D}$ onto the disk

$$\left|q(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}.$$

It is evident that for the disk $|w-a| < r$, the following inequality holds true:

$$|\arg w| \leq \sin^{-1}\left(\frac{r}{a}\right).$$

Therefore,

$$|\arg q(z)| \leq \sin^{-1}\left(\frac{\frac{A-B}{1-B^2}}{\frac{1-AB}{1-B^2}}\right) = \sin^{-1}\left(\frac{A-B}{1-AB}\right).$$

Further in view of (5.14), it follows that

$$\sin^{-1}\left(\frac{A-B}{1-AB}\right) \leq \frac{\pi}{4}$$

which implies

$$|\arg q(z)| \leq \frac{\pi}{4}$$

or

$$|\arg q^2(z)| \leq \frac{\pi}{2}$$

which shows

$$\operatorname{Re}\left(q^2(z)\right) > 0. \tag{5.21}$$

Therefore, it is clear that from (5.20) and (5.21)

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0.$$

By Lemma 2.1, if p is analytic and $p(z) \neq 0$ in $\mathbb{D}$, $p(0) = 1$ and

$$p(z) + \frac{zp'(z)}{p^2(z)} \prec q(z) + \frac{zq'(z)}{q^2(z)},$$

then $p \in \mathcal{P}[A, B]$ and q is the best dominant. This complete the proof. ∎

Analogous to the condition in Lemma 5.3, the following result considers the expression $p^2(z) + zp'(z)/p(z)$ in order to obtain a sufficient condition for an analytic function to be in the class $\mathcal{P}[A, B]$.

Lemma 5.4 *Let $-1 < B < A \leq 1$ and*

$$\frac{A-B}{1-AB} \leq \frac{1}{\sqrt{2}}.$$

Let $q(z) = (1+Az)/(1+Bz)$. If p is analytic and $p(z) \neq 0$ in $\mathbb{D}$, $p(0) = 1$ and

$$p^2(z) + \frac{zp'(z)}{p(z)} \prec q^2(z) + \frac{zq'(z)}{q(z)}, \tag{5.22}$$

then $p \in \mathcal{P}[A, B]$ and q is the best dominant.

Proof. Similar to the proof of Lemma 5.3, let $q(z) = (1+Az)/(1+Bz)$ and by using logarithmic differentiation on q it yields

$$\frac{zq'(z)}{q(z)} = \frac{z(A-B)}{(1+Az)(1+Bz)}.$$

Another computations show that,

$$q^2(z) + \frac{zq'(z)}{q(z)} = \frac{1 + (4A-B)z + (3A^2 + AB - B^2)z^2 + A^3z^3}{(1+Az)(1+Bz)^2}. \tag{5.23}$$

Let the functions ϑ and φ defined by

$$\vartheta(w) = w^2, \qquad \varphi(w) = \frac{1}{w}.$$

Since $w \neq 0$ the functions ϑ and φ are analytic in $\mathbb{C}$ and $\varphi(w) \neq 0$. Analogue in the previous theorems, let $Q, h : \mathbb{D} \to \mathbb{C}$ be the functions defined by

$$Q(z) := zq'(z)\varphi(q(z)) = \frac{zq'(z)}{q(z)} \tag{5.24}$$

and

$$h(z) := \vartheta(q(z)) + Q(z) = q^2(z) + \frac{zq'(z)}{q(z)}. \tag{5.25}$$

A computation from

$$Q(z) = \frac{z(A-B)}{(1+Az)(1+Bz)}$$

shows that

$$\frac{zQ'(z)}{Q(z)} = \frac{1-ABz^2}{(1+Az)(1+Bz)}. \tag{5.26}$$

Thus

$$\begin{aligned}
\operatorname{Re}\left(\frac{1-ABz^2}{(1+Az)(1+Bz)}\right) &= \operatorname{Re}\left(\frac{1}{1+Az}+\frac{1}{1+Bz}-1\right)\\
&> \frac{1}{1+|A|}+\frac{1}{1+|B|}-1\\
&= \frac{1-|AB|}{(1+|A|)(1+|B|)} > 0 \quad (-1 < B < A \leq 1).
\end{aligned}$$

Therefore

$$\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) > 0. \tag{5.27}$$

Therefore Q is starlike univalent in $\mathbb{D}$. Now (5.24) and (5.25) yield

$$h(z) = q^2(z) + Q(z) = 2q^2(z) + \frac{zQ'(z)}{Q(z)}.$$

From the proof of Theorem 5.3, it is clear that $\operatorname{Re}(q^2(z)) > 0$ and (5.27) show that $\operatorname{Re}\left(\frac{zQ'(z)}{Q(z)}\right) \geq 0$. Therefore, $\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$. By Lemma 2.1, it is proven that, if p is analytic and $p(z) \neq 0$ in $\mathbb{D}$, $p(0) = 1$ and

$$p^2(z) + \frac{zp'(z)}{p(z)} \prec q^2(z) + \frac{zq'(z)}{q(z)},$$

then $p \in \mathcal{P}[A, B]$ and q is the best dominant. This complete the proof. ∎

5.3 Sufficient Conditions for Starlikeness

By making use of Lemma 5.1(iii) and Lemma 5.2, the following result is proved.

Theorem 5.2 *Let A, B, C, D, α and β be real numbers satisfying $|D| \leq 1$, $C \neq D$, $|B| \leq 1$, $A \neq B$ and $\beta \neq 0$. Let $b = -4KM$, $c = 2\big(IJ - L(K+M)\big)$ and $d = I^2 + J^2 - L^2 - (M-K)^2$, where $I = (A-B)(1+\alpha+\beta)$,*
$J = (A-B)(A\alpha + B)$, $K = C - D$, $L = 2CB - DA(1+\alpha+\beta) - DB(1-\alpha-\beta)$ and $M = CB^2 - DA\big(B + \alpha(A-B)\big)$. Let

$$\frac{\alpha+\beta+1}{\beta} \geq \left|\frac{(1-\alpha-\beta)B + 2\alpha A}{\beta}\right|.$$

Further, suppose one of the following conditions hold:

(i) If $b > 0$ and $|c| < 2b$, then $c^2 \leq 4bd$,

(ii) If $b \leq 0$ or $|c| \geq 2b$, then $|c| \leq b + d$.

If an analytic function p in $\mathbb{D}$ satisfies $p(0) = 1$ and the subordination

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z) \prec \frac{1+Cz}{1+Dz},$$

then $p \in \mathcal{P}[A, B]$.

Proof. Let q be the convex function given by $q(z) = (1+Az)/(1+Bz)$. By Lemma 5.2, it follows that the subordination

$$(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z) \prec (1-\alpha)q(z) + \alpha q^2(z) + \beta z q'(z)$$

implies

$$p(z) \prec q(z).$$

The result is proved if it could be shown that

$$g(z) := \frac{1+Cz}{1+Dz} \prec (1-\alpha)q(z) + \alpha q^2(z) + \beta z q'(z) =: h(z) \tag{5.28}$$

Since g is univalent, the subordination $g(z) \prec h(z)$ is equivalent to the subordination

$$z \prec g^{-1}(h(z)) =: H(z).$$

The proof will be completed by showing that $|H(e^{i\theta})| \geq 1$ for all $\theta \in [0, 2\pi]$. First note that from (5.12) we have

$$h(z) = \frac{(1-\alpha)(1+Az)(1+Bz) + \alpha(1+Az)^2 + \beta z(A-B)}{(1+Bz)^2}. \tag{5.29}$$

Furthermore

$$q(z) = \frac{1+Cz}{1+Dz}$$

yields

$$g^{-1}(w) = \frac{w-1}{C-Dw}. \tag{5.30}$$

Replacing $w = h(z)$ in $g^{-1}(w)$, we obtain

$$\begin{aligned} H(z) &= g^{-1}(h(z)) \\ &= \frac{h(z)-1}{C-Dh(z)} \\ &= \frac{(A-B)(1+\alpha+\beta)z + (A-B)(A\alpha+B)z^2}{\begin{array}{l} C-D+\big(2CB-DA(1+\alpha+\beta)-DB(1-\alpha-\beta)\big)z \\ +\Big(CB^2-DA\big(B+\alpha(A-B)\big)\Big)z^2 \end{array}} \end{aligned}$$

By writing,

$$\begin{aligned} I &= (A-B)(1+\alpha+\beta), \\ J &= (A-B)(A\alpha+B), \\ K &= C-D, \\ L &= 2CB-DA(1+\alpha+\beta)-DB(1-\alpha-\beta), \end{aligned}$$

and

$$M = CB^2 - DA\big(B + \alpha(A - B)\big)$$

we see that

$$\begin{aligned} H(z) = g^{-1}(h(z)) &= \frac{Iz + Jz^2}{K + Lz + Mz^2} \\ &= \frac{z(I + Jz)}{z(Kz^{-1} + L + Mz)} \\ &= \frac{I + Jz}{Kz^{-1} + L + Mz}. \end{aligned}$$

Replacing $z = e^{i\theta}$ in $|H(z)|$ we get

$$|H(e^{i\theta})| = \left| \frac{I + Je^{i\theta}}{Ke^{-i\theta} + L + Me^{i\theta}} \right|$$

$$= \left| \frac{I + J\cos\theta + iJ\sin\theta}{K\cos\theta - iK\sin\theta + L + M\cos\theta + iM\sin\theta} \right|$$

$$= \left| \frac{I + J\cos\theta + iJ\sin\theta}{L + (K + M)\cos\theta + i(M - K)\sin\theta} \right|.$$

Further computations show that,

$$|H(e^{i\theta})|^2 \geq 1$$

or

$$\frac{(I + J\cos\theta)^2 + J^2\sin^2\theta}{\big(L + (K + M)\cos\theta\big)^2 + (M - K)^2\sin^2\theta} \geq 1$$

or

$$\frac{I^2 + 2IJ\cos\theta + J^2\cos^2\theta + J^2(1 - \cos^2\theta)}{L^2 + 2L(K + M)\cos\theta + (K + M)^2\cos^2\theta + (M - K)^2(1 - \cos^2\theta)} \geq 1$$

or

$$I^2 + J^2 - L^2 - (M-K)^2 + 2\big(IJ - L(K+M)\big)\cos\theta - 4KM\cos^2\theta \geq 0. \quad (5.31)$$

Let

$$\begin{aligned} b &= -4KM, \\ c &= 2\big(IJ - L(K+M)\big), \\ d &= I^2 + J^2 - L^2 - (M-K)^2, \end{aligned}$$

and

$$t = \cos\theta.$$

Then (5.31) can be written as,

$$bt^2 + ct + d \geq 0.$$

Now $|H(e^{i\theta})|^2 \geq 1$ provided $bt^2 + ct + d \geq 0$ or equivalently

$$\min_{|t|\leq 1}\Big(bt^2 + ct + d\Big) \geq 0.$$

By using Lemma 5.1(iii), we see that

$$\min_{|t|\leq 1}\Big(bt^2 + ct + d\Big) = \begin{cases} \frac{4bd-c^2}{4b} & (b > 0 \quad and \quad |c| < 2b), \\ \\ b + d - |c| & (b \leq 0 \quad or \quad |c| \geq 2b). \end{cases}$$

Therefore, $|H(e^{i\theta})| \geq 1$ provided that when $b > 0$ and $|c| < 2b$

$$\frac{4bd - c^2}{4b} \geq 0$$

or equivalently $c^2 \leq 4bd$, or when $b \leq 0$ or $|c| \geq 2b$

$$|c| \leq b + d.$$

This complete the proof. ∎

An application of Theorem 5.2 to the Bernardi's integral operator on the class $\mathcal{R}[C, D]$ yields the following result.

