INCLUSION PROPERTIES OF LINEAR OPERATORS AND ANALYTIC FUNCTIONS

by

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SYMBOLS

Symbol	Description	page
$A(f)$	Alexander operator	28
$\mathcal{A}_b$	Class of analytic functions f with fixed second coefficient	172
	of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad \lvert a_2 \rvert = 2b \quad (z \in \mathcal{U})$	
$\mathcal{A}_m$	Class of m-valent analytic functions f of the form	2
	$f(z) = z^m + \sum_{k=1+m}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	
$\mathcal{A}$	Class of analytic functions f of the form	2
	$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathcal{U})$	
$\mathcal{B}(\alpha, \rho, \lambda)$	α-Bazilevič function of order ρ and type λ in $\mathcal{A}$	66
$\mathcal{B}(\alpha)$	α-Bazilevič function in $\mathcal{A}$	67
$(a)_n$	Pochhammer symbol	29
$\mathbb{C}$	Complex plane	1
$C^*(m, a, h)$	$\left\{ f \in \Sigma : -\frac{z(k_a * f)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(k_a * g_j)(z)} \prec h(z), \right.$	
	$\left. z \in \mathcal{U}^*, \ g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m, a, h) \right\}$	50
$\mathcal{CCV}$	Class of close-to-convex functions in $\mathcal{A}$	5
$\mathcal{C}_a(h)$	$\left\{ f \in \mathcal{A} : \ \frac{z(k_a * f)'(z)}{(k_a * \psi)(z)} \prec h(z), \ \psi \in \mathcal{S}_a(h), \ z \in \mathcal{U} \right\}$	37
$\mathcal{CCV}(g, h)$	$\left\{ f \in \mathcal{A} : \ \frac{z(g * f)'(z)}{(g * \psi)(z)} \prec h(z), \ \psi \in \mathcal{ST}(g, h), \ z \in \mathcal{U} \right\}$	37
$\mathcal{CCV}_m(h)$	$\left\{ \hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : \frac{mzf_k'(z)}{\sum_{j=1}^{m} \psi_j(z)} \prec h(z), \right.$	
	$\left. z \in \mathcal{U}, \ \hat{\psi} \in \mathcal{ST}_m(h), \ f_k \in \mathcal{A}, \ k = 1, \cdots, m \right\}$	39
$\mathcal{CV}$	Class of convex functions in $\mathcal{A}$	4

$\mathcal{CV}(\alpha)$	Class of convex functions of order α in $\mathcal{A}$	5
$\mathcal{CV}_\alpha$	$\left\{f \in \mathcal{A} : \left\|\frac{zf''(z)}{f'(z)}\right\| < 1-\alpha, z \in \mathcal{U}\right\}$	37
$\mathcal{CV}(g,h)$	$\left\{f \in \mathcal{A} : 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathcal{CV}_m(h)$	$\bigg\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle :$	
	$\frac{m(zf_k')'(z)}{\sum_{j=1}^{m} f_j'(z)} \prec h(z),\ f_k \in \mathcal{A},\ z \in \mathcal{U},\ k = 1, \cdots, m\bigg\}$	38
$\mathcal{CV}(h)$	$\left\{f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	36
$\overline{\text{co}}(D)$	The closed convex hull of a set D	4
D_λ^n	Generalized Sălăgean operator for meromorphic functions	49
$f * g$	Convolution or Hadamard product of functions f and g	17
$\mathcal{H}(\mathcal{U})$	Class of analytic functions in $\mathcal{U}$	2
$H_{a,b,c}(f)$	Hohlove operator	30
$\prec$	Subordinate to	24
$\mathcal{I}_{\lambda,\mu}^{\beta}$	$\mathcal{I}_{\lambda,\mu}^{\beta} f(z) = (f * p_\mu * q_{\beta,\lambda})(z)$	48
k	Koebe function $k(z) = z/(1-z)^2$	2
k_a	$k_a(z) := \frac{z}{(1-z)^a}, \quad a > 0$	37
$\mathcal{K}_a(h)$	$\left\{f \in \mathcal{A} : \frac{z(f*k_a)''(z)}{(f*k_a)'(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathbb{N}$	$\mathbb{N} := \{1, 2, \cdots\}$	2
$k - \mathcal{UCV}$	Class of k-uniformly convex functions in $\mathcal{A}$	120
$L(f)$	Libera operator	28
$L_\gamma(f)$	Bernardi-Libera-Livingston operator	29
$\mathcal{L}(\alpha,\beta)$	$\bigg\{f \in \mathcal{A} : \alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\beta)z}{1-z},$	

	$z \in \mathcal{U},\ \beta \in \mathbb{R} \setminus \{1\},\ \alpha \geq 0 \Big\}$	173
M	Möbius transformations	6
p_μ	$p_\mu(z) := \frac{1}{z(1-z)^\mu}$	48
$\mathcal{P}_\gamma$	$\left\{ f \in \mathcal{A} : \left\| \arg\left((1-\gamma)\dfrac{f(z)}{z} + \gamma f'(z) \right) \right\| < \dfrac{\pi}{2},\ z \in \mathcal{U} \right\}$	74
$\mathcal{P}(\beta)$	$\Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0,\ z \in \mathcal{U} \Big\}$	22
$\mathcal{P}_\alpha(\beta)$	$\Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with}$ $\operatorname{Re} e^{i\phi}\Big((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta \Big) > 0,\ z \in \mathcal{U} \Big\}$	77
$\mathcal{PST}$	Class of parabolic starlike functions in $\mathcal{A}$	9
$\mathcal{PST}(\alpha)$	Class of parabolic starlike functions of order α in $\mathcal{A}$	8
Q	$\Big\{ q : \text{analytic and injective on } \bar{\mathcal{U}} \backslash E(q),$ $q'(\zeta) \neq 0,\ \zeta \in \partial\mathcal{U} \backslash E(q) \Big\}$	24
$q_{\beta,\lambda}$	$q_{\beta,\lambda}(z) := \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\lambda}{k+1+\lambda} \right)^\beta z^k$	48
$Q(m,n,\lambda,h)$	$\Big\{ f \in \Sigma : -\frac{z(D_\lambda^n f(z))'}{\frac{1}{m}\sum_{j=1}^m D_\lambda^n g_j(z)} \prec h(z),$ $z \in \mathcal{U}^*, g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m,n,\lambda,h) \Big\}$	49
$\mathcal{QCV}$	Class of quasi-convex functions in $\mathcal{A}$	5
$\mathcal{QCV}(g,h)$	$\Big\{ f \in \mathcal{A} :\ \frac{z(g*f)'(z)}{(g*\phi)(z)} \prec h(z),\ \phi \in \mathcal{CV}(g,h),\ z \in \mathcal{U} \Big\}$	37
$\mathcal{QCV}_m(h)$	$\Big\{ \hat{f} := \langle f_1, f_2, \ldots, f_m \rangle : \frac{m(zf_k')'(z)}{\sum_{j=1}^m \phi_j'(z)} \prec h(z),$ $z \in \mathcal{U},\ \hat{\varphi} \in \mathcal{CV}_m(h),\ f_k \in \mathcal{A},\ k = 1, \cdots, m \Big\}$	39
$\mathbb{R}$	Set of all real numbers	21
Re	Real part of a complex number	4

$\mathcal{R}_\alpha$	Class of prestarlike functions of order α in $\mathcal{A}$	19
$\mathcal{R}^\alpha$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + \alpha z f''(z)\right) > 0,\ z \in \mathcal{U}\right\}$	68
$\mathcal{R}(\beta)$	$\left\{f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + z f''(z)\right) > \beta,\ z \in \mathcal{U}\right\}$	115
$\mathcal{R}_\alpha(\beta)$	$\{f \in \mathcal{A} : \exists \phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\left(f'(z) + \alpha z f''(z) - \beta\right) > 0,\ z \in \mathcal{U}\Big\}$	77
$\mathcal{S}$	Class of all normalized univalent functions f of the form $f(z) = z + a_2 z^2 + \cdots,\ z \in \mathcal{U}$	2
$S(f, z)$	Schwarzian derivative of analytic function f	6
$\mathcal{S}_a(h)$	$\left\{f \in \mathcal{A} : \frac{z(f*k_a)'(z)}{(f*k_a)(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
Σ	Class of all normalized meromorphic functions f of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathcal{U}^*)$	10
$\Sigma^\beta_{\lambda,\mu}(m, h)$	$\left\{g := \{g_1, g_2, \ldots, g_m\} : -\frac{z(\mathcal{I}^\beta_{\lambda,\mu} g_k(z))'}{\frac{1}{m}\sum_{j=1}^m \mathcal{I}^\beta_{\lambda,\mu} g_j(z)} \prec h(z),\ z \in \mathcal{U}^*,\ g_k \in \Sigma,\ k = 1, \cdots, m\right\}$	48
$\Sigma(m, n, \lambda, h)$	$\left\{f = \{f_1, f_2, \cdots, f_m\} : -\frac{z(D^n_\lambda f_i(z))'}{\frac{1}{m}\sum_{j=1}^m D^n_\lambda f_j(z)} \prec h(z),\ z \in \mathcal{U}^*,\ f_i \in \Sigma,\ i = 1, \cdots, m\right\}$	49
$\Sigma(m, a, h)$	$\left\{g = \{g_1, g_2, \cdots, g_m\} : -\frac{z(k_a * g_i)'(z)}{\frac{1}{m}\sum_{j=1}^m (k_a * g_j)(z)} \prec h(z),\ z \in \mathcal{U}^*,\ g_i \in \Sigma,\ i = 1, \cdots, m\right\}$	49
$\Sigma^{ccv}_m(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle : -\frac{m z f'_k(z)}{\sum_{j=1}^m \psi_j(z)} \prec h(z),\ \hat{\psi} \in \Sigma^{st}_m(h),\ z \in \mathcal{U}^*,\ f_k \in \Sigma,\ k = 1, \cdots, m\right\}$	51
$\Sigma^{cv}(\alpha)$	$\left\{f \in \Sigma :\ -\operatorname{Re}\frac{(zf'(z))'}{f'(z)} > \alpha,\ z \in \mathcal{U}^*\right\}$	11