Theorem 5.3 [17, Theorem 3.2] *Let the conditions of Theorem 5.2 hold with $\alpha = 0$ and $\beta = 1/(\mu+1)$. If $f \in \mathcal{R}[C, D]$, then F_μ given by the Bernardi's integral (5.1) is in $\mathcal{R}[A, B]$.*

Proof. It follows from (5.1) that

$$(\mu + 1)f(z) = zF'_\mu(z) + \mu F_\mu(z),$$

and so

$$f'(z) = F'_\mu(z) + \frac{zF''_\mu(z)}{\mu + 1}.$$

The result now follows from Theorem 5.2 with $p(z) = F'_\mu(z), \alpha = 0$ and $\beta = 1/(\mu + 1)$. ∎

For $A = \gamma$, $B = -\gamma$, $C = \delta$ and $D = -\delta\,(0 < \gamma, \delta \leq 1)$, then $I = 2\gamma\big((\mu+2)/(\mu+1)\big)$, $J = -2\gamma^2$, $K = 2\delta$, $L = -2\mu\delta\gamma/(\mu+1)$ and $M = 0$. Since $KM = 0$, condition (ii) in Theorem 5.2 holds provided

$$2\gamma(\mu + 1)\left|\gamma^2(\mu + 2) + \delta^2\mu\right| \leq \gamma^2 \left(\mu^2(1 - \delta^2) + 4(\mu + 1)\right) + (\gamma^4 - \delta^2)(\mu + 1)^2.$$

Thus, Theorem 5.3 yields the following result (see also [17, Remark 3.3]):

Corollary 5.3.1 *Let $\mu > -1$, $0 < \gamma, \delta \leq 1$ and*

$$2\gamma(\mu+1)\left|\gamma^2(\mu+2)+\delta^2\mu\right| \leq \gamma^2\left(\mu^2(1-\delta^2)+4(\mu+1)\right)+(\gamma^4-\delta^2)(\mu+1)^2.$$

If $f \in \mathcal{R}[\delta]$, then F_μ given by the Bernardi's integral (5.1) lies in $\mathcal{R}[\gamma]$.

Note that Corollary 5.3.1 extends Theorem 5.1 from the case $\mu \geq 0$ to $\mu > -1$.

For $A = 1-\gamma$, $B = 0$, $C = 1-\delta$ and $D = 0\,(0 < \gamma, \delta \leq 1)$, then $I = (1-\gamma)\big((\mu+2)/(\mu+1)\big)$, $J = 0$, $K = 1-\delta$, $L = 0$ and $M = 0$. Since $KM = 0$, condition (ii) in Theorem 5.2 holds provided $(1-\delta)/(1-\gamma) \leq (\mu+2)/(\mu+1)$. Thus, Theorem 5.3 yields the following result (also see [17, Corollary 3.3]):

Corollary 5.3.2 *Let $\mu > -1$, $0 < \gamma, \delta \leq 1$ and*

$$\frac{1-\delta}{1-\gamma} \leq \frac{\mu+2}{\mu+1}.$$

If $f \in \mathcal{R}_\delta$, then F_μ given by Bernardi's integral (5.1) is in $\mathcal{R}_\gamma$.

An application of Theorem 5.2 yield the sufficient conditions for normalized analytic function to be Janowski starlike.

Theorem 5.4 *Let the conditions of Theorem 5.2 holds. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)}\left(\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\alpha\frac{zf'(z)}{f(z)}+(1-\alpha)\right) \prec \frac{1+Cz}{1+Dz}$$

then $f \in \mathcal{ST}[A,B]$.

Proof. With $p(z) = \frac{zf'(z)}{f(z)}$, a computation shows that

$$\begin{aligned}&(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)\\ &= (1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(\frac{zf'(z)}{f(z)}\right)^2+\beta\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\end{aligned}$$

$$= \frac{zf'(z)}{f(z)}\left(\beta\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)+\alpha\frac{zf'(z)}{f(z)}+(1-\alpha)\right).$$

The result now follows from Theorem 5.2. ∎

The following sufficient condition for starlikeness of order γ is an application of Theorem 5.4.

Corollary 5.3.3 *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{zf'(z)}{f(z)}\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)-1\right| < \frac{1-\gamma}{2} \quad (0 \leq \gamma < 1),$$

then $f \in \mathcal{ST}(\gamma)$

Proof. The result follows from Theorem 5.4 by taking $\alpha = 0$, $\beta = 1$, $A = 1-2\gamma$, $B = -1$, $C = (1-\gamma)/2$ and $D = 0\,(0 \leq \gamma < 1)$. ∎

An application of Lemma 5.1 and Lemma 5.3 gives the following result.

Theorem 5.5 *Let $-1 < B < A \leq 1$, $-1 \leq D < C \leq 1$ and $(A-B)/(1-AB) \leq 1/\sqrt{2}$. Let $a = -8LR$, $b = 4IK - 4(LN+RM)$, $c = 2J(I+K) - 2(NM+LM+NR-3LR)$ and $d = J^2+(K-I)^2-(M-R)^2-(L-N)^2$, where $I = 2(A-B)$, $J = (2A+B)(A-B)$, $K = A^2(A-B)$, $L = (C-D)$, $M = \big(C(2A+B)-D(4A-B)\big)$, $N = \big(AC(A+2B)-D(3A^2+AB-B^2)\big)$, and $R = A^2(BC-DA)$. Further, suppose one of the following conditions hold:*

(1) Let $a \neq 0$.

(i) If

$$b^2 \geq 3ac \quad \text{and} \quad |a+c| < b+d-\frac{1}{27a^2}\left(2b^3-9abc+27da^2-2(b^2-3ac)^{\frac{3}{2}}\right),$$

then

$$2b^3-9abc+27da^2-2(b^2-3ac)^{\frac{3}{2}} \geq 0,$$

(ii) If

$$b^2 < 3ac \quad or \quad |a+c| \geq b+d-\frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2-3ac)^{\frac{3}{2}}\right),$$

then

$$|a+c| \leq b+d.$$

(2) Let $a = 0$.

(i) If $b > 0$ and $|c| < 2b$, then $c^2 \leq 4bd$,

(ii) If $b \leq 0$ or $|c| \geq 2b$, then $|c| \leq b+d$.

If p is analytic and $p(z) \neq 0$ in $\mathbb{D}$, $p(0) = 1$ and satisfies the subordination

$$p(z) + \frac{zp'(z)}{p^2(z)} \prec \frac{1+Cz}{1+Dz},$$

then $p \in \mathcal{P}[A,B]$.

Proof. Let the function q be defined by $q(z) = (1+Az)/(1+Bz)$. Similar to the proof of Theorem 5.2, by Lemma 5.3 it follows that the subordination

$$p(z) + \frac{zp'(z)}{p^2(z)} \prec q(z) + \frac{zq'(z)}{q^2(z)}$$

implies $p(z) \prec q(z)$. In light of differential chain, the result is proved if it could be shown that

$$g(z) := \frac{1+Cz}{1+Dz} \prec q(z) + \frac{zq'(z)}{q^2(z)} =: h(z). \tag{5.32}$$

Since g is univalent, the subordination $g(z) \prec h(z)$ is equivalent to the subordination

$$z \prec g^{-1}(h(z)) =: H(z).$$

The proof will be completed by showing that $|H(e^{i\theta})| \geq 1$ for all $\theta \in [0, 2\pi]$. First note that from (5.17) we have

$$h(z) = \frac{1 + (4A - B)z + (3A^2 - B^2 + BA)z^2 + A^3 z^3}{(1 + Bz)(1 + Az)^2}. \tag{5.33}$$

Now (5.33) and (5.30) yield

$$\begin{aligned}
H(z) &= g^{-1}(h(z)) \\
&= \frac{h(z) - 1}{C - Dh(z)} \\
&= \frac{2(A - B)z + (2A + B)(A - B)z^2 + A^2(A - B)z^3}{\begin{aligned} &C - D + \big((C(2A + B) - D(4A - B)\big)z \\ &+ \left(AC(A + 2B) - D(3A^2 + AB - B^2)\right) z^2 + A^2(BC - DA)z^3 \end{aligned}}.
\end{aligned}$$

By writing,

$$\begin{aligned}
I &= 2(A - B), \\
J &= (2A + B)(A - B), \\
K &= A^2(A - B), \\
L &= C - D, \\
M &= \big(C(2A + B) - D(4A - B)\big), \\
N &= \Big(AC(A + 2B) - D(3A^2 + AB - B^2)\Big),
\end{aligned}$$

and

$$R = A^2(BC - DA)$$

we see that

$$\begin{aligned}
H(z) &= \frac{Iz + Jz^2 + Kz^3}{L + Mz + Nz^2 + Rz^3} \\
&= \frac{z\left(Iz^{-1} + J + Kz\right)}{Lz^{-1} + M + Nz + Rz^2}.
\end{aligned} \tag{5.34}$$