$\Sigma_m^{cv}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle : -\frac{m(zf_k')'(z)}{\sum_{j=1}^m f_j'(z)} \prec h(z),\ f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	50
$\Sigma_m^{qcv}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle : -\frac{m(zf_k')'(z)}{\sum_{j=1}^m \varphi_j'(z)} \prec h(z),\ \hat{\varphi} \in \Sigma_m^{cv}(h),\ f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	51
$\Sigma^{st}(\alpha)$	$\left\{f \in \Sigma :\ -\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha,\ z \in \mathcal{U}^*\right\}$	11
$\Sigma_m^{st}(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle :\ -\frac{mzf_k'(z)}{\sum_{j=1}^m f_j(z)} \prec h(z), f_k \in \Sigma,\ z \in \mathcal{U}^*,\ k = 1, \cdots, m\right\}$	50
$\mathcal{SST}(\alpha)$	Class of strongly starlike functions of order α in $\mathcal{A}$	59
$\mathcal{ST}$	Class of starlike functions in $\mathcal{A}$	4
$\mathcal{ST}[A, B]$	$\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U},\ -1 \leq B < A \leq 1\}$	15
$\mathcal{ST}(\alpha)$	Class of starlike functions of order α in $\mathcal{A}$	5
$\mathcal{ST}_\alpha$	$\left\{f \in \mathcal{A} : \left\lvert\frac{zf'(z)}{f(z)} - 1\right\rvert < 1 - \alpha,\ z \in \mathcal{U}\right\}$	115
$\mathcal{ST}(h)$	$\left\{f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	36
$\mathcal{ST}(g, h)$	$\left\{f \in \mathcal{A} : \frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z),\ z \in \mathcal{U}\right\}$	37
$\mathcal{ST}_m(h)$	$\left\{\hat{f} := \langle f_1, f_2, \ldots, f_m\rangle :\ \frac{mzf_k'(z)}{\sum_{j=1}^m f_j(z)} \prec h(z),\ f_k \in \mathcal{A},\ z \in \mathcal{U},\ k = 1, \cdots, m\right\}$	38
$\mathcal{T}$	Class of analytic functions with negative coefficients	12
$\mathcal{TCV}(\alpha)$	Class of convex functions of order α in $\mathcal{T}$	12
$\mathcal{TL}(\alpha, \beta)$	$\left\{f \in \mathcal{T} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\left(\alpha\frac{zf''(z)}{f'(z)} + 1\right)\right) > \beta,\ z \in \mathcal{U}\right\}$	128
$\mathcal{TST}(\alpha)$	Class of starlike functions of order α in $\mathcal{T}$	12

$\mathcal{U}$	Open unit disk $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$	1
$\mathcal{U}^*$	punctured unit disk $\{z \in \mathbb{C} : 0 < \lvert z\rvert < 1\}$	10
$\mathcal{UCV}$	Class of uniformly convex functions in $\mathcal{A}$	8
$\mathcal{UST}$	Class of uniformly starlike functions in $\mathcal{A}$	9
$V_\lambda(f)$	$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt$	22
$\mathcal{V}^*$	The dual set of $\mathcal{V}$	20
$\mathcal{V}^{**}$	The second dual of $\mathcal{V}$	20
$\mathcal{W}_\beta(\alpha,\gamma)$	$\Big\{f \in \mathcal{A} : \exists\phi \in \mathbb{R}$ with $\operatorname{Re} e^{i\phi}\Big((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta\Big) > 0, z \in \mathcal{U}\Big\}$	23
$\Psi_m[\Omega, q]$	Class of admissible functions	138
$\Psi'_m[\Omega, q]$	Class of admissible functions	155

SIFAT-SIFAT RANGKUMAN PENGOPERASIAN LINEAR DAN FUNGSI ANALISIS

ABSTRAK

Tesis ini mengkaji kelas $\mathcal{A}$ terdiri daripada fungsi analisis ternormalkan di dalam cakera unit terbuka $\mathcal{U}$ pada satah kompleks. Kelas fungsi meromorfi di dalam cakera unit berliang tidak termasuk titik asalan turut dikaji. Secara keseluruhannya, tesis ini merangkumi enam permasalahan kajian. Pertama, subkelas fungsi-fungsi bak-bintang, cembung, hampir cembung dan kuasi cembung diitlakkan dengan memperkenalkan subkelas baru fungsi-fungsi analisis dan meromorfi. Sifat tutupan kelas-kelas baru ini akan dikaji dan akan dibuktikan bahawa konvolusi kelas-kelas ini dengan fungsi pra bak-bintang dan pengoperasi kamiran Bernardi-Libera-Livingston adalah bersifat tertutup.

Keunivalenan fungsi $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ dikaji dengan menyarankan terbitan Schwarzian $S(f, z)$ dan pekali kedua a_2 fungsi f memenuhi ketaksamaan tertentu. Kriteria baru untuk fungsi analisis menjadi α-Bazilevič kuat tertib tak negatif dibangunkan dalam sebutan terbitan Schwarzian dan pekali kedua. Juga syarat-syarat serupa untuk pekali kedua dan terbitan Schwarzian $S(f, z)$ bagi f diperoleh yang menjamin fungsi f tersebut terkandung di dalam subkelas tertentu untuk $\mathcal{S}$. Untuk suatu fungsi analisis $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ yang memenuhi ketaksamaan $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$, batas tajam β diperoleh supaya f sama ada bak-bintang atau cembung tertib α. Batas tajam untuk η juga diperoleh agar fungsi f yang memenuhi $\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq 1 - \beta$ adalah bak bintang atau cembung tertib α. Beberapa ketaksamaan pekali lain berkaitan dengan subkelas-subkelas tertentu juga dikaji. Andaikan $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analisis pada $\mathcal{U}$ dengan pekali kedua a_2 memenuhi $|a_2| = 2b$, $0 \leq b \leq 1$, dan katakan f memenuhi sama ada $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ atau $|a_n| \leq c/n$ $(c > 0)$

untuk $n \geq 3$. Jejari tajam bak-bintang Janowski dan beberapa jejari berkaitan untuk fungsi sedemikian juga diperoleh.

Sifat kecembungan pengoperasi kamiran umum $V_\lambda(f)(z) := \int_0^1 \lambda(t)f(tz)/tdt$ pada suatu subkelas fungsi analisis yang mengandung beberapa subkelas tersohor akan dikaji. Beberapa aplikasi menarik dengan pilihan λ berbeza akan dibincang. Sifat-sifat geometrik untuk pengoperasi kamiran teritlak berbentuk $\mathcal{V}_\lambda(f) = \rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$ akan juga diterangkan. Akhir sekali, sifat subordinasi dan superordinasi untuk pengoperasi linear teritlak yang memenuhi suatu hubungan jadi semula pembeza peringkat pertama telah dikaji. Suatu kelas fungsi teraku yang sesuai telah dipertimbangkan untuk mendapatkan syarat cukup bagi domainan dan subordinan terbaik. Keputusan yang diperoleh menyatukann hasil kajian terdahulu.

INCLUSION PROPERTIES OF LINEAR OPERATORS AND ANALYTIC FUNCTIONS

ABSTRACT

This thesis studies the class $\mathcal{A}$ of normalized analytic functions in the open unit disk $\mathcal{U}$ of the complex plane. The class of meromorphic functions in the punctured unit disk which does not include the origin is also studied. This thesis investigates six research problems. First, the classical subclasses of starlike, convex, close-to-convex and quasi-convex functions are extended by introducing new subclasses of analytic and meromorphic functions. The closure properties of these newly defined classes are investigated and it is shown that these classes are closed under convolution with prestarlike functions and the Bernardi-Libera-Livingston integral operator.

The univalence of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ is investigated by requiring the Schwarzian derivative $S(f,z)$ and the second coefficient a_2 of f to satisfy certain inequalities. New criterion for analytic functions to be strongly α-Bazilevič of nonnegative order is established in terms of the Schwarzian derivatives and the second coefficients. Also, similar conditions on the second coefficient of f and its Schwarzian derivative $S(f,z)$ are obtained that would ensure the function f belongs to particular subclasses of $\mathcal{S}$. For an analytic function $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$, a sharp bound on β is determined so that f is either starlike or convex of order α. A sharp bound on η is obtained that ensures functions f satisfying $\sum_{n=2}^{\infty} \left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n| \leq 1-\beta$ is either starlike or convex of order η. Several other coefficient inequalities related to certain subclasses are also investigated. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in the unit disk $\mathcal{U}$ with the second coefficient a_2 satisfying $|a_2| = 2b$, $0 \leq b \leq 1$, and let f satisfy either $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$

for $n \geq 3$. Sharp radius of Janowski starlikeness for such functions is obtained. Several related radii are also obtained.

The convexity property of a general integral operator $V_\lambda(f)(z) := \int_0^1 \lambda(t)f(tz)/tdt$ on a new class of analytic functions which includes several well-known classes is investigated. Several interesting applications for different choices of λ are discussed. The geometric properties of the generalized integral operator of the form $\mathcal{V}_\lambda(f) = \rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$ are also inquired. Finally, subordination and superordination properties of general linear operators satisfying a certain first-order differential recurrence relation are investigated. An appropriate class of admissible functions is considered to determine sufficient conditions for best dominant and best subordinant. The results obtained unify earlier works.

CHAPTER 1

INTRODUCTION

Geometric function theory is a remarkable area in complex analysis. This field is more often associated with geometric properties of analytic functions. Geometric function theory has raised the interest of many researchers since the beginning of the 20th century. The purpose of this chapter is to review and assemble for references, relevant definitions and known results in geometric function theory which underlie the theory of univalent functions.

1.1 Univalent Functions

A function f is *analytic* at z_0 in a domain D if it is differentiable in some neighborhood of z_0, and it is *analytic on a domain* D if it is analytic at all points in D. An analytic function f is said to be *univalent* in a domain D of the complex plane $\mathbb{C}$ if it is one-to-one in D. It is *locally univalent* in D if f is univalent in some neighborhood of each point $z_0 \in D$. It is known that a function f is locally univalent in D provided $f'(z) \neq 0$ for any $z \in D$ [48, p. 5]. In 1851, Riemann proved that any simply connected domain which is not the entire plane and the unit disk $\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\}$ are conformally equivalent.

Theorem 1.1 (Riemann Mapping Theorem) [48, p. 11] *Let D be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in D. Then there is a unique univalent analytic function f which maps D onto the unit disk $\mathcal{U}$ satisfying $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

Therefore, the study of conformal mappings on simply connected domains may be confined to the study of functions that are univalent on the unit disk $\mathcal{U}$. The Riemann Mapping Theorem shows that there is a one-to-one correspondence between proper simply connected domains (geometric objects) and suitably normalized univalent functions (analytic objects).

Let $\mathcal{H}(\mathcal{U})$ denote the set of all analytic functions defined in the unit disk $\mathcal{U}$. Let $\mathcal{A}$ be the class of normalized analytic functions f defined in $\mathcal{U}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

More generally, let $\mathcal{A}_m$ denote the subclass of $\mathcal{A}$ consisting of normalized analytic functions f of the form

$$f(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad (m \in \mathbb{N} := \{1, 2, \cdots\}).$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions. The class $\mathcal{S}$ are treated extensively in the books [48, 61, 151]. Bernardi [33] provided a comprehensive list of papers on univalent functions theory published before 1981.

The *Koebe function* defined by

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n,$$

and its rotations $e^{-i\beta} k(e^{i\beta} z)$, play an important role in the class $\mathcal{S}$. The Koebe function maps $\mathcal{U}$ in a one-to-one manner onto a domain D consisting of the entire complex plane except for a slit along the negative real axis from $w = -\infty$ to $w = -1/4$. A significant problem in the theory of univalent functions is the Bieberbach's conjecture which asserts that the Koebe function has the largest coefficients in $\mathcal{S}$.

Theorem 1.2 (Bieberbach's Conjecture) [48] *If* $f = \sum_{n=1}^{\infty} a_n z^n \in \mathcal{S}$, *then*

$$|a_n| \leq n \quad (n \geq 2).$$

Equality occurs only for the Koebe function and its rotations.

In 1916, Bieberbach [36] proved the inequality for $n = 2$, and conjectured that it is true for any n. In 1985, de Branges [37] proved this conjecture for all coefficients $n \geq 2$. Before de Branges's proof, the Bieberbach's conjecture was known for $n \leq 6$. Löwner [101] developed parametric representation of slit mapping and used it to prove the Bieberbach's conjecture for $n = 3$. The cases $n = 4, 5, 6$ were proved by Garabedian and Schiffer [57], Pederson and Schiffer [147], and Pederson [146]. In 1925, Littlewood [95] showed that the coefficients of each function $f \in \mathcal{S}$ satisfy $|a_n| \leq en$ $(n \geq 2)$. Duren [48], Goodman [61] and Pommerenke [151] provided the history of this problem.

As an application, a famous covering theorem due to Koebe can be proved by Bieberbach's conjecture for the second coefficient. This theorem states that if $f \in \mathcal{S}$, then the image of $\mathcal{U}$ under f must cover an open disk centered at the origin with radius 1/4.

Theorem 1.3 (Koebe One-Quarter Theorem) [61, p. 62] *The range of every function $f \in \mathcal{S}$ contains the disk $\{w : \ |w| < 1/4\}$.*

The Koebe function and its rotations are the only functions in $\mathcal{S}$ which omit a value of modulus 1/4. The sharp upper and lower bounds for $|f(z)|$ and $|f'(z)|$ where $f \in \mathcal{S}$ are a consequence of the Bieberbach's conjecture for the second coefficient.

Theorem 1.4 (Distortion and Growth Theorem) [61] *Let $f \in \mathcal{S}$. Then for each $z = re^{i\theta} \in \mathcal{U}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3},$$

and

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}.$$

The above inequalities are sharp with equality occurring for the Koebe function and its rotations.

1.2 Subclasses of Univalent Functions

The long gap between the formulation of the Bieberbach's conjecture (1916) and its proof by de Branges (1985) motivated researchers to investigate its validity on several subclasses of $\mathcal{S}$. These classes are defined by geometric conditions, and include the class of starlike functions, convex functions, close-to-convex functions, and quasi-convex functions. A set D in the plane is said to be *starlike* with respect to an interior point w_0 in D if the line segment joining w_0 to every other point w in D lies entirely in D. A set D in the plane is *convex* if it is starlike with respect to each of its points; that is, if the line segment joining any two points of D lies entirely in D. The *closed convex hull* of a set D in $\mathbb{C}$ is the closure of intersection of all convex sets containing D. It is the smallest closed convex set containing D and is denoted by $\overline{\text{co}}(D)$.

A function $f \in \mathcal{A}$ is *starlike* if $f(\mathcal{U})$ is a starlike domain with respect to the origin, and f is *convex* if $f(\mathcal{U})$ is a convex domain. Analytically, these are respectively equivalent to the conditions

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \quad \text{and} \quad \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathcal{U}).$$

In 1915, Alexander [4] showed that there is a close connection between convex and starlike functions.

Theorem 1.5 (Alexander Theorem) [4] *Suppose that $f'(z) \neq 0$ in $\mathcal{U}$. Then f is convex in $\mathcal{U}$ if and only if zf' is starlike in $\mathcal{U}$.*

Denote the classes of starlike and convex functions by $\mathcal{ST}$ and $\mathcal{CV}$ respectively.

More generally, for $\alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ be subclasses of $\mathcal{A}$ consisting respectively of starlike functions of order α and convex functions of order α. For $0 \leq \alpha < 1$, these functions are known to be univalent [48, p. 40], and are defined analytically by

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\}, \tag{1.2}$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}. \tag{1.3}$$

Clearly, $\mathcal{ST} = \mathcal{ST}(0)$ and $\mathcal{CV} = \mathcal{CV}(0)$.

In 1952, Kaplan [77] introduced the class of close-to-convex functions. A function $f \in \mathcal{A}$ is *close-to-convex* in $\mathcal{U}$ if there is a starlike function ψ and a real number α such that

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{\psi(z)} > 0 \quad (z \in \mathcal{U}). \tag{1.4}$$

The class of all such functions is denoted by $\mathcal{CCV}$. Geometrically, f is close-to-convex if and only if the image of $|z| = r$ has no large hairpin turns; that is, there is no sections of the curve $f(C_r)$ in which the tangent vector turns backward through an angle greater than π. Starlike functions are evidently close-to-convex.

Another subclass of $\mathcal{S}$ is the class of quasi-convex functions. A function $f \in \mathcal{A}$ is said to be *quasi-convex* in $\mathcal{U}$ if there is a function ϕ in $\mathcal{CV}$ such that

$$\operatorname{Re} \frac{(zf'(z))'}{\phi'(z)} > 0 \quad (z \in \mathcal{U}).$$

This set of functions denoted by $\mathcal{QCV}$ was introduced by Noor and Thomas [129]. Note that $\mathcal{CV} \subset \mathcal{QCV}$ where $\phi(z) \equiv f(z)$. Every close-to-convex function is univalent. This can be inferred from the following simple but important criterion for univalence proved by Noshiro [130] and Warschawski [207].

Theorem 1.6 (Noshiro-Warschawski Theorem) [61, p. 47] *If f is analytic* *in a convex domain D and* $\operatorname{Re} f'(z) > 0$ *there, then f is univalent in D.*

The subclasses of $\mathcal{S}$ consisting of starlike, convex and close-to-convex functions satisfy the following chain:

$$\mathcal{CV} \subset \mathcal{ST} \subset \mathcal{CCV} \subset \mathcal{S}.$$

There are many criteria for functions to be univalent. In 1949, Nehari [123] obtained univalence criterion which involves the Schwarzian derivative. Let $S(f,z)$ denote the *Schwarzian derivative* of a locally univalent analytic function f defined by

$$S(f,z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1.5}$$

A *Möbius transformation* M is defined by

$$M(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0). \tag{1.6}$$

The function M is univalent on the closed complex plane containing the point at ∞. A function of the form (1.6) always maps "circles" onto "circles" where a "circle" means a straight line or a circle [61, p. 10]. It can be shown that the Schwarzian derivative is invariant under Möbius transformations, that is, $S(M \circ f, z) = S(f,z)$. Also, the Schwarzian derivative of an analytic function f is identically zero if and only if it is a Möbius transformation [48, p. 259].

The following univalence criterion was given by Nehari.

Theorem 1.7 [123] *If $f \in \mathcal{S}$, then*

$$|S(f,z)| \leq \frac{6}{(1-|z|^2)^2}. \tag{1.7}$$

Conversely, if an analytic function f in $\mathcal{U}$ satisfies

$$|S(f,z)| \le \frac{2}{(1-|z|^2)^2}, \tag{1.8}$$

then f is univalent in $\mathcal{U}$. The results are sharp.

The preceding result was first proved by Kraus [85] but had been forgotten for a long time. Nehari re-discovered and proved Theorem 1.7. The Koebe function satisfies (1.7) and shows that the constant 6 is sharp. Also, the function

$$L(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) \tag{1.9}$$

which maps $\mathcal{U}$ univalently onto the parallel strip $|\operatorname{Im} w| < \pi/2$ satisfies (1.8) and shows that the constant 2 is sharp. Nehari [125] also showed that inequality (1.8) holds if f is convex and this result is sharp for the function L defined by (1.9).