Replacing $z = e^{i\theta}$ in $|H(z)|$ we get

$$|H(e^{i\theta})| = \left| \frac{e^{i\theta}\left(Ie^{-i\theta} + J + Ke^{i\theta}\right)}{Le^{-i\theta} + M + Ne^{i\theta} + Re^{i2\theta}} \right|$$

$$= \left| \frac{J + (I+K)\cos\theta + i(K-I)\sin\theta}{M + (L+N)\cos\theta + R\cos 2\theta + i[(N-L)\sin\theta + R\sin 2\theta]} \right|.$$

A computation shows that,

$$|H(e^{i\theta})|^2 \geq 1$$

or

$$\frac{\big(J + (I+K)\cos\theta\big)^2 + (K-I)^2\sin^2\theta}{\big(M + (L+N)\cos\theta + R\cos 2\theta\big)^2 + \big((N-L)\sin\theta + R\sin 2\theta\big)^2} \geq 1 \tag{5.35}$$

From the numerator of (5.35), computations show that

$$\begin{aligned}&\big(J + (I+K)\cos\theta\big)^2 + (K-I)^2\sin^2\theta \\ &= J^2 + (K-I)^2 + 2J(I+K)\cos\theta + 4IK\cos^2\theta.\end{aligned}$$

Similarly, from the denominator of (5.35), computations show that

$$\begin{aligned}&\big(M + (L+N)\cos\theta + R\cos 2\theta\big)^2 \\ &= (M-R)^2 + 2(M-R)(L+N)\cos\theta + \Big((L+N)^2 + 4R(M-R)\Big)\cos^2\theta \\ &\quad + 4R(L+N)\cos^3\theta + 4R^2\cos^4\theta,\end{aligned}$$

and

$$\begin{aligned}&\big((N-L)\sin\theta + R\sin 2\theta\big)^2 \\ &= (L-N)^2 - 4R(L-N)\cos\theta + \Big(4R^2 - (L-N)^2\Big)\cos^2\theta \\ &\quad + 4R(L-N)\cos^3\theta - 4R^2\cos^4\theta.\end{aligned}$$

Therefore

$$
\begin{aligned}
&(M+(L+N)\cos\theta+R\cos 2\theta)^2+((N-L)\sin\theta+R\sin 2\theta)^2\\
&=(M-R)^2+(L-N)^2+2(NM+LM+NR-3LR)\cos\theta\\
&\qquad+4(LN+RM)\cos^2\theta+8LR\cos^3\theta.
\end{aligned}
$$

Now (5.35) can be written as

$$
\frac{J^2+(K-I)^2+2J(I+K)\cos\theta+4IK\cos^2\theta}{\begin{array}{l}(M-R)^2+(L-N)^2+2(NM+LM+NR-3LR)\cos\theta\\ +4(LN+RM)\cos^2\theta+8LR\cos^3\theta\end{array}}\geq 1,
$$

or equivalently

$$
\begin{aligned}
&J^2+(K-I)^2-(M-R)^2-(L-N)^2\\
&\qquad+\big(2J(I+K)-2(NM+LM+NR-3LR)\big)\cos\theta\\
&\qquad\qquad+\big(4IK-4(LN+RM)\big)\cos^2\theta-8LR\cos^3\theta\geq 0.
\end{aligned}
\tag{5.36}
$$

Let

$$
\begin{aligned}
a&=-8LR,\\
b&=4IK-4(LN+RM),\\
c&=2J(I+K)-2(NM+LM+NR-3LR),\\
d&=J^2+(K-I)^2-(M-R)^2-(L-N)^2,
\end{aligned}
$$

and

$$t=\cos\theta.$$

Then (5.36) becomes,

$$at^3+bt^2+ct+d\geq 0.$$

To show that $|H(e^{i\theta})|^2 \geq 1$ we have to prove $at^3 + bt^2 + ct + d \geq 0$ or equivalently

$$\min_{|t|\leq 1} \left(at^3 + bt^2 + ct + d \right) \geq 0.$$

Thus, in view of Lemma 5.1 $|H(e^{i\theta})| \geq 1$ if one of the following condition holds:

(1) Let $a \neq 0$.

(i) If

$$b^2 \geq 3ac \quad \text{and} \quad \frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}}\right) < b+d-|a+c|,$$

then

$$\min_{|t|\leq 1} \left(at^3 + bt^2 + ct + d \right)$$
$$= \frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}}\right) \geq 0$$

which implies

$$2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}} \geq 0.$$

(ii) If

$$b^2 < 3ac \quad \text{or} \quad \frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}}\right) \geq b+d-|a+c|,$$

then

$$\min_{|t|\leq 1} \left(at^3 + bt^2 + ct + d \right) = b + d - |a + c| \geq 0$$

which implies

$$|a + c| \leq b + d.$$

(2) Let $a = 0$.

(i) If $b > 0$ and $|c| < 2b$, then

$$\min_{|t|\leq 1} \left(bt^2 + ct + d\right) = \frac{4bd - c^2}{4b} \geq 0$$

which implies $c^2 \leq 4bd$,

(ii) If $b \leq 0$ or $|c| \geq 2b$, then

$$\min_{|t|\leq 1} \left(bt^2 + ct + d\right) = b + d - |c| \geq 0$$

which implies $|c| \leq b + d$.

This complete the proof. ∎

The following theorem which gives sufficient condition for Janowski starlikeness is a consequence of Theorem 5.5.

Theorem 5.6 *Let the conditions of Theorem 5.5 holds. If $f \in \mathcal{A}$ satisfies*

$$\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{zf'(z)}{f(z)} - 1 \prec \frac{1 + Cz}{1 + Dz}$$

then $f \in \mathcal{ST}[A, B]$.

Proof. With $p(z) = \frac{zf'(z)}{f(z)}$, logarithmic differentiation yields

$$p(z) + \frac{zp'(z)}{p^2(z)} = \frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{zf'(z)}{f(z)} - 1.$$

The result now follows from Theorem 5.5. ∎

Lemma 5.1 and Lemma 5.4 will be use to prove the following result.

Theorem 5.7 *Let* $-1 < B < A \leq 1$, $-1 \leq D < C \leq 1$ *and* $(A-B)/(1-AB) \leq 1/\sqrt{2}$. *Let* $a = -8LR$, $b = 4IK - 4(LN + RM)$, $c = 2J(I+K) - 2(NM + LM + NR - 3LR)$ *and* $d = J^2 + (K-I)^2 - (M-R)^2 - (L-N)^2$, *where* $I = 3(A-B)$, $J = (3A+2B)(A-B)$, $K = A(A^2 - B^2)$, $L = (C-D)$, $M = \big(C(A+2B) - D(4A-B)\big)$, $N = \big(BC(2A+B) - D(3A^2 + AB - B^2)\big)$, *and* $R = A(CB^2 - DA^2)$. *Further, suppose one of the following conditions hold:*

(1) *Let* $a \neq 0$.

(i) *If*

$$b^2 \geq 3ac \quad \text{and} \quad |a+c| < b+d-\frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2-3ac)^{\frac{3}{2}}\right),$$

then

$$2b^3 - 9abc + 27da^2 - 2(b^2 - 3ac)^{\frac{3}{2}} \geq 0,$$

(ii) *If*

$$b^2 < 3ac \quad \text{or} \quad |a+c| \geq b+d-\frac{1}{27a^2}\left(2b^3 - 9abc + 27da^2 - 2(b^2-3ac)^{\frac{3}{2}}\right),$$

then

$$|a+c| \leq b+d.$$

(2) *Let* $a = 0$.

(i) *If* $b > 0$ *and* $|c| < 2b$, *then* $c^2 \leq 4bd$,

(ii) *If* $b \leq 0$ *or* $|c| \geq 2b$, *then* $|c| \leq b + d$.

If p *is analytic and* $p(z) \neq 0$ *in* $\mathbb{D}$, $p(0) = 1$ *and satisfies the subordination*

$$p^2(z) + \frac{zp'(z)}{p(z)} \prec \frac{1+Cz}{1+Dz},$$

then $p \in \mathcal{P}[A, B]$.

Proof. Defined $q(z) = (1 + Az)/(1 + Bz)$. Analogue to the proof of Theorem 5.5, it is sufficient to prove

$$g(z) := \frac{1 + Cz}{1 + Dz} \prec q^2(z) + \frac{zq'(z)}{q(z)} =: h(z)$$

or equivalently by noting that

$$g(z) \prec h(z) \Rightarrow z \prec g^{-1}(h(z)) =: H(z)$$

we show that $|H(e^{i\theta})| \geq 1$ for all $\theta \in [0, 2\pi]$. From (5.23) we have

$$h(z) = \frac{1 + (4A - B)z + (3A^2 - B^2 + BA)z^2 + A^3z^3}{(1 + Az)(1 + Bz)^2}. \tag{5.37}$$

Now (5.37) and (5.30) yield

$$\begin{aligned} H(z) &= g^{-1}(h(z)) \\ &= \frac{h(z) - 1}{C - Dh(z)} \\ &= \frac{3(A - B)z + (3A + 2B)(A - B)z^2 + A(A^2 - B^2)z^3}{\begin{array}{l} C - D + \big(C(A + 2B) - D(4A - B)\big)z \\ + \big(BC(2A + B) - D(3A^2 + AB - B^2)\big)\, z^2 + A(CB^2 - DA^2)z^3 \end{array}}. \end{aligned}$$

By writing,

$$\begin{aligned} I &= 3(A - B), \\ J &= (3A + 2B)(A - B), \\ K &= A(A^2 - B^2), \\ L &= (C - D), \end{aligned}$$

$$M = \big(C(A+2B) - D(4A-B)\big),$$
$$N = \big(BC(2A+B) - D(3A^2+AB-B^2)\big),$$

and

$$R = A(CB^2 - DA^2)$$

we see that
$$H(z) = g^{-1}(h(z)) = \frac{Iz + Jz^2 + Kz^3}{L + Mz + Nz^2 + Rz^3}.$$

The proof of the remaining parts run along similar lines with the proof of Theorem 5.5, p. 131. Therefore it is omitted. ∎

Theorem 5.8 *Let the conditions of Theorem 5.7 holds. If $f \in \mathcal{A}$ satisfies*

$$1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1+Cz}{1+Dz}$$

then $f \in \mathcal{ST}[A,B]$.