By considering two particular positive functions, Nehari [123] obtained a bound on the Schwarzian derivative that ensures univalence of an analytic function in $\mathcal{A}$. In fact, the following theorem was proved.

Theorem 1.8 [123, Theorem II, p. 549] *If $f \in \mathcal{A}$ satisfies*

$$|S(f,z)| \le \frac{\pi^2}{2} \quad (z \in \mathcal{U}),$$

then $f \in \mathcal{S}$. The result is sharp for the function f given by $f(z) = (\exp(i\pi z) - 1)/i\pi$.

The problem of finding similar bounds on the Schwarzian derivatives that would imply univalence, starlikeness or convexity of functions was investigated by a number of authors including Gabriel [55], Friedland and Nehari [54], and

Ozaki and Nunokawa [139]. Chiang [41] investigated convexity of functions f by requiring the Schwarzian derivative $S(f,z)$ and the second coefficient a_2 of f to satisfy certain inequalities. In Chapter 3, it is assumed that the second coefficient of an analytic function f is small enough and that the Schwarzian derivative $S(f,z)$ satisfies a certain inequality. Under these assumptions, it is shown that f is univalent. Also, similar conditions on the second coefficient of f and its Schwarzian derivative $S(f,z)$ are obtained that would ensure the function f belongs to particular subclasses of $\mathcal{S}$.

Various subclasses of $\mathcal{ST}$ and $\mathcal{CV}$ were later introduced that possess certain geometric features. Goodman [62] introduced the class of uniformly convex functions $\mathcal{UCV}$. Geometrically, a function $f \in \mathcal{S}$ is *uniformly convex* if it maps every circular arc γ contained in $\mathcal{U}$ with center $\zeta \in \mathcal{U}$ onto a convex arc. Goodman [62] gave a two-variable analytic characterization for the class $\mathcal{UCV}$, that is,

$$\mathcal{UCV} := \left\{ f \in \mathcal{S} : \ \operatorname{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \ \zeta, z \in \mathcal{U} \right\},$$

while Rønning [167], and Ma and Minda [103] independently gave a one-variable characterization for $f \in \mathcal{UCV}$ by using the minimum principle for harmonic functions:

$$f \in \mathcal{UCV} \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (z \in \mathcal{U}). \tag{1.10}$$

For $0 \le \alpha < 1$, let Ω_α be the parabolic region in the right-half plane defined by

$$\Omega_\alpha = \{w = u + iv : \ v^2 < 4(1-\alpha)(u-\alpha)\} = \{w : \ |w-1| < 1 - 2\alpha + \operatorname{Re} w\}.$$

The class $\mathcal{PST}(\alpha)$ of *parabolic starlike functions of order* α is the subclass of $\mathcal{A}$ consisting of functions f such that $zf'(z)/f(z) \in \Omega_\alpha$, $z \in \mathcal{U}$. Thus $f \in \mathcal{PST}(\alpha)$

if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\alpha + \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U}). \tag{1.11}$$

The class $\mathcal{PST}$, called *parabolic starlike functions*, was introduced by Rønning [167]. Analytically, $f \in \mathcal{PST}$ if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \quad (z \in \mathcal{U}).$$

Rønning [167] also showed that

$$f \in \mathcal{UCV} \Leftrightarrow zf' \in \mathcal{PST}(1/2) = \mathcal{PST}.$$

Closely related is the class $\mathcal{UST}$ of uniformly starlike functions introduced by Goodman [63]. A function $f \in \mathcal{S}$ is *uniformly starlike* if it maps every circular arc γ contained in $\mathcal{U}$ with center $\zeta \in \mathcal{U}$ onto a starlike domain with respect to $f(\zeta)$. A two-variable analytic characterization of the class $\mathcal{UST}$ is given by

$$\mathcal{UST} := \left\{ f \in \mathcal{S} : \operatorname{Re}\left(\frac{(z-\zeta)f'(z)}{f(z) - f(\zeta)}\right) > 0,\ \zeta, z \in \mathcal{U} \right\}. \tag{1.12}$$

Goodman [62] showed that the classical Alexander relation (Theorem 1.5) does not hold between $\mathcal{UST}$ and $\mathcal{UCV}$. Such a question between $\mathcal{UST}$ and $\mathcal{UCV}$ is in fact equivalent to $\mathcal{UST} = \mathcal{PST}$, and it was shown in [62, 168] that there is no inclusion between $\mathcal{UST}$ and $\mathcal{PST}$:

$$\mathcal{UST} \not\subset \mathcal{PST}, \quad \mathcal{PST} \not\subset \mathcal{UST}.$$

Several authors have studied the above classes, amongst which include the works of [62, 102–104, 165, 179]; surveys on the classes $\mathcal{UCV}$, $\mathcal{UST}$ and $\mathcal{PST}$ can be found in [14] by Ali and Ravichandran, and in [166] by Rønning.

The class of meromorphic functions is yet another subclass of univalent functions that will be discussed in the thesis. Let Σ denote the class of normalized *meromorphic* functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \tag{1.13}$$

that are analytic in the punctured unit disk $\mathcal{U}^* := \{z : 0 < |z| < 1\}$ except for a simple pole at 0. In 1914, Gronwall [65] proved the following Area Theorem.

Theorem 1.9 (Area Theorem) *If f is univalent function of the form*

$$f(\xi) = \xi + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{\xi^n} \quad (|\xi| > 1), \tag{1.14}$$

then $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$.

The interest of the class Σ arose from an application of the Area Theorem in the proof of the Bieberbach's conjecture for the second coefficient. A function f of the form (1.14) and $g \in \Sigma$ are related by the transformation $f(1/z) = g(z)$.

The transformation

$$f(\xi) = \frac{1}{g(1/\xi)} \quad (|\xi| > 1) \tag{1.15}$$

takes each g in $\mathcal{S}$ into a function f of the form (1.14). By the transformation (1.15), the Koebe function takes a particularly simple form

$$\phi(\xi) = \frac{1}{k(1/\xi)} = \xi - 2 + \frac{1}{\xi}$$

which maps the exterior of unit disk $\{\xi \in \mathbb{C} : 1 < |\xi| < \infty\}$ onto the domain consisting of the entire complex plane minus the slit $-4 \leq w \leq 0$.

A function $f \in \Sigma$ is said to be *starlike* if it is univalent and the complement of $f(\mathcal{U})$ is a starlike domain with respect to the origin where $f(z) \neq 0$ for $z \in \mathcal{U}^*$. Denote by Σ^{st} the class of meromorphically starlike functions. Analytically, it is known that $f \in \Sigma^{st}$ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < 0 \quad (z \in \mathcal{U}^*).$$

Note that $f \in \Sigma^{st}$ implies $f(z) \neq 0$ for $z \in \mathcal{U}^*$. Similarly, a function $f \in \Sigma$ is *convex*, denoted by $f \in \Sigma^{cv}$, if it is univalent and the complement of $f(\mathcal{U})$ is a convex domain. Analytically, $f \in \Sigma^{cv}$ if and only if

$$\operatorname{Re} \frac{(zf'(z))'}{f'(z)} < 0 \quad (z \in \mathcal{U}^*).$$

In general, for $0 \leq \alpha < 1$, the classes of *meromorphic starlike functions of order* α and *meromorphic convex functions of order* α respectively are defined by

$$\Sigma^{st}(\alpha) := \left\{ f \in \Sigma : \operatorname{Re} \frac{zf'(z)}{f(z)} < \alpha \right\},$$
$$\Sigma^{cv}(\alpha) := \left\{ f \in \Sigma : \operatorname{Re} \frac{(zf'(z))'}{f'(z)} < \alpha \right\}.$$

These classes have been studied by several authors [23, 24, 88, 116, 117, 191, 192, 205]. We assembled geometric features and analytic expressions of the well-known subclasses of univalent functions to apply for future convenience.

1.3 Function with Negative Coefficients

The following simple result follows from an application of the Noshiro-Warschawski Theorem (Theorem 1.6).

Theorem 1.10 *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{U}$, *and* $\sum_{n=2}^{\infty} |a_n| \leq 1$. *Then* $f \in \mathcal{S}$.

If $a_n \leq 0$ for all n, then the condition above is also a necessary condition for f to be univalent. In 1961, Merkes *et al.* [105] obtained a sufficient condition for $f \in \mathcal{A}$ to be starlike of order α, which is also necessary in the event $a_n \leq 0$.

Theorem 1.11 [105, Thorem 2, p. 961] *Let* $0 \leq \alpha < 1$, *and* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$. *Then* $f \in \mathcal{ST}(\alpha)$ *if*

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq 1-\alpha. \tag{1.16}$$

If $a_n \leq 0$ *for all* n, *then* (1.16) *is a necessary condition for* $f \in \mathcal{ST}(\alpha)$.

This motivated the investigation of functions whose coefficients are negative. The class of functions with negative coefficients in $\mathcal{A}$, denoted by $\mathcal{T}$, consists of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{1.17}$$

Denote by $\mathcal{TST}(\alpha)$ and $\mathcal{TCV}(\alpha)$ the respective subclasses of functions with negative coefficients in $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$. For starlike and convex functions of order α with negative coefficients, Silverman [182] determined the distortion theorem, covering theorem, and coefficients inequalities and extreme points. Silverman [182] also provided a survey, some open problems, and conjectures on analytic functions with negative coefficients. In 2003, the classes $\mathcal{TST}$ and $\mathcal{TCV}$ were generalized in terms of subordination by Ravichandran [158]. The subordination concept and its applications will be treated in Section 1.8.