Proof. With $p(z) = \frac{zf'(z)}{f(z)}$, a computation shows that

$$p^2(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\left(\frac{zf'(z)}{f(z)} - 1\right).$$

The result now follows from Theorem 5.8. ∎

CHAPTER 6

HARMONIC FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

6.1 Introduction

In the well-established theory of analytic univalent functions, there are several studies on hypergeometric functions associated with classes of analytic functions (See for example [30, 36, 59, 71, 89, 98, 117, 124, 136, 137]) investigating univalence, starlikeness and other properties of these functions. On the other hand only some corresponding studies on connections of hypergeometric functions with harmonic mappings have been done [5,6,22,77]. Pursuing this line of study, results that bring out connections of hypergeometric functions with a class of harmonic univalent functions considered in [145] are investigated in this chapter.

Recall that, for $f = h + \bar{g} \in \mathcal{S}_H$ the series expansion for the analytic functions h and g are expressed as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n, \ |b_1| < 1. \tag{6.1}$$

If ϕ_1 and ϕ_2 are analytic and $f = h + \bar{g}$ is in $\mathcal{S}_H$, the convolution or the Hadamard product is defined by

$$f * (\phi_1 + \overline{\phi_2}) = h * \phi_1 + \overline{g * \phi_2}.$$

Let a, b and c be any complex numbers with $c \neq 0, -1, -2, -3, \cdots$. As defined in Subsection 1.3.2, p. 23, the Gaussian hypergeometric function is defined by

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{6.2}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by (1.19).

Since the hypergeometric series in (6.2) converges absolutely in $\mathbb{D}$, it follows that $F(a, b; c; z)$ defines a function which is analytic in $\mathbb{D}$, provided that c is neither zero nor a negative integer. In fact, $F(a, b; c; 1)$ converges for $\operatorname{Re}(c - a - b > 0)$ and is related to the gamma given by

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \; c \neq 0, 1, 2, \ldots \tag{6.3}$$

In particular, the incomplete beta function, related to the Gaussian hypergeometric function $\varphi(a, c; z)$, is defined by

$$\varphi(a, c; z) = zF(a, 1; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1}, z \in \mathcal{U}, \; c \neq 0, 1, 2, \ldots. \tag{6.4}$$

Throughout this thesis, let $G(z) = \phi_1(z) + \overline{\phi_2(z)}$ be a function where $\phi_1(z)$ and $\phi_2(z)$ are the hypergeometric functions defined by

$$\phi_1(z) := zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n, \tag{6.5}$$

$$\phi_2(z) := F(a_2, b_2; c_2 : z) - 1 = \sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} z^n, \quad |a_2 b_2| < |c_2|. \tag{6.6}$$

Based on the study in [145], for $\alpha \geq 0$ and $0 \leq \beta < 1$, let $HP(\alpha, \beta)$ denote the class of harmonic functions of the form (6.1) satisfying the condition

$$\operatorname{Re}\Big(\alpha z\big(h''(z) + g''(z)\big) + \big(h'(z) + g'(z)\big)\Big) > \beta.$$

Also denote by $HT(\alpha, \beta) = HP(\alpha, \beta) \bigcap \mathcal{T}_H$ where $\mathcal{T}_H$ [123], is the class of harmonic functions f such that

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n - \overline{\sum_{n=1}^{\infty} |b_n| z^n}, \quad |b_1| < 1. \tag{6.7}$$

Note that when $p = 1$, the Dziok-Srivastava operator which is defined in Subsection 1.3.2, p. 24, is written as

$$H_{(1)}^{l,m}[\alpha_1] = H^{l,m}[\alpha_1]. \tag{6.8}$$

Throughout this chapter, the Dziok-Srivastava operator will be denoted by (6.8). The Dziok-Srivastava operator when extended to the harmonic function $f = h + \overline{g}$ is defined by

$$H^{l,m}[\alpha_1]f(z) = H^{l,m}[\alpha_1]h(z) + \overline{H^{l,m}[\alpha_1]g(z)}. \tag{6.9}$$

Motivated by earlier works of [22, 26, 52–54, 56, 76, 112, 123, 133] on harmonic functions, we introduce here a new subclass $G_H([\alpha_1], \gamma)$ of $\mathcal{S}_H$ using the Dziok-Srivastava operator extended to harmonic functions.

Let $G_H([\alpha_1], \gamma)$ denote the subfamily of starlike harmonic functions $f \in \mathcal{S}_H$ of the form (6.1) such that

$$\operatorname{Re}\left(1 + (1 + e^{i\psi})\frac{\left(\begin{array}{l} z^2(H^{l,m}[\alpha_1]h(z))'' \\ + \overline{2z(H^{l,m}[\alpha_1]g(z))' + z^2(H^{l,m}[\alpha_1]g(z))''} \end{array}\right)}{z(H^{l,m}[\alpha_1]h(z))' - \overline{z(H^{l,m}[\alpha_1]g(z))'}}\right) \geq \gamma \tag{6.10}$$

where $H^{l,m}[\alpha_1]f(z)$ is defined by (6.9) $0 \leq \gamma < 1$, $z \in \mathbb{D}$ and ψ real.

Also let $\mathcal{T}_H([\alpha_1], \gamma) = G_H([\alpha_1], \gamma) \bigcap \mathcal{T}_H$.

Lemma 6.1 *If $f = h + \bar{g}$ is given by (6.1) and*

$$\sum_{n=1}^{\infty} n\big(\alpha(n-1) + 1\big)(|a_n| + |b_n|) \leq 2 - \beta, \quad 0 \leq |b_1| < 1 - \beta, \tag{6.11}$$

where $a_1 = 1, \alpha \geq 0$ and $0 \leq \beta < 1$, then f is harmonic univalent and sense preserving in $\mathbb{D}$ and $f \in HP(\alpha, \beta)$.

Proof. For $|z_1| \leq |z_2| < 1$, we have

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&= \left|(z_1 - z_2) + \sum_{n=2}^{\infty} a_n(z_1^n - z_2^n)\right| - \left|\sum_{n=1}^{\infty} b_n(z_1^n - z_2^n)\right| \\
&\geq |z_1 - z_2| - \sum_{n=2}^{\infty} |a_n||z_1^n - z_2^n| - \sum_{n=1}^{\infty} |b_n||z_1^n - z_2^n| \\
&= |z_1 - z_2| \left(1 - |b_1| - \sum_{n=2}^{\infty} (|a_n| + |b_n|) \left|\frac{z_1^n - z_2^n}{z_1 - z_2}\right|\right) \\
&\geq |z_1 - z_2| \left(1 - |b_1| - \sum_{n=2}^{\infty} n(|a_n| + |b_n|)|z_2|^{n-1}\right) \\
&\geq |z_1 - z_2| \left(1 - |b_1| - |z_2| \sum_{n=2}^{\infty} n(\alpha(n-1)+1)(|a_n| + |b_n|)\right) \\
&\geq |z_1 - z_2| \left(1 - \beta - |b_1| - |z_2|(1 - \beta - |b_1|)\right) \\
&= |z_1 - z_2|(1 - \beta - |b_1|)(1 - |z_2|) > 0.
\end{aligned}$$

Hence, $|f(z_1) - f(z_2)| > 0$ and f is univalent in $\mathbb{D}$. To prove f is locally univalent and sense-preserving in $\mathbb{D}$, it is enough to show that $|h'(z)| > |g'(z)|$.

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \\
&> 1 - \sum_{n=2}^{\infty} n|a_n| \\
&\geq 1 - \beta - \sum_{n=2}^{\infty} n(\alpha(n-1)+1)|a_n| \\
&\geq \sum_{n=1}^{\infty} n(\alpha(n-1)+1)|b_n| \\
&> \sum_{n=1}^{\infty} n|b_n||z|^{n-1} = |g'(z)|.
\end{aligned}$$

Using the fact that $\operatorname{Re} w > \beta$ if and only if $|1 - \beta + w| > |1 + \beta - w|$, it is sufficient to show that

$$|1-\beta+\alpha z(h''(z)+g''(z))+h'(z)+g'(z)|-|1+\beta-\alpha z(h''(z)+g''(z))-h'(z)-g'(z)|>0 \tag{6.12}$$

in proving $f \in HP(\alpha,\beta)$. Substituting for $h(z)$ and $g(z)$ in (6.12) yields,

$$\begin{aligned}
&\left|2-\beta+\sum_{n=2}^{\infty} n(\alpha(n-1)+1)a_n z^{n-1}+\sum_{n=1}^{\infty} n(\alpha(n-1)+1)b_n z^{n-1}\right| \\
&-\left|\beta-\sum_{n=2}^{\infty} n(\alpha(n-1)+1)a_n z^{n-1}-\sum_{n=1}^{\infty} n(\alpha(n-1)+1)b_n z^{n-1}\right| \\
&\geq (2-\beta)-\sum_{n=2}^{\infty} n(\alpha(n-1)+1)|a_n||z|^{n-1}-\sum_{n=1}^{\infty} n(\alpha(n-1)+1)|b_n||z|^{n-1} \\
&\quad -\beta-\sum_{n=2}^{\infty} n(\alpha(n-1)+1)|a_n||z|^{n-1}-\sum_{n=1}^{\infty} n(\alpha(n-1)+1)|b_n||z|^{n-1} \\
&\geq 2\left(1-\beta-|z|\left(\sum_{n=2}^{\infty} n(\alpha(n-1)+1)|a_n|+\sum_{n=1}^{\infty} n(\alpha(n-1)+1)|b_n|\right)\right) \\
&\geq 2(1-\beta)(1-|z|)>0
\end{aligned}$$

by the condition (6.11). ∎

6.2 Coefficient Condition for Gaussian Hypergeometric Function

Theorem 6.1 *If $a_j, b_j > 0$ and $c_j > a_j + b_j + 2$ for $j = 1, 2$, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in $\mathbb{D}$ and $G \in HP(\alpha,\beta)$, is that*

$$\begin{aligned}
&\left(\frac{\alpha(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2}+\frac{a_1b_1(2\alpha+1)}{c_1-a_1-b_1-1}+1\right)F(a_1,b_1;c_1;1) \\
&\quad +\left(\frac{\alpha(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2}+\frac{a_2b_2}{c_2-a_2-b_2-1}\right)F(a_2,b_2;c_2;1)\leq 2-\beta
\end{aligned} \tag{6.13}$$

where $\alpha \geq 0$ *and* $0 \leq \beta < 1$.