As in the case with the Bieberbach's conjecture, there are several easily stated questions related to the class $\mathcal{T}$ that appear difficult to solve. Related works to analytic functions with negative coefficients include [10, 11, 26, 89, 118, 119, 136, 155, 156, 175]. Merkes *et al.* [105] proved Theorem 1.11 based on a method used by Clunie and Keogh [46], which was later applied to obtain sufficient conditions for functions f to be in certain subclasses of analytic functions. For instance, the following lemma is a sufficient coefficient condition for functions $f \in \mathcal{A}$ to satisfy

$$\operatorname{Re}\left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \beta \quad (\alpha \geq 0,\ \beta < 1,\ z \in \mathcal{U}). \tag{1.18}$$

Lemma 1.1 [97] *Let $\beta < 1$, and $\alpha \geq 0$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq 1 - \beta, \tag{1.19}$$

then f satisfies (1.18). *If $a_n \leq 0$ for all n, then* (1.19) *is a necessary condition for functions f to satisfy* (1.18).

Geometric properties of analytic functions satisfying (1.18) will be investigated in Chapters 5. Sălăgean [176] obtained several interesting implications for analytic functions with negative coefficients. Motivated by the investigation of Sălăgean [176], several implications are investigated for functions $f \in \mathcal{A}$ satisfying (1.18). The largest bound β for analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$ are determined that will ensure f to be either starlike or convex of some positive order. For $f \in \mathcal{TST}(\alpha)$, and $f \in \mathcal{TCV}(\alpha)$, the largest value is obtained that bounds each coefficient inequality of the form $\sum n a_n$, $\sum n(n-1)a_n$, $\sum (n-1)a_n$ and $\sum n^2 a_n$. The results obtained will be applied to ensure the hypergeometric functions $zF(a, b; c; z)$ satisfy (1.18). The hypergeometric functions will be treated in Section 1.9.

1.4 Univalent Functions with Fixed Second Coefficient

Certain properties of analytic functions are influenced by their second coefficient. In 1920, Gronwall [66] extended the distortion and growth theorems for an analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with a pre-assigned second coefficient. Corresponding results for convex functions with a pre-assigned second coefficient were also obtained [66].

Let the class $\mathcal{A}_b$ consist of functions $f \in \mathcal{A}$ with a fixed second coefficient a_2 with $|a_2| = 2b$, $0 \leq b \leq 1$. Each $f \in \mathcal{A}_b$ has the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|a_2| = 2b).$$

Let $\mathcal{CV}_b(\alpha)$ denote the class of convex functions of order α and $\mathcal{ST}_b(\alpha)$ denote the class of starlike functions of order α where $f \in \mathcal{A}_b$. Also denote by $\mathcal{ST}_b := \mathcal{ST}_b(0)$ and $\mathcal{CV}_b := \mathcal{CV}_b(0)$ the class of starlike functions and the class of convex functions with $|a_2| = 2b$ respectively. Finkelstein [52] obtained distortion and growth theorems for the classes $\mathcal{ST}_b$ and $\mathcal{CV}_b$. The results obtained in [52] were generalized to the class $\mathcal{ST}_b(\alpha)$ by Tepper [199] and the class $\mathcal{CV}_b(\alpha)$ by Padmanabhan [140]. Later in 2001, Padmanabhan [141] investigated the problem for general classes of functions defined by subordination.

Silverman [181] investigated the influence of the second coefficient on the class of close-to-convex functions. Here, a function $f \in \mathcal{A}_b$ is close-to-convex of order β and type α, denoted by $f \in \mathcal{CCV}_b(\alpha, \beta)$, if there is a function $\psi \in \mathcal{CV}_b(\alpha)$ such that

$$\operatorname{Re} \frac{f'(z)}{\psi'(z)} > \beta \quad (\beta \geq 0).$$

Silverman [181] proved distortion and covering theorems for $f \in \mathcal{CCV}_b(\alpha, \beta)$. The theory of differential subordination for functions $f \in \mathcal{A}_b$ was discussed in [13,122]. Ali *et al.* provided a brief history of these works in [9].

Lewandowski *et al.* [92] proved that an analytic function f satisfying

$$\operatorname{Re} \left(\frac{z^2 f''(z)}{f(z)} + \frac{z f'(z)}{f(z)} \right) > 0 \quad (z \in \mathcal{U}) \tag{1.20}$$

is starlike. The class of such functions was extended to the form (1.18) and has subsequently been investigated by Ramesha *et al.* [157], Nunokawa *et al.* [133], Obradović and Joshi [134], Padmanabhan [142], Ravichandran [160, 162], and Liu *et al.* [97]. For $-\alpha/2 \leq \beta < 1$, Li and Owa [93] proved that functions satisfying (1.18) are starlike. In 2002, the class of analytic functions satisfying

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < \beta \quad (\beta > 1,\ z \in \mathcal{U})$$

was considered by Owa and Nishiwaki [128], while its subclasses were earlier investigated by Uralegaddi *et al.* [204, 206], Owa and Srivastava [138]. Liu *et al.* [96] investigated the class of functions satisfying

$$\operatorname{Re} \left(\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \right) < \beta \quad (\alpha \geq 0,\ \beta > 1,\ z \in \mathcal{U}). \tag{1.21}$$

In Chapter 7, the class of functions satisfying (1.18) and (1.21) will be put in a generl form

$$\mathcal{L}(\alpha, \beta) \cap \mathcal{A}_b := \left\{ f \in \mathcal{A}_b : \alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2\beta)z}{1 - z},\ \beta \in \mathbb{R} \setminus \{1\},\ \alpha \geq 0 \right\}. \tag{1.22}$$

Also, the well-known class of analytic functions introduced by Janowski [73] defined by

$$\mathcal{ST}[A, B] \cap \mathcal{A}_b = \left\{ f \in \mathcal{A}_b : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1 \right\}$$

will be considered. The radius properties for functions $f \in \mathcal{L}(\alpha, \beta) \cap \mathcal{A}_b$ and $f \in \mathcal{ST}[A, B] \cap \mathcal{A}_b$ are investigated in Chapter 7. The radius problems will be treated in the next section.

1.5 Radius Problems

Let $\mathcal{M}$ be a set of functions and $\mathcal{P}$ be a property which functions in $\mathcal{M}$ may or may not possess in a disk $|z| < r$. The least upper bound of all numbers r such that every function $f \in \mathcal{M}$ has the property $\mathcal{P}$ in the disk $\mathcal{U}_r = \{z : |z| < r\}$ is the radius for the property $\mathcal{P}$ in the set $\mathcal{M}$. Every univalent analytic function is univalent, but every univalent function is not always convex. However, every univalent analytic mapping maps a sufficiently small disk into a convex domain. The largest radius of the disk with this property is the radius of convexity. It is known that the radius of convexity for the set $\mathcal{S}$ is $2-\sqrt{3}$ and is attained by the Koebe function [127]. Grunsky [67] proved that the radius of starlikeness for the set $\mathcal{S}$ is $\tanh(\pi/4)$. The radius of close-to-convexity for the set $\mathcal{S}$ was determined by Krzyż [87]. A list of such radius problems was provided by Goodman [61, Chapter 13].

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, de Branges [37] proved the Bieberbach's conjecture that $|a_n| \leq n$ $(n \geq 2)$ (Theorem 1.2). However, the inequality $|a_n| \leq n$ $(n \geq 2)$ does not imply f is univalent; for example, $f(z) = z + 2z^2$ satisfies the coefficient inequality but f is not a member of $\mathcal{S}$ as $f'(-1/4) = 0$. In view of this, it is interesting to investigate the radius of univalence, starlikeness, and other geometric properties of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ when the Taylor coefficients of f satisfy $|a_n| \leq cn + d$ $(n \geq 2)$.

The inequality $|a_n| \leq M$ holds for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|f(z)| \leq M$, and for these functions, Landau [90] proved that the radius of univalence is $M - \sqrt{M^2 - 1}$. For functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying the inequality $|a_n| \leq n$ $(n \geq 2)$, Gavrilov [58] showed that the radius of univalence is the real root $r_0 \approx 0.1648$ of the equation $2(1-r)^3 - (1+r) = 0$, and for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|a_n| \leq M$ $(n \geq 2)$, the radius of univalence is $1 - \sqrt{M/(1+M)}$. Yamashita [209] showed that the radius of univalence obtained by Gavrilov is the radius of starlikeness as well. Indeed, Gavrilov [58, Theorem 1] estimated the radius of univalence to be $0.125 < r_0 < 0.130$, while Yamashita [209] obtained $r_0 \approx 0.1648$. Yamashita also determined the radius of convexity for functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ to be the real root of $(M+1)(1-r)^3 - M(1+r) = 0$.

Recently Kalaj *et al.* [74] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying similar coefficient inequalities.

In [161], Ravichandran obtained the sharp radii of starlikeness and convexity of order α for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$ $(M > 0)$, $n \geq 3$. The radius constants for uniform convexity and parabolic starlikeness for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$, $n \geq 3$ were also obtained. Ravichandran [161] determined the radius of positivity for the real part of the functions $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ satisfying the inequality $|c_n| \leq 2M$ $(M > 0)$, $n \geq 3$ with $|c_2| = 2b$, $0 \leq b \leq 1$.

Let $f = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfy either $|a_n| \leq cn + d$ $(c,\ d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$ for $n \geq 3$. In Chapter 7, sharp $\mathcal{L}(\alpha, \beta)$-radius and sharp $\mathcal{ST}[A, B]$- radius for these classes are obtained. The radius constants obtained by Ravichandran [161] and Yamashita [209] are shown to be special cases of the results obtained in Chapter 7.