Proof. Let

$$\begin{aligned} G(z) &= \phi_1(z) + \overline{\phi_2(z)} \\ &= z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n + \sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^n. \end{aligned}$$

When the condition (6.13) holds for the coefficients of $G = \phi_1 + \overline{\phi_2}$, it is enough to prove that

$$\sum_{n=1}^{\infty} n(\alpha(n-1)+1) \left(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \right) \leq 2 - \beta. \tag{6.14}$$

Write the left side of equality (6.14) as

$$\begin{aligned} &\alpha \sum_{n=1}^{\infty} n(n-1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \alpha \sum_{n=1}^{\infty} n(n-1) \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\ &\quad + \sum_{n=1}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\ &= \alpha \sum_{n=1}^{\infty} [(n-1)^2 + (n-1)] \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \alpha \sum_{n=1}^{\infty} (n^2 - n) \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\ &\quad + \sum_{n=1}^{\infty} (n-1+1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\ &= \alpha \sum_{n=1}^{\infty} n^2 \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + (\alpha+1) \sum_{n=1}^{\infty} n \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \sum_{n=0}^{\infty} \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} \\ &\quad + \alpha \sum_{n=1}^{\infty} n^2 \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} - (\alpha-1) \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\ &= \alpha \left(\frac{(a_1)_2(b_1)_2}{(c_1 - a_1 - b_1 - 2)_2} + \frac{a_1 b_1}{c_1 - a_1 - b_1 - 1} \right) F(a_1, b_1; c_1; 1) \\ &\quad + \frac{(\alpha+1) a_1 b_1 F(a_1, b_1; c_1; 1)}{c_1 - a_1 - b_1 - 1} + F(a_1, b_1; c_1; 1) \\ &\quad + \alpha \left(\frac{(a_2)_2(b_2)_2}{(c_2 - a_2 - b_2 - 2)_2} + \frac{a_2 b_2}{c_2 - a_2 - b_2 - 1} \right) F(a_2, b_2; c_2; 1) \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{(\alpha-1)a_2b_2F(a_2,b_2;c_2;1)}{c_2-a_2-b_2-1} \\
&= \left(\frac{\alpha(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{a_1b_1(2\alpha+1)}{c_1-a_1-b_1-1} + 1\right) F(a_1,b_1;c_1;1) \\
&\quad + \left(\frac{\alpha(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{a_2b_2}{c_2-a_2-b_2-1}\right) F(a_2,b_2;c_2;1),
\end{aligned}
$$

by an application of Lemma 2.2. This yields (6.13) . In order to prove that G is locally univalent and sense-preserving in $\mathbb{D}$, it is sufficient to show that $|\phi_1'(z)| > |\phi_2'(z)|$, .

$$
\begin{aligned}
|\phi_1'(z)| &= \left|1 + \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^{n-1}\right| \\
&> 1 - \sum_{n=2}^{\infty} n \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |z|^{n-1} \\
&> 1 - \sum_{n=2}^{\infty} (n-1) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \\
&= 1 - \sum_{n=1}^{\infty} n \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} - \sum_{n=1}^{\infty} \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} \\
&= 1 - \left(\frac{ab}{c-a-b-1}\right) F(a,b;c;1) - (F(a,b;c;1)-1) \\
&= 2 - \left(\frac{a_1b_1}{c_1-a_1-b_1-1} + 1\right) F(a_1,b_1;c_1;1) \\
&\geq 2 - \beta - \left(\frac{\alpha(a_1)_2(b_1)_2}{(c_1-a_1-b_1-2)_2} + \frac{a_1b_1(2\alpha+1)}{c_1-a_1-b_1-1} + 1\right) F(a_1,b_1;c_1;1) \\
&\geq \left(\frac{\alpha(a_2)_2(b_2)_2}{(c_2-a_2-b_2-2)_2} + \frac{a_2b_2}{c_2-a_2-b_2-1}\right) F(a_2,b_2;c_2;1) \quad \text{(by (6.13))} \\
&\geq \left(\frac{a_2b_2}{c_2-a_2-b_2-1}\right) F(a_2,b_2;c_2;1) \\
&= \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\
&> \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} |z|^{n-1} \qquad (|z|<1) \\
&= \left|\sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^{n-1}\right| = |\phi_2'(z)|.
\end{aligned}
$$

In fact, for $|z_1| \leq |z_2| < 1$, we have

$$
\begin{aligned}
&|G(z_1) - G(z_2)| \\
&\geq |\phi_1(z_1) - \phi_1(z_2)| - |\phi_2(z_1) - \phi_2(z_2)| \\
&= \left|(z_1 - z_2) + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}(z_1^n - z_2^n)\right| - \left|\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}(z_1^n - z_2^n)\right| \\
&\geq |z_1 - z_2| \left(1 - \frac{a_2 b_2}{c_2} - \sum_{n=2}^{\infty} n \left(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right) |z_2|^{n-1}\right) \\
&\geq |z_1 - z_2| \Bigg(1 - \beta - \frac{a_2 b_2}{c_2} - |z_2| \sum_{n=2}^{\infty} n(\alpha(n-1)+1) \Bigg(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \\
&\qquad\qquad + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \Bigg) \Bigg) \\
&\geq |z_1 - z_2| \left(1 - \beta - \frac{a_2 b_2}{c_2} - |z_2| \left(1 - \beta - \frac{a_2 b_2}{c_2}\right)\right) \quad \text{(by (6.14))} \\
&\geq |z_1 - z_2| \left(1 - \beta - \frac{a_2 b_2}{c_2}\right)(1 - |z_2|) > 0
\end{aligned}
$$

Hence, $|G(z_1) - G(z_2)| > 0$ which shows that G is univalent in $\mathbb{D}$. ∎

Lemma 6.2 *If $f = h + \bar{g}$ is given by (6.7), then $f \in HT(\alpha, \beta)$ if and only if*

$$\sum_{n=1}^{\infty} n\big(\alpha(n-1)+1\big)(|a_n| + |b_n|) \leq 2 - \beta, \quad 0 \leq |b_1| < 1 - \beta,$$

where $a_1 = 1, \alpha \geq 0$ and $0 \leq \beta < 1$.

The sufficiency of this result is from Lemma 6.1 and the proof of necessity is on lines similar to the the proof of Theorem 2.2 in [145]. Define

$$
\begin{aligned}
G_1(z) &= z\left(2 - \frac{\phi_1(z)}{z}\right) - \overline{\phi_2(z)} \\
&= z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} z^n - \overline{\sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} z^n}
\end{aligned}
$$

on using (6.5) and (6.6). Clearly $G_1 \in \mathcal{T}_H$.

Theorem 6.2 *Let $\alpha \geq 0, 0 \leq \beta < 1, a_j, b_j > 0, c_j > a_j + b_j + 2$, for $j = 1, 2$ and $a_2 b_2 < c_2$. G_1 is in $HT(\alpha, \beta)$ if and only if (6.13) holds.*

Proof. In view of Theorem 6.1, sufficiency of (6.13) is clear. We only need to show the necessity of (6.13). If $G_1 \in HT(\alpha, \beta)$, then G_1 satisfies (6.14) by Lemma 6.2 and hence (6.13) holds. ∎

Theorem 6.3 *Let $0 \leq \beta < 1, a_j, b_j > 0, c_j > a_j + b_j + 1$, for $j = 1, 2$ and $a_2 b_2 < c_2$. A necessary and sufficient condition such that $f * (\phi_1 + \overline{\phi_2}) \in HT(\alpha, \beta)$ for $f \in HT(\alpha, \beta)$ is that*

$$F(a_1, b_1; c_1 : 1) + F(a_2, b_2; c_2 : 1) \leq 3 \tag{6.15}$$

where ϕ_1, ϕ_2 are as defined, respectively, by (6.5) and (6.6).

Proof. Let $f = h + \bar{g} \in HT(\alpha, \beta)$, where h and g are given by (6.7). Then

$$\begin{aligned}\big(f * (\phi_1 + \overline{\phi_2})\big)(z) &= h(z) * \phi_1(z) + \overline{g(z) * \phi_2(z)} \\ &= z - \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} a_n z^n - \overline{\sum_{n=1}^{\infty} \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} b_n z^n}.\end{aligned}$$

In view of Lemma (6.2), we need to prove that $(f * (\phi_1 + \overline{\phi_2})) \in HT(\alpha, \beta)$ if and only if

$$\sum_{n=1}^{\infty} n(\alpha(n-1)+1) \left(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} a_n + \frac{(a_2)_n (b_2)_n}{(c_2)_n (1)_n} b_n \right) \leq 2 - \beta. \tag{6.16}$$

As an application of Lemma (6.2), we have

$$\sum_{n=1}^{\infty} n\big(\alpha(n-1)+1\big)(a_n + b_n) \leq 2 - \beta$$

or

$$\sum_{n=2}^{\infty} n\big(\alpha(n-1)+1\big) a_n + \sum_{n=1}^{\infty} n\big(\alpha(n-1)+1\big) b_n \leq 1 - \beta$$

which implies

$$n\big(\alpha(n-1)+1\big)a_n \leq 1-\beta \quad \text{and} \quad n\big(\alpha(n-1)+1\big)b_n \leq 1-\beta.$$

Hence,

$$a_n \leq \frac{1-\beta}{n\big(\alpha(n-1)+1\big)}, n = 2, 3, \cdots, \quad \text{and} \quad b_n \leq \frac{1-\beta}{n\big(\alpha(n-1)+1\big)}, n = 1, 2, \cdots. \tag{6.17}$$

Rewriting (6.16) we get

$$\sum_{n=1}^{\infty} n(\alpha(n-1)+1)\left(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}a_n + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}b_n\right) \leq 2-\beta$$

or

$$\sum_{n=2}^{\infty} n\big(\alpha(n-1)+1\big)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}a_n + \sum_{n=1}^{\infty} n\big(\alpha(n-1)+1\big)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}b_n \leq 1-\beta. \tag{6.18}$$

By applying (6.17), the left hand side of (6.18) is bounded above by

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1-\beta)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty}(1-\beta)\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\
&= (1-\beta)\left(\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right) \\
&= (1-\beta)\big(F(a_1, b_1; c_1 : 1) + F(a_2, b_2; c_2 : 1) - 2\big).
\end{aligned}$$

The last expression is bounded above by $(1-\beta)$ if and only if (6.15) is satisfied. This proves (6.16) and the results follows. ∎