1.6 Convolution

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be analytic in the unit disk $\mathcal{U}$. The Hadamard product of f and g is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n \quad (z \in \mathcal{U}).$$

The alternative representation as a convolution integral

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f\left(\frac{z}{\zeta}\right) g(\zeta) \frac{d\zeta}{\zeta} \quad (|z| < \rho < 1),$$

is the reason $f * g$ is also called the convolution of f and g where R_f and R_g are the radii of convergence for f and g respectively [172, p. 11]. Since f and g are analytic in $\mathcal{U}$, $R_f \geq 1$ and $R_g \geq 1$. Thus,

$$\frac{1}{R_{f*g}} = \limsup |a_n b_n|^{\frac{1}{n}} \leq \left(\limsup |a_n|^{\frac{1}{n}}\right) \left(\limsup |b_n|^{\frac{1}{n}}\right) = \frac{1}{R_f} \frac{1}{R_g} \leq 1,$$

where R_{f*g} is the radius of convergence for $f * g$. Hence $f * g$ is analytic in $|z| < R_f R_g$. Mandelbrojt and Schiffer [150] conjectured univalence is preserved under integral convolution; namely if $f, g \in \mathcal{S}$, then

$$G(z) = \int_0^z \frac{(f * g)(t)}{t} dt \in \mathcal{S}.$$

Epstein and Schöenberg [50], Hayman [70], and Loewner and Netanyahu [100] proved counterexamples to the Mandelbrojt and Schiffer conjecture. In 1958, Pólya and Schöenberg [150] conjectured that

$$\mathcal{CV} * \mathcal{CV} \subset \mathcal{CV}.$$

Suffridge [195] proved that the convolution of every pair of convex functions is close-to-convex. In 1973, the Polya and Schöenberg's conjecture was proved by Ruscheweyh and Sheil-Small [173]. They also proved that the class of starlike functions and close-to-convex functions are closed under convolution with convex functions. However, it turns out that the class of univalent functions is not closed under convolution. In fact, $\mathcal{ST} * \mathcal{ST}$ is not even contained in the family $\mathcal{S}$. For example, let $f = g = k \in \mathcal{ST}$, where k is the Koebe function. Then $f * g \notin \mathcal{S}$

because $a_n = n^2 > n$. Further details about related works can be found in [48].

A subclass of analytic functions considered by Ruscheweyh [172] known as prestarlike functions was applied to the basic convolution results.

For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of *prestarlike functions of order* α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} :\ f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$. In particular,

$$f \in \mathcal{R}_{1/2} \Leftrightarrow \operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{1}{2} \quad (z \in \mathcal{U}),$$

$$f \in \mathcal{R}_0 \Leftrightarrow \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > 0 \quad (z \in \mathcal{U}). \tag{1.23}$$

Therefore, $\mathcal{R}_{1/2} = \mathcal{ST}_{1/2}$ and $\mathcal{R}_0 = \mathcal{CV}$. It is a known result [172] that the classes of starlike functions of order α and convex functions of order α are closed under convolution with prestarlike functions of order α. Prestarlike functions have a number of interesting geometric properties. Ruscheweyh [172] and Sheil-Small [180] investigated the significance of prestarlike functions. The results and techniques of Ruscheweyh and Sheil-Small developed in [173] in connection with their proof of the Polya-Schöenberg conjecture have been applied in many convolution articles. The convex hull method is based on the following convolution result for prestarlike and starlike functions.

Theorem 1.12 [172, Theorem 2.4] *Let $\alpha \leq 1$, $\phi \in \mathcal{R}_\alpha$ and $f \in \mathcal{ST}(\alpha)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\operatorname{co}}(H(\mathcal{U})),$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$, where $\overline{\operatorname{co}}(H(\mathcal{U}))$ denotes the closed convex *hull of $H(\mathcal{U})$.*

In Chapter 2, the classical subclasses of starlike, convex, close-to-convex and quasi-convex functions are extended to new subclasses of analytic functions. Using the method of convex hull and the theory of differential subordinations discussed later in Section 1.8, convolution properties of these newly defined subclasses of analytic functions are investigated. It is shown that these classes are closed under convolution with prestarlike functions. Also, new subclasses for meromorphic functions are similarly introduced, and the convolution features of these subclasses are investigated. It is proved that these classes are also closed under convolution with prestarlike functions. It is shown that the Bernardi-Libera-Livingston integral operator preserve all these subclasses of analytic and meromorphic functions. It would be evident that various earlier works, for example those of [3, 35, 44, 120, 148, 159], are special instances of the results obtained.

1.7 Dual Set and Duality for Convolution

Let $\mathcal{A}_0$ be the set of all functions $f \in \mathcal{H}(\mathcal{U})$ satisfying $f(0) = 1$. For $V \subset \mathcal{A}_0$, define the dual set

$$V^* := \big\{f \in \mathcal{A}_0 : \; (f * g)(z) \neq 0 \text{ for all } g \in V, z \in \mathcal{U}\big\}.$$

The second dual V^{**} is defined as $V^{**} = (V^*)^*$. It is of interest to investigate the relations between V and V^{**}. In general, V^{**} is much bigger than V, but many properties of V remain valid in V^{**}. Let Λ be the set of continuous linear functionals on $\mathcal{H}(\mathcal{U})$ and $\lambda(V) := \{\lambda(f) : f \in V\}$. In 1975, Ruscheweyh [170] proved the following fundamental result, known as the *Duality Principle*.

Theorem 1.13 (Duality Principle) [170] *Let $V \subset \mathcal{A}_0$ have the following properties:*

(1) V is compact,

(2) $f \in V$ implies $f(xz) \in V$ for all $|x| \leq 1$.

Then $\lambda(V) = \lambda(V^{**})$ for all $\lambda \in \Lambda$ on $\mathcal{A}$, and $\overline{\mathrm{co}}(V) = \overline{\mathrm{co}}(V^{**})$.

The Duality Principle has numerous applications to the class of functions possessing certain geometric properties like bounded real part, convexity, starlikeness, close-to-convexity and univalence. The monograph of Ruscheweyh [172], and also the paper [170] in which many of the results of this topic were first published have become basic references for duality theory. As an application of Duality Principle, the following corollary was shown by Ruscheweyh [172]. The result is false with V^{**} replaced by $\overline{\mathrm{co}}(V)$.

Corollary 1.1 [172, Corollary 1.1. p. 17] *Let $V \subset \mathcal{A}_0$ satisfy the conditions in Theorem 1.13. Let $\lambda_1, \lambda_2 \in \Lambda$ with $0 \notin \lambda_2(V)$. Then for any $f \in V^{**}$ there exists a function $g \in V$ such that*

$$\frac{\lambda_1(f)}{\lambda_2(f)} = \frac{\lambda_1(g)}{\lambda_2(g)}.$$

Ruscheweyh determined a big class of sets in $\mathcal{A}_0$ in which the above result was applicable.

Theorem 1.14 [170, Theorem 1, p. 68] *If*

$$V_\beta = \left\{(1-\beta)\frac{1+xz}{1+yz} + \beta : \ |x| = |y| = 1, \ \beta \in \mathbb{R}, \ \beta \neq 1\right\},$$

then

$$V_\beta^{*} = \left\{f \in A_0 : \mathrm{Re}\, f(z) > \frac{1-2\beta}{2(1-\beta)}\right\},$$

and

$$V_\beta^{**} = \left\{f \in A_0 : \exists \phi \in \mathbb{R} \text{ with } \mathrm{Re}\, e^{i\phi}(f(z) - \beta) > 0, \ z \in \mathcal{U}\right\}.$$

Singh and Singh [187] proved the Bernardi integral operator

$$F_c(z) = (c+1)\int_0^1 t^{c-1} f(tz)dt \quad (c > -1)$$

is starlike for $-1 < c \leq 0$, where $\operatorname{Re} f'(z) > 0$ in $\mathcal{U}$. In 1986, Mocanu proved that

$$\operatorname{Re} f'(z) > 0 \Rightarrow F_1 \in \mathcal{ST},$$

and the result was later improved by Nunokawa [131]. Singh and Singh [186] also proved

$$\operatorname{Re} f'(z) > -\frac{1}{4} \Rightarrow F_0(z) \in \mathcal{ST}.$$

Such problems were earlier handled using the theory of subordination which will be discussed in Section 1.8. In 1975, Fournier and Ruscheweyh [53] used the Duality Principle [172] to find the sharp bound for β such that $F_c(\mathcal{P}(\beta)) \subset \mathcal{ST}$ where $\mathcal{P}(\beta)$ is given by

$$\mathcal{P}(\beta) := \Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi}\left(f'(z) - \beta\right) > 0, \quad z \in \mathcal{U} \Big\}, \tag{1.24}$$

and $-1 < c \leq 2$.

Indeed, Fournier and Ruscheweyh [53] investigated starlikeness properties of a general operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt. \tag{1.25}$$

over functions f in the class $\mathcal{P}(\beta)$ given by (1.24), where λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. Ali and Singh [21] found a sharp estimate of the parameter β that ensures $V_\lambda(f)$ is convex over $\mathcal{P}(\beta)$.

The duality theory of convolutions developed by Ruscheweyh [172] is now popularly used by several authors to discuss similar problems, among which include the works of [27, 27–31, 45, 47, 83, 152–154]. As a consequence of these works, several interesting results on integral operators for special choices of λ were derived. A survey on integral transforms in geometric function theory was provided by Kim [81]. Integral operators will be treated again in Section 1.9.

The class $\mathcal{W}_\beta(\alpha, \gamma)$ defined by

$$\begin{aligned}\mathcal{W}_\beta(\alpha,\gamma) := \Big\{ & f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with} \\ & \operatorname{Re} e^{i\phi} \left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta \right) > 0,\ z \in \mathcal{U} \Big\},\end{aligned} \tag{1.26}$$

for $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$ was recently introduced by Ali *et al.* [12]. Ali *et al.* [7] investigated the starlikeness of integral transform (4.1) over the class $\mathcal{W}_\beta(\alpha, \gamma)$ by applying the Duality Principle.