6.3 Integral Operators

Theorem 6.4 *If $a_j, b_j > 0$ and $c_j > a_j + b_j + 1$ for $j = 1, 2$, then a sufficient condition for a function*

$$G_2(z) = \int_0^z F(a_1, b_1; c_1; t)dt + \overline{\int_0^z \big(F(a_2, b_2; c_2; t) - 1\big)dt}$$

to be in $HP(\alpha, \beta)$ is that

$$\left(\frac{\alpha(a_1 b_1)}{c_1 - a_1 - b_1 - 1} + 1\right) F(a_1, b_1; c_1; 1) + \left(\frac{\alpha(a_2 b_2)}{c_2 - a_2 - b_2 - 1} + 1\right) F(a_2, b_2; c_2; 1) \leq 3 - \beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Proof. In view of Lemma 6.1, the function

$$G_2(z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} z^n + \overline{\sum_{n=2}^{\infty} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n} z^n}$$

is in $HP(\alpha, \beta)$ if

$$\sum_{n=2}^{\infty} n(\alpha(n-1)+1)\left(\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_n} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}\right) \leq 1 - \beta. \qquad (6.19)$$

By a computation we can write the left hand side of (6.19) as

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\alpha n + 1)\left(\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\right) \\
&= \alpha \sum_{n=1}^{\infty} n \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \alpha \sum_{n=1}^{\infty} n \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} + \sum_{n=1}^{\infty} \frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n} + \sum_{n=1}^{\infty} \frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n} \\
&= \frac{\alpha a_1 b_1}{c_1 - a_1 - b_1 - 1} F(a_1, b_1; c_1; 1) + \frac{\alpha a_2 b_2}{c_2 - a_2 - b_2 - 1} F(a_2, b_2; c_2; 1) \\
&\quad + (F(a_1, b_1; c_1; 1) - 1) + (F(a_2, b_2; c_2; 1) - 1)
\end{aligned}$$

$$= \left(\frac{\alpha(a_1b_1)}{c_1-a_1-b_1-1}+1\right) F(a_1,b_1;c_1;1)$$
$$+ \left(\frac{\alpha(a_2b_2)}{c_2-a_2-b_2-1}+1\right) F(a_2,b_2;c_2;1) - 2.$$

The last expression is bounded above by $(1-\beta)$ and this yields the results. ∎

Theorem 6.5 *If $a_1, b_1 > -1, c_1 > 0, a_1b_1 < 0, a_2 > 0, b_2 > 0,$ and $c_j > a_j + b_j + 2, j = 1, 2,$ then*

$$G_3(z) = \int_0^z F(a_1,b_1;c_1;t)dt - \overline{\int_0^z \big(F(a_2,b_2;c_2;t)-1\big)dt}$$

to be in $HT(\alpha,\beta)$ if and only if

$$\left(\frac{\alpha(a_1b_1)}{c_1-a_1-b_1-1}+1\right) F(a_1,b_1;c_1;1)$$
$$- \left(\frac{\alpha(a_2b_2)}{c_2-a_2-b_2-1}+1\right) F(a_2,b_2;c_2;1) + 1 \geq \beta$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Proof. We write

$$G_3(z) = z - \frac{|a_1b_1|}{c_1}\sum_{n=2}^{\infty}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n}z^n - \overline{\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}z^n}.$$

In view of Lemma (6.2) it is sufficient to show that

$$\sum_{n=2}^{\infty} n(\alpha(n-1)+1)\left(\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-2}(b_1+1)_{n-2}}{(c_1+1)_{n-2}(1)_n} + \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_n}\right) \leq 1-\beta. \tag{6.20}$$

By a routine computation (6.20) can be written as

$$\alpha\sum_{n=0}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_n} + \alpha\sum_{n=1}^{\infty} n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$

$$+\sum_{n=1}^{\infty}\frac{|a_1b_1|}{c_1}\frac{(a_1+1)_{n-1}(b_1+1)_{n-1}}{(c_1+1)_{n-1}(1)_n}+\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\leq(1-\beta)$$

which implies

$$\alpha\sum_{n=0}^{\infty}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_n}+\frac{\alpha c_1}{|a_1b_1|}\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\sum_{n=0}^{\infty}\frac{(a_1+1)_n(b_1+1)_n}{(c_1+1)_n(1)_{n+1}}+\frac{c_1}{|a_1b_1|}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\leq\frac{c_1(1-\beta)}{|a_1b_1|}.$$

But, this is equivalent to

$$\frac{\alpha c_1}{a_1b_1}\sum_{n=1}^{\infty}n\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{\alpha c_1}{|a_1b_1|}\sum_{n=1}^{\infty}n\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}$$
$$+\frac{c_1}{a_1b_1}\sum_{n=1}^{\infty}\frac{(a_1)_n(b_1)_n}{(c_1)_n(1)_n}+\frac{c_1}{|a_1b_1|}\sum_{n=1}^{\infty}\frac{(a_2)_n(b_2)_n}{(c_2)_n(1)_n}\leq\frac{c_1(1-\beta)}{|a_1b_1|}.$$

which yields

$$\left(\frac{\alpha(a_1b_1)}{c_1-a_1-b_1-1}+1\right)F(a_1,b_1;c_1;1)$$
$$-\left(\frac{\alpha(a_2b_2)}{c_2-a_2-b_2-1}+1\right)F(a_2,b_2;c_2;1)+1\geq\beta.$$

This completes the proof. ∎

6.4 Coefficient Conditions for Incomplete Beta Function

In particular, the results parallel to Theorems 6.1, 6.3 ,6.4 and 6.5 may also be obtained for the incomplete beta function $\varphi(a,c;z)$ as defined by (6.4). Let

$$\phi_1(z)=\varphi(a_1,c_1;z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}}{(c_1)_{n-1}}z^n,$$

$$\phi_2(z) = \frac{1}{z}\varphi(a_2, c_2; z) - 1 = \sum_{n=1}^{\infty} \frac{(a_2)_n}{(c_2)_n} z^n, \ |a_2| < |c_2|.$$

Making use of

$$F(a_1, 1; c_1; 1) = \frac{c_1 - 1}{c_1 - a_1 - 1} \quad \text{and} \quad F(a_2, 1; c_2; 1) - 1 = \frac{a_2}{c_2 - a_2 - 1}$$

the following theorems are obtained.

Theorem 6.6 *If $a_j > 0$ and $c_j > a_j + 3$ for $j = 1, 2$, then a sufficient condition for $G = \phi_1 + \overline{\phi_2}$ to be harmonic univalent in $\mathbb{D}$ with $\phi_1 + \overline{\phi_2} \in HP(\alpha, \beta)$, is that*

$$\begin{aligned}\left(\frac{2\alpha(a_1)_2}{(c_1 - a_1 - 3)_2} + \frac{2\alpha a_1 + c_1 - 2}{c_1 - a_1 - 2}\right) \frac{c_1 - 1}{c_1 - a_1 - 1} \\ + \left(\frac{2\alpha(a_2)_2}{(c_2 - a_2 - 3)_2} + \frac{a_2}{c_2 - a_2 - 2}\right) \frac{c_2 - 1}{c_2 - a_2 - 1} \leq 2 - \beta \qquad (6.21)\end{aligned}$$

where $\alpha \geq 0$ and $0 \leq \beta < 1$.

Note that the condition (6.21) is necessary and sufficient for $G = \phi_1 + \overline{\phi_2}$ to be in $HT(\alpha, \beta)$.

Theorem 6.7 *Let $0 \leq \beta < 1, a_j > 0, c_j > a_j + 2$, for $j = 1, 2$ and $a_2 < c_2$. A necessary and sufficient condition such that $f * (\phi_1 + \overline{\phi_2}) \in HT(\alpha, \beta)$ for $f \in HT(\alpha, \beta)$ is that*

$$\frac{c_1 - 1}{c_1 - a_1 - 1} + \frac{c_2 - 1}{c_2 - a_2 - 1} \leq 3 - \beta.$$

Theorem 6.8 *If $a_j > 0$ and $c_j > a_j + 2$ for $j = 1, 2$, then sufficient condition for*

$$\int_0^z \varphi(a_1, c_1; t)dt + \overline{\int_0^z \big(\varphi(a_2, c_2; t) - 1\big)dt}$$

is in $HP(\alpha, \beta)$ is

$$\left(\frac{\alpha a_1}{c_1 - a_1 - 2} + 1\right) \frac{c_1 - 1}{c_1 - a_1 - 1} + \left(\frac{\alpha a_2}{c_2 - a_2 - 2} + 1\right) \frac{c_2 - 1}{c_2 - a_2 - 1} \leq 3 - \beta$$

where $\alpha \geq 0$ *and* $0 \leq \beta < 1$.

Theorem 6.9 *If* $a_1 > -1, c_1 > 0, a_1 < 0, a_2 > 0,$ *and* $c_j > a_j + 3, j = 1, 2,$ *then*

$$\int_0^z \varphi(a_1, c_1; t)dt - \overline{\int_0^z \big(\varphi(a_2, c_2; t) - 1\big)dt}$$

is in $HT(\alpha, \beta)$ *if and only if*

$$\left(\frac{\alpha a_1}{c_1 - a_1 - 2} + 1\right)\frac{c_1 - 1}{c_1 - a_1 - 1} - \left(\frac{\alpha a_2}{c_2 - a_2 - 2} + 1\right)\frac{c_2 - 1}{c_2 - a_2 - 1} + 1 \geq \beta$$

where $\alpha \geq 0$ *and* $0 \leq \beta < 1$.

6.5 Coefficient Condition for Dziok-Srivastava Operator

A sufficient coefficient condition for functions belonging to the class $G_H([\alpha_1], \gamma)$ is now derived.

Theorem 6.10 *Let* $f = h + \overline{g}$ *be given by (6.1). If*

$$\sum_{n=1}^{\infty} n\left(\frac{2n - 1 - \gamma}{1 - \gamma}|a_n| + \frac{2n + 1 + \gamma}{1 - \gamma}|b_n|\right)\Gamma_n \leq 2. \tag{6.22}$$

$0 \leq \gamma < 1$, *then* $f \in G_H([\alpha_1], \gamma)$.

Proof. When the condition (6.22) holds for the coefficients of $f = h + \overline{g}$, it is shown that the inequality (6.10) is satisfied. Write the left side of inequality (6.10) as

$$\operatorname{Re}\left(\frac{\begin{array}{c} z(H^{l,m}[\alpha_1]h(z))' + (1 + e^{i\psi})z^2(H^{l,m}[\alpha_1]h(z))'' \\ + (1 + 2e^{i\psi})\overline{z(H^{l,m}[\alpha_1]g(z))'} + (1 + e^{i\psi})\overline{z^2(H^{l,m}[\alpha_1]g(z))''} \end{array}}{z(H^{l,m}[\alpha_1]h(z))' - \overline{z(H^{l,m}[\alpha_1]g(z))'}}\right)$$

$$= \operatorname{Re}\frac{A(z)}{B(z)},$$

where

$$A(z) = z(H^{l,m}[\alpha_1]h(z))' + (1+e^{i\psi})z^2(H^{l,m}[\alpha_1]h(z))''$$
$$+ (1+2e^{i\psi})\overline{z(H^{l,m}[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(H^{l,m}[\alpha_1]g(z))''}$$

and

$$B(z) = z(H^{l,m}[\alpha_1]h(z))' - \overline{z(H^{l,m}[\alpha_1]g(z))'}.$$

Since Re $(w) \geq \gamma$ if and only if $|1-\gamma+w| \geq |1+\gamma-w|$, it is sufficient to show that

$$|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0. \tag{6.23}$$

A computation shows that

$$\begin{aligned}
&A(z) + (1-\gamma)B(z)\\
&= (2-\gamma)z(H^{l,m}[\alpha_1]h(z))' + (1+e^{i\psi})z^2(H^{l,m}[\alpha_1]h(z))''\\
&\quad + (2e^{i\psi}+\gamma)\overline{z(H^{l,m}[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(H^{l,m}[\alpha_1]g(z))''}\\
&= (2-\gamma)z + \sum_{n=2}^{\infty}\big((2-\gamma)n + (1+e^{i\psi})(n^2-n)\big)\Gamma_n a_n z^n\\
&\quad + \sum_{n=1}^{\infty}\big((2e^{i\psi}+\gamma)n + (1+e^{i\psi})n(n-1)\big)\Gamma_n b_n \overline{z}^n
\end{aligned}$$

and

$$\begin{aligned}
&A(z) - (1+\gamma)B(z)\\
&= -\gamma z(H^{l,m}[\alpha_1]h(z))' + (1+e^{i\psi})z^2(H^{l,m}[\alpha_1]h(z))''\\
&\quad + (2+2e^{i\psi}+\gamma)\overline{z(H^{l,m}[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(H^{l,m}[\alpha_1]g(z))''}\\
&= -\gamma z + \sum_{n=2}^{\infty}\big((n^2-n-\gamma n) + e^{i\psi}(n^2-n)\big)\Gamma_n a_n z^n\\
&\quad + \sum_{n=1}^{\infty}\big((n^2+n+\gamma n) + e^{i\psi}(n^2+n)\big)\Gamma_n b_n \overline{z}^n.
\end{aligned}$$

Hence,

$$
\begin{aligned}
&|A(z)+(1-\gamma)B(z)|-|A(z)-(1+\gamma)B(z)| \\
&\geq |(2-\gamma)z| - \left|\sum_{n=2}^{\infty}\big((2-\gamma)n+(1+e^{i\psi})(n^2-n)\big)\Gamma_n a_n z^n\right| \\
&\quad - \left|\sum_{n=1}^{\infty}\big((2e^{i\psi}+\gamma)n+(1+e^{i\psi})n(n-1)\big)\Gamma_n b_n \overline{z}^n\right| \\
&\quad -|-\gamma z| - \left|\sum_{n=2}^{\infty}\big((n^2-n-\gamma n)+e^{i\psi}(n^2-n)\big)\Gamma_n a_n z^n\right| \\
&\quad - \left|\sum_{n=1}^{\infty}\big((n^2+n+\gamma n)+e^{i\psi}(n^2+n)\big)\Gamma_n b_n \overline{z}^n\right| \\
&\geq 2(1-\gamma)|z| - \sum_{n=2}^{\infty} 2n(2n-1-\gamma)\Gamma_n |a_n||z|^n \\
&\quad - \sum_{n=1}^{\infty} 2n(2n+1+\gamma)\Gamma_n |b_n||z|^n \\
&\geq 2(1-\gamma)|z|\left(1-\sum_{n=2}^{\infty} n\frac{2n-1-\gamma}{1-\gamma}\Gamma_n|a_n| - \sum_{n=1}^{\infty} n\frac{2n+1+\gamma}{1-\gamma}\Gamma_n|b_n|\right) \\
&\geq 2(1-\gamma)|z|\left(1+|a_1|-\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n|+\frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n\right) \\
&\geq 0
\end{aligned}
$$

by the inequality

$$
\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n|+\frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n \leq 1+|a_1|. \tag{6.24}
$$

By the fact that $1+|a_1| \leq 2$, (6.24) yield (6.22) , which implies that $f \in G_H([\alpha_1],\gamma)$. ∎

Now we obtain the necessary and sufficient condition for the function $f=h+\overline{g}$ given by (6.7) to be in $\mathcal{T}_H([\alpha_1],\gamma)$.

Theorem 6.11 *Let $f=h+\overline{g}$ be given by (6.7). Then $f \in \mathcal{T}_H([\alpha_1],\gamma)$ if and only* *if*

$$\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n| + \frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n \leq 2 \tag{6.25}$$

where $0 \leq \gamma < 1$.

Proof. Since $\mathcal{T}_H([\alpha_1],\gamma) \subset G_H([\alpha_1,\beta_1])$, we only need to prove the necessary part of the theorem. Assume that $f \in \mathcal{T}_H([\alpha_1],\gamma)$, then by virtue of (6.10), we obtain

$$\operatorname{Re}\left(\frac{\begin{array}{c}(1-\gamma)z(H^{l,m}[\alpha_1]h(z))' + (1+e^{i\psi})z^2(H^{l,m}[\alpha_1]h(z))'' \\ +(2e^{i\psi}+\gamma+1)\overline{z(H^{l,m}[\alpha_1]g(z))'} + (1+e^{i\psi})\overline{z^2(H^{l,m}[\alpha_1]g(z))''}\end{array}}{z(H^{l,m}[\alpha_1]h(z))' - \overline{z(H^{l,m}[\alpha_1]g(z))'}}\right) \geq 0. \tag{6.26}$$

By taking the numerator of the left hand side of (6.26), a computation shows that

$$\begin{aligned}
&(1-\gamma)z\left(1-\sum_{n=2}^{\infty} n\Gamma_n|a_n|z^{n-1}\right) + (1+e^{i\psi})z^2\left(-\sum_{n=2}^{\infty} n(n-1)\Gamma_n|a_n|z^{n-2}\right) \\
&+(2e^{i\psi}+\gamma+1)\overline{z}\left(-\sum_{n=1}^{\infty} n\Gamma_n|b_n|\overline{z}^{n-1}\right) + (1+e^{i\psi})\overline{z}^2\left(-\sum_{n=1}^{\infty} n(n-1)\Gamma_n|b_n|\overline{z}^{n-2}\right) \\
&= (1-\gamma)z - \sum_{n=2}^{\infty} n\big((1-\gamma)+(1+e^{i\psi})(n-1)\big)\Gamma_n|a_n|z^n \\
&\quad - \sum_{n=1}^{\infty} n\big((2e^{i\psi}+\gamma+1)+(1+e^{i\psi})(n-1)\big)\Gamma_n|b_n|\overline{z}^n.
\end{aligned}$$

Therefore, by rewriting (6.26) it follows that

$$\operatorname{Re}\left(\frac{\begin{array}{l}(1-\gamma)-\displaystyle\sum_{n=2}^{\infty} n\big(n(1+e^{i\psi})-e^{i\psi}-\gamma\big)\Gamma_n|a_n|z^{n-1}\\ -\dfrac{\overline{z}}{z}\displaystyle\sum_{n=1}^{\infty} n\big(n(1+e^{i\psi})+e^{i\psi}+\gamma\big)\Gamma_n|b_n|\overline{z}^{n-1}\end{array}}{1-\displaystyle\sum_{n=2}^{\infty} n\Gamma_n|a_n|z^{n-1}+\dfrac{\overline{z}}{z}\sum_{n=1}^{\infty} n\Gamma_n|b_n|\overline{z}^{n-1}}\right)\geq 0. \tag{6.27}$$

This inequality (6.27) must hold for all values of $z \in \mathbb{D}$ and for real ψ, so that on taking $z = r < 1$ and $\psi = 0$, (6.27) reduces to

$$\frac{(1-\gamma)-\left(\displaystyle\sum_{n=2}^{\infty} n(2n-1-\gamma)\Gamma_n|a_n|r^{n-1}+\sum_{n=1}^{\infty} n(2n+1+\gamma)\Gamma_n|b_n|r^{n-1}\right)}{1-\displaystyle\sum_{n=2}^{\infty}\Gamma_n|a_n|r^{n-1}+\sum_{n=1}^{\infty}\Gamma_n|b_n|r^{n-1}}\geq 0. \tag{6.28}$$

Letting $r \to 1^-$ through real values, it follows that

$$\frac{(1-\gamma)-\left(\displaystyle\sum_{n=2}^{\infty} n(2n-1-\gamma)\Gamma_n|a_n|+\sum_{n=1}^{\infty} n(2n+1+\gamma)\Gamma_n|b_n|\right)}{1-\displaystyle\sum_{n=2}^{\infty}\Gamma_n|a_n|+\sum_{n=1}^{\infty}\Gamma_n|b_n|}\geq 0$$

or

$$(1-\gamma)-\left(\sum_{n=2}^{\infty} n(2n-1-\gamma)\Gamma_n|a_n|+\sum_{n=1}^{\infty} n(2n+1+\gamma)\Gamma_n|b_n|\right)\geq 0. \tag{6.29}$$

A computation shows that

$$(1-\gamma)-\left(\sum_{n=2}^{\infty} n(2n-1-\gamma)\Gamma_n|a_n|+\sum_{n=1}^{\infty} n(2n+1+\gamma)\Gamma_n|b_n|\right)$$

$$\geq (1-\gamma)\left(1+|a_1| - \sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n| + \frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n\right).$$

Hence, (6.29) is equivalent to

$$(1-\gamma)\left(1+|a_1| - \sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n| + \frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n\right) \geq 0 \quad (6.30)$$

which holds true if

$$\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n| + \frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n \leq 1+|a_1|. \quad (6.31)$$

By the fact that $1+|a_1| \leq 2$, (6.31) yields (6.25). This completes the proof. ∎

6.6 Extreme Points and Inclusion Results

We determine the extreme points of closed convex hulls of $\mathcal{T}_H([\alpha_1],\gamma)$ denoted by $\overline{co}\mathcal{T}_H([\alpha_1],\gamma)$.