In Chapter 4, the Duality Principle is used to determine the best value of $\beta < 1$ that ensures the integral operator $V_\lambda(f)$ in (1.25) maps the class $\mathcal{W}_\beta(\alpha, \gamma)$ defined in (1.26) into the class of convex functions. Simple necessary and sufficient condition for $V_\lambda(f)$ to be convex are obtained. For specific choices of the admissible function λ, several applications are investigated. As an important consequence, it is shown that a function f satisfying the third-order differential equation

$$\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \right) > \beta$$

is convex in $\mathcal{U}$ where $\beta > -0.629445$. Also, the smallest value of $\beta < 1$ is obtained such that the generalized integral operator of the form $\rho z + (1-\rho)V_\lambda(f)$, $\rho < 1$, over the class of $\mathcal{W}_\beta(\alpha, \gamma)$ is starlike. Corresponding result for $\rho z + (1-\rho)V_\lambda(f)$,

$\rho < 1$, to be convex is also derived.

1.8 Differential Subordination

In this section, the basic definitions and theorems in the theory of subordination and certain applications of differential subordinations are described. A function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, if there exists a Schwarz function w, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. If g is univalent in $\mathcal{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. The following concepts and terminologies were introduced by Miller and Mocanu in [111].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$, and h be univalent in $\mathcal{U}$. If an analytic function p satisfies the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.27}$$

then p is called a *solution* of the differential subordination. The univalent function q is called a *dominant of the solution* of the differential subordination, or more simply, *dominant*, if $p(z) \prec q(z)$ for all p satisfying (1.27). A dominant q_1 satisfying $q_1(z) \prec q(z)$ for all dominants q of (1.27) is said to be the *best dominant* of (1.27). The best dominant is unique up to a rotation of $\mathcal{U}$. Miller and Mocano provided a comprehensive discussion on differential subordination in [111].

Let $\psi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$, and $h(z)$ be analytic in $\mathcal{U}$. Let p and $\psi(p(z), zp'(z), z^2p''(z); z)$ be univalent in $\mathcal{U}$. If p satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.28}$$

then p is called a *solution* of the differential superordination. An analytic function q is called a *subordinant* of the solution of the differential superordination, or more simply, *subordinant*, if $q(z) \prec p(z)$ for all p satisfying (1.28). A univalent subordinant q_1 satisfying $q(z) \prec q_1(z)$ for all subordinants q of (1.28) is said to be the *best subordinant* of (1.28). The best subordinant is unique up to a rotation of $\mathcal{U}$. Further discussion on the differential superordination can be found in Miller and Mocano [112].

In 1935, the simple first-order differential subordination

$$zp'(z) \prec h(z)$$

was considered by Goluzin [60]. He showed that if h is convex, then

$$p(z) \prec q(z) = \int_0^z h(t)t^{-1}dt,$$

and this q is the best dominant. Suffridge [196, p. 777] proved that Goluzin's result even holds if h is starlike.

In 1947, Robinson [164, p. 22] considered the differential subordination

$$p(z) + zp'(z) \prec h(z).$$

He proved that if h and $q(z) = z^{-1}\int_0^z h(t)dt$ are univalent, then q is the best dominant, at least for $|z| < 1/5$. The differential subordination

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z) \quad (\gamma \neq 0, \operatorname{Re}\gamma \geq 0),$$

was considered by Hallenbeck and Rusheweyh [69, p. 192] in 1975. They showed that if h is convex, then

$$p(z) \prec q(z) = \gamma z^{-\gamma} \int_0^z h(t)t^{\gamma-1}dt,$$

and this is the best dominant.

In 1978, Miller and Mocanu [109] reformulated the above mentioned results by considering the differential subordination implication

$$\{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathcal{U}\} \subset \Omega \Rightarrow p(\mathcal{U}) \subset q(\mathcal{U}),$$

where $\Omega \subset \mathbb{C}$, $p \in \mathcal{H}(\mathcal{U})$, $q \in \mathcal{S}$, and $\psi(r, s, t; z)$ an analytic function from $\mathbb{C}^3 \times \mathcal{U}$ to $\mathbb{C}$.

Denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathcal{U}} \backslash E(q)$, where

$$E(q) = \{\zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty\}$$

and $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \backslash E(q)$. Let $\mathcal{Q}(a)$ be the subclass of $\mathcal{Q}$ for which $q(0) = a$. Further, let $\mathcal{Q}_0 := \mathcal{Q}(0)$ and $\mathcal{Q}_1 := \mathcal{Q}(1)$.

The subordination methodology is applied to an appropriate class of admissible functions. The following class of admissible functions was given by Miller and Mocanu [111].

Definition 1.1 [111, Definition 2.3a, p. 27] *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and m be a positive integer. The class of admissible functions $\Psi_m[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever $r = q(\zeta), s = k\zeta q'(\zeta)$ and*

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathcal{U}, \zeta \in \partial\mathcal{U} \backslash E(q)$ and $k \geq m$. Denote by $\Psi[\Omega, q] := \Psi_1[\Omega, q]$.

The next theorem is the foundation result in the theory of first and second-order differential subordinations.

Theorem 1.15 [111, Theorem 2.3b, p. 28] *Let* $\psi \in \Psi_m[\Omega, q]$ *with* $q(0) = a$. *If* $p(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots$ *satisfies*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) \in \Omega,$$

then $p(z) \prec q(z)$.

Analogous to the case of subordination, a suitable class of admissible functions is required so that the differential superordination implication

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) : z \in \mathcal{U}\} \Rightarrow \Delta \subset p(\mathcal{U}), \tag{1.29}$$

holds. In 2003, Miller and Mocanu [112] considered the following class of admissible functions related to differential superordination.

Definition 1.2 [112, Definition 3, p. 817] *Let* Ω *be a set in* $\mathbb{C}$ *and* $q(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots$ *with* $q'(z) \neq 0$. *The class of admissible functions* $\Psi'_m[\Omega, q]$ *consists of functions* $\psi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$ *satisfying the admissibility condition* $\psi(r, s, t; \zeta) \in \Omega$ *whenever* $r = q(z), s = zq'(z)/k$ *and*

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{k} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathcal{U}, \zeta \in \partial\mathcal{U}$ *and* $k \geq m \geq 1$. *Denote by* $\Psi'[\Omega, q] := \Psi'_1[\Omega, q]$.

Theorem 1.16 [112, Theorem 1, p. 818] *Let* $\psi \in \Psi'_m[\Omega, q]$ *with* $q(0) = a$. *If* $p \in \mathcal{Q}(a)$ *and* $\psi\big(p(z), zp'(z), z^2p''(z); z\big)$ *is univalent in* $\mathcal{U}$, *then*

$$\Omega \subset \{\psi\big(p(z), zp'(z), z^2p''(z); z\big), z \in \mathcal{U}\}$$

implies $q(z) \prec p(z)$.

The monograph by Miller and Mocanu [111] gives a good introduction to the theory of differential subordination, while the book by Bulboaca [39] investigates both subordination and superordination. In recent years, the theory of differential subordination was applied by various authors to deal with many important problems in the field. Works in this direction include those of [8, 16–20, 22, 43, 208].

In Chapter 6, subordination and superordination results for general linear operators satisfying a certain first-order differential recurrence relation are investigated. By considering an appropriate class of admissible functions, sufficient conditions for the best dominant and the best subordinant are determined. Several sandwich-type results are obtained. As application of the results obtained, various examples of differential inequalities and subordinations are presented. Additionally, the admissibility conditions for functions with positive real part are studied. The results obtained in Chapter 6 unify earlier works in this direction including those of [8, 16–19, 19, 20, 121].

1.9 Linear Operators

In the theory of univalent functions, a variety of linear and nonlinear integral operators have been considered. The Alexander operator [4] is defined by

$$A(f)(z) = \int_0^z \frac{f(t)}{t} dt,$$

while the Libera operator [94] is defined by

$$L(f)(z) = \frac{2}{z} \int_0^z f(t) dt.$$

The operator given by

$$L_\gamma(f)(z) = \frac{1+\gamma}{z^\gamma}\int_0^z f(t)t^{\gamma-1}dt \quad (\gamma > -1), \tag{1.30}$$

is called the generalized Bernardi-Libera-Livingston integral operator [98].

One interesting problem in geometric function theory is to determine operators from $\mathcal{A}$ to $\mathcal{A}$ which map a given $C \subset \mathcal{A}$ into itself; in particular, the operators that preserve the geometric properties of the domain. An example of such results are those proved by Pascu [145], and Lewandowski *et al.* [92] as follows.

Theorem 1.17 [92, 145] *If $L_\gamma : \mathcal{A} \to \mathcal{A}$ is the integral operator defined by* (1.30) *and* $\operatorname{Re}\gamma \geq 0$*, then*

(1) $L_\gamma[\mathcal{ST}] \subset \mathcal{ST}$,

(2) $L_\gamma[\mathcal{CV}] \subset \mathcal{CV}$,

(3) $L_\gamma[\mathcal{CCV}] \subset \mathcal{CCV}$.

In fact, there is a vast number of articles dealing with integral operators between classes of analytic functions. Further general operators were studied in the survey articles by Miller and Mocanu [110] and by Srivastava [189], where a long list of other references can be found.

For $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} z^n = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots,$$

where the Pochhammer symbol $(\lambda)_n$ is given by $(\lambda)_n = \lambda(\lambda+1)_{n-1},\ (\lambda)_0 = 1$. The series converges absolutely in $\mathcal{U}$. It also converges on $|z| = 1$ when $\operatorname{Re}(c-a-b) > 0$.