Theorem 6.12 *A function $f(z) \in \overline{co}\mathcal{T}_H([\alpha_1],\gamma)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z))$$

where

$$h_1(z) = z, h_n(z) = z - \frac{1-\gamma}{n(2n-1-\gamma)\Gamma_n} z^n; \quad (n \geq 2),$$
$$g_n(z) = z - \frac{1-\gamma}{n(2n+1+\gamma)\Gamma_n} \overline{z}^n; \quad (n \geq 2),$$
$$\sum_{n=1}^{\infty}(X_n + Y_n) = 1, \quad X_n \geq 0 \quad \textit{and} \quad Y_n \geq 0.$$

In particular, the extreme points of $\mathcal{T}_H([\alpha_1],\gamma)$ are $\{h_m\}$ and $\{g_m\}$.

Proof. First, we note that for f as in theorem above, we may write

$$\begin{aligned}
f(z) &= \sum_{n=1}^{\infty} \left(X_n h_n(z) + Y_n g_n(z)\right) \\
&= z\sum_{n=1}^{\infty}(X_n + Y_n) + (h_1(z) - z)X_1 + \sum_{n=2}^{\infty}(h_n(z) - z)X_n \\
&\quad + \sum_{n=1}^{\infty}(g_n(z) - z)Y_n \\
&= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{n(2n-1-\gamma)\Gamma_n} X_n z^n - \sum_{n=1}^{\infty} \frac{1-\gamma}{n(2n+1+\gamma)\Gamma_n} Y_n \overline{z}^n \\
&= z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=1}^{\infty} B_n \overline{z}^n
\end{aligned}$$

where

$$A_n = \frac{1-\gamma}{n(2n-1-\gamma))\Gamma_n} X_n \quad \text{and} \quad B_n = \frac{1-\gamma}{n(2n+1+\gamma)\Gamma_n} Y_n.$$

Therefore, in view of (6.25)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma} A_n + \frac{2n+1+\gamma}{1-\gamma} B_n\right)\Gamma_n \\
&= X_1 + \sum_{n=2}^{\infty} \frac{n(2n-1-\gamma)\Gamma_n}{1-\gamma}\left(\frac{1-\gamma}{n(2n-1-\gamma))\Gamma_n} X_n\right) \\
&\quad + \sum_{n=1}^{\infty} \frac{n(2n+1+\gamma)\Gamma_n}{1-\gamma}\left(\frac{1-\gamma}{n(2n+1+\gamma)\Gamma_n} Y_n\right) \\
&= X_1 + \left(-X_1 + \sum_{n=1}^{\infty} X_n\right) + \sum_{n=1}^{\infty} Y_n \\
&= 1 < 2
\end{aligned}$$

and hence $f(z) \in \overline{co}\mathcal{T}_H([\alpha_1], \gamma)$.

Conversely, suppose that $f(z) \in \overline{co}\mathcal{T}_H([\alpha_1], \gamma)$. Set

$$X_n = \frac{n(2n-1-\gamma)\Gamma_n}{1-\gamma} A_n, \quad (n \geq 2), \quad Y_n = \frac{n(2n+1+\gamma)\Gamma_n}{1-\gamma} B_n, \quad (n \geq 1)$$

where $\sum_{n=1}^{\infty}(X_n + Y_n) = 1$. Then

$$\begin{aligned} f(z) &= z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=1}^{\infty} B_n \overline{z}^n, \quad A_n,\ B_n \geq 0. \\ &= z - \sum_{n=2}^{\infty} \frac{1-\gamma}{n(2n-1-\gamma)\Gamma_n} X_n z^n - \sum_{n=1}^{\infty} \frac{1-\gamma}{n(2n+1+\gamma)\Gamma_n} Y_n \overline{z}^n \\ &= \sum_{n=1}^{\infty} (X_n h_n(z) + Y_n g_n(z)) \end{aligned}$$

as required. ∎

Now we show that $\mathcal{T}_H([\alpha_1], \gamma)$ is closed under convex combinations of its members.

Theorem 6.13 *The family $\mathcal{T}_H([\alpha_1], \gamma)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i \in \mathcal{T}_H([\alpha_1], \gamma)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} |a_{i,n}| z^n - \sum_{n=1}^{\infty} |b_{i,n}| \overline{z}^n.$$

Then, by inequality (6.25)

$$\sum_{n=1}^{\infty} n \left(\frac{2n-1-\gamma}{1-\gamma} |a_{i,n}| + \frac{2n+1+\gamma}{1-\gamma} |b_{i,n}| \right) \Gamma_n \leq 2 \tag{6.32}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i,n}| \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{i,n}| \right) \overline{z}^n.$$

Using the inequality (6.25), it follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}\left(\sum_{i=1}^{\infty} t_i|a_{i,n}|\right)+\frac{2n+1+\gamma}{1-\gamma}\left(\sum_{i=1}^{\infty} t_i|b_{i,n}|\right)\right)\Gamma_n \\
&=\sum_{i=1}^{\infty} t_i\left(\sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_{i,n}|+\frac{2n+1+\gamma}{1-\gamma}|b_{i,n}|\right)\Gamma_n\right) \\
&\leq 2
\end{aligned}$$

and therefore $\sum_{i=1}^{\infty} t_i f_i \in \mathcal{T}_H([\alpha_1],\gamma)$. ∎

Theorem 6.14 *For* $0 \leq \delta \leq \gamma < 1$, *let* $f(z) \in \mathcal{T}_H([\alpha_1],\gamma)$ *and* $F(z) \in \mathcal{T}_H([\alpha_1],\delta)$. *Then* $f(z) * F(z) \in G_H([\alpha_1],\gamma) \subset G_H([\alpha_1],\delta)$.

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n - \sum_{n=1}^{\infty} |b_n|\overline{z}^n \in \mathcal{T}_H([\alpha_1],\gamma)$$

and

$$F(z) = z - \sum_{n=2}^{\infty} |A_n| z^n - \sum_{n=1}^{\infty} |B_n|\overline{z}^n \in \mathcal{T}_H([\alpha_1],\delta).$$

Then

$$f(z) * F(z) = z + \sum_{n=2}^{\infty} |a_n||A_n| z^n + \sum_{n=1}^{\infty} |b_m||B_m|\overline{z}^n.$$

Note that $|A_n| \leq 1$ and $|B_n| \leq 1$ for $n \geq 2$. Now we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(\frac{2n-1-\delta}{1-\delta}|a_n||A_n|+\frac{2n+1+\delta}{1-\delta}|b_n||B_n|\right)\Gamma_n \\
&\leq \sum_{n=1}^{\infty} n\left(\frac{2n-1-\delta}{1-\delta}|a_n|+\frac{2n+1+\delta}{1-\delta}|b_n|\right)\Gamma_n \\
&\leq \sum_{n=1}^{\infty} n\left(\frac{2n-1-\gamma}{1-\gamma}|a_n|+\frac{2n+1+\gamma}{1-\gamma}|b_n|\right)\Gamma_n \\
&\leq 2
\end{aligned}$$

using Theorem 6.11 since $f(z) \in \mathcal{T}_H([\alpha_1],\gamma)$ and $0 \leq \delta \leq \gamma < 1$. This proves that

$f(z) * F(z) \in G_H([\alpha_1], \gamma) \subset G_H([\alpha_1], \delta)$. ∎

Now, we examine a closure property of the class $\mathcal{T}_H([\alpha_1], \gamma)$ under the generalized Bernardi-Libera-Livingston integral operator $F(z)$

$$F(z) := (\mu + 1) \int_0^1 t^{\mu-1} f(tz) dt \quad (\mu > -1)$$

which was defined earlier in Section 1.3.2, p. 22.

Theorem 6.15 *Let $f(z) \in \mathcal{T}_H([\alpha_1], \gamma)$. Then $F(z) \in \mathcal{T}_H([\alpha_1], \gamma)$*

Proof. Let

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n - \overline{\sum_{n=1}^{\infty} |b_n| z^n}.$$

Then

$$\begin{aligned} F(z) &= (\mu + 1) \int_0^1 t^{\mu-1} \left((tz) - \sum_{n=2}^{\infty} |a_n| (tz)^n - \overline{\sum_{n=1}^{\infty} |b_n| (tz)^n} \right) dt \\ &= z - \sum_{n=2}^{\infty} A_n z^n - \overline{\sum_{n=1}^{\infty} B_n z^n} \end{aligned}$$

where

$$A_n = \frac{\mu + 1}{\mu + n} |a_n| \quad \text{and} \quad B_n = \frac{\mu + 1}{\mu + n} |b_n|.$$

Therefore, since $f(z) \in \mathcal{T}_H([\alpha_1], \gamma)$,

$$\begin{aligned} &\sum_{n=1}^{\infty} n \left(\frac{2n - 1 - \gamma}{1 - \gamma} \left(\frac{\mu + 1}{\mu + n} |a_n| \right) + \frac{2n + 1 + \gamma}{1 - \gamma} \left(\frac{\mu + 1}{\mu + n} |b_n| \right) \right) \Gamma_n \\ &\leq \sum_{n=1}^{\infty} n \left(\frac{2n - 1 - \gamma}{1 - \gamma} |a_n| + \frac{2n + 1 + \gamma}{1 - \gamma} |b_n| \right) \Gamma_n \\ &\leq 2. \end{aligned}$$

Therefore by Theorem 6.11, $F(z) \in \mathcal{T}_H([\alpha_1], \gamma)$. ∎

CONCLUSION

A modest attempt has been made in this thesis to introduce and study certain classes of analytic and harmonic functions defined on the open unit disk $\mathbb{D}$ and certain classes of meromorphic functions defined on the open punctured unit disk $\mathbb{D}^*$ using subordination, superordination and convolution.

- Closure properties under convolution of general classes of meromorphic multivalent functions with respect to n-ply symmetric, conjugate and symmetric conjugate points are obtained.

- Differential subordination, differential superordination and sandwich-type results are obtained for multivalent meromorphic functions associated with the Dziok-Srivastava and Liu-Srivastava linear operator by investigating appropriate classes of admissible functions.

- Sufficient conditions for Janowski starlikeness of analytic functions by making use of the theory of differential subordination which are also applied to Bernardi's integral operator are obtained.

- New subclasses of harmonic functions defined through hypergeometric functions are introduced and coefficient bounds, extreme points, inclusion results and closure under an integral operator are discussed.

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