For $\mathrm{Re}(c-a-b) > 0$, the value of the hypergeometric function $F(a,b;c;z)$ at $z = 1$ is related to the Gamma function by the Gauss summation formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, \cdots).$$

The theory of hypergeometric functions developed with many applications and generalizations after their use by de Branges [37] in the proof of the Bieberbach's conjecture. Hohlov [79] applied hypergeometric functions to define a generalized integral operator by

$$H_{a,b,c}(f)(z) := zF(a,b;c;z) * f(z).$$

Carlson and Shaffer [40] studied this operator in the special case $a = 1$. The Hohlov operator reduces to the well-known operators as follows: $A(f)(z) := H_{1,1,2}(f)(z)$, $L(f)(z) := H_{1,2,3}(f)(z)$, and $L_\gamma(f)(z) := H_{1,\gamma+1,\gamma+2}(f)(z)$.

Let $\lambda : [0,1] \to \mathbb{R}$ be a nonnegative function with

$$\int_0^1 \lambda(t)dt = 1.$$

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [53] introduced the general operator V_λ given by (1.25). Various known integral operators are of the form (1.25) for specific choices of λ. For example,

$$\lambda(t) := (1+c)t^c \quad (c > -1),$$

gives the Bernardi integral operator, while the choice

$$\lambda(t) := \frac{(a+1)^p}{\Gamma(p)} t^a (\log \frac{1}{t})^{p-1} \quad (a > -1,\ p \geq 0),$$

yields the Komatu operator [84]. Clearly for $p = 1$ the Komatu operator is the Bernardi operator. For a certain choice of λ given by

$$\lambda(t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)} t^{b-1}(1-t)^{c-a-b} F\left(c-a, 1-a; c-a-b+1; 1-t\right),$$

the integral operator V_λ in (4.1) is the convolution between a function f and the Gaussian hypergeometric function $F(a, b; c; z)$, which is related to the general Hohlov operator.

In [21], Ali and Singh generalized the operator (1.25) to the case

$$\mathcal{V}_\lambda(f)(z) := \rho z + (1-\rho)V_\lambda(f) = z \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * f(z) \quad (\rho < 1).$$

The order of starlikeness and convexity of the operator $\mathcal{V}_\lambda(f)$ where $f \in \mathcal{P}(\beta)$ defined by (1.24) were obtained by Ali and Singh [21].

In Chapter 4, convexity properties of the integral operator $V_\lambda(f)$ is investigated over the class $\mathcal{W}_\beta(\alpha, \gamma)$ defined by (1.26). As an application of the results obtained, convexity of various known integral operators for different choices of λ is investigated. Also, sufficient conditions are obtained that would ensure the integral operator $\mathcal{V}_\lambda(f)$ over the class $\mathcal{W}_\beta(\alpha, \gamma)$ is starlike or convex. The results obtained extended and improved earlier works by several authors.

Linear operators on the class of normalized analytic functions satisfying a certain first-order differential recurrence relation are yet another object that will be discussed in this thesis. Let $\alpha_k \in \mathbb{C}$ $(k = 1, 2, \cdots, l)$ and $\beta_k \in \mathbb{C}\backslash\mathbb{Z}_0^- := \{0, -1, -2, \cdots\}$ $(k = 1, 2, \cdots, j)$. The generalized hypergeometric function ${}_lF_j(\alpha_1,$

$\cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)$ is defined by

$$ {}_lF_j(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_j)_n} \frac{z^n}{n!} \quad (l \leq j+1). \tag{1.31} $$

Note that the series in (1.31) converges absolutely for $|z| < \infty$ if $l < j+1$, and for $z \in \mathcal{U}$ if $l = j+1$. Let

$$ w = \sum_{k=1}^{m} \beta_k - \sum_{k=1}^{l} \alpha_k. $$

The series in (1.31), with $l = j+1$, is absolutely convergent for $|z| = 1$ if $\operatorname{Re} w > 0$, and conditionally convergent for $|z| = 1$ if $-1 < \operatorname{Re} w \leq 0$ [193, p. 34].

Let

$$ h_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) := z^m \; {}_lF_j(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z). $$

The convolution operator $H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) : \mathcal{A}_m \to \mathcal{A}_m$ was introduced by Dziok and Srivastava [49, 190] by means of Hadamard product involving the generalized hypergeometric functions

$$ \begin{aligned} H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) f(z) &:= h_m(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) * f(z) \\ &= z^m + \sum_{k=m+1}^{\infty} \frac{(\alpha_1)_{k-m} \cdots (\alpha_l)_{k-m}}{(\beta_1)_{k-m} \cdots (\beta_j)_{k-m}} \frac{a_k z^k}{(k-m)!}. \end{aligned} \tag{1.32} $$

It is known [49] that

$$ \begin{aligned} \alpha_1 H_m^{l,j}(\alpha_1 + 1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) f(z) &= z[H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) f(z)]' \\ &\quad + (\alpha_1 - m) H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z) f(z). \end{aligned} \tag{1.33} $$

Special cases of the Dziok-Srivastava linear operator include the Hohlov linear operator [72], the Carlson-Shaffer linear operator [40], the Ruscheweyh derivative operator [169], the generalized Bernardi-Libera-Livingston linear integral operator [34, 94, 98], and the Srivastava-Owa fractional derivative operator [135, 137].

The multiplier transformation $I_p(n, \lambda)$ on $\mathcal{A}_m$ defined by

$$I_m(n, \lambda)f(z) := z^m + \sum_{k=m+1}^{\infty} \left(\frac{k+\lambda}{m+\lambda}\right)^n a_k z^k \tag{1.34}$$

satisfies

$$(m+\lambda)I_m(n+1, \lambda)f(z) = z[I_m(n, \lambda)f(z)]' + \lambda I_m(n, \lambda)f(z). \tag{1.35}$$

Certain important properties of these linear operators depend on the differential recurrence relation, for example as that given by (1.33) and (1.35). In Chapter 6, a general class consisting of such operators satisfying a certain recurrence relation is introduced. Differential subordination and superordination results for multivalent functions defined by the general linear operator are obtained. Chapter 6 aims to show that the class of functions defined through each of the general linear operators can be given a unified treatment. The unified operator includes as special cases the Dziok-Srivastava operator defined in (1.32) and multiplier transform defined by (1.34) as well as several other operators introduced by various authors [17–19].

1.10 Scope of the Thesis

This thesis will discuss six research problems. In Chapter 2, general classes of analytic functions defined by convolution with a fixed analytic function are introduced. Convolution properties of these classes which include the classical classes of starlike, convex, close-to-convex, and quasi-convex analytic functions are investigated. These classes are shown to be closed under convolution with prestarlike functions and the Bernardi-Libera integral operator. Similar results are also obtained for the classes consisting of meromorphic functions in the punctured unit disk.

In Chapter 3, a normalized analytic function f is shown to be univalent in the open unit disk $\mathcal{U}$ if its second coefficient is sufficiently small and relates to its Schwarzian derivative through a certain inequality. New criteria for analytic functions to be in certain subclasses of functions are established in terms of the Schwarzian derivatives and the second coefficients. These include obtaining a sufficient condition for functions to be strongly α-Bazilevič of order β.

In Chapter 4, for λ satisfying a certain admissibility criteria, sufficient conditions are obtained for the integral transform

$$V_\lambda(f)(z) := \int_0^1 \lambda(t)\frac{f(tz)}{t}dt$$

to map normalized analytic functions f satisfying

$$\operatorname{Re} e^{i\phi}\left((1-\alpha+2\gamma)\frac{f(z)}{z}+(\alpha-2\gamma)f'(z)+\gamma zf''(z)-\beta\right)>0$$

into the class of convex functions. Several interesting applications for different choices of λ are discussed. In particular, the smallest value $\beta<1$ is obtained that ensures a function f satisfying $\operatorname{Re}\left(f'(z)+\alpha zf''(z)+\gamma z^2f'''(z)\right)>\beta$ is convex.

The aim of Chapter 5 is to determine a sharp bound on β so that an analytic function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ satisfying the inequality $\sum_{n=2}^{\infty}n(n-1)|a_n|\leq\beta$ is either starlike or convex of order α. Several other coefficient inequalities related to certain subclasses are also investigated.

In Chapter 6, a general class consisting of the operators satisfying a certain first-order differential recurrence relation is introduced. For any operator in this class, certain second-order differential subordination and superordination implications are investigated on analytic functions generated by the operator. Several sandwich-type results are also obtained. The results obtained unify earlier works.

In the final chapter, radius properties of analytic functions $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ with fixed second coefficient are investigated, where a_2 satisfies $|a_2| = 2b$, $0 \leq b \leq 1$. Sharp radius of Janowski starlikeness and radius constant of $\mathcal{L}(\alpha, \beta)$ are obtained when $|a_n| \leq cn + d$ $(c, d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$ for $n \geq 3$.

CHAPTER 2

CONVOLUTION OF ANALYTIC AND MEROMORPHIC FUNCTIONS

2.1 Introduction and Definitions

The theory of differential subordination has been applied in numerous areas of univalent function theory. Ma and Minda [102] used differential subordination to give a unified presentation of various subclasses of starlike and convex analytic functions.

For $0 \leq \alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ be subclasses of $\mathcal{A}$ consisting respectively of starlike functions of order α and convex functions of order α. These functions are defined analytically by

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\},$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

Let h be an analytic function in $\mathcal{U}$ with positive real part, $h(0) = 1$, and $h'(0) > 0$. Further let h map the unit disk $\mathcal{U}$ onto a region starlike with respect to 1. Ma and Minda [102] introduced the classes

$$\begin{aligned} \mathcal{ST}(h) &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec h(z) \right\}, \\ \mathcal{CV}(h) &= \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec h(z) \right\}, \end{aligned} \tag{2.1}$$

and obtained growth, distortion and covering theorems. For $h(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$, $\mathcal{ST}(h)$ reduces to the familiar class $\mathcal{ST}(\alpha)$ of starlike functions and $\mathcal{CV}(h)$ to the class $\mathcal{CV}(\alpha)$ of convex functions.

Let g be a fixed function in $\mathcal{A}$ and h be a convex function with positive real part in $\mathcal{U}$ normalized by $h(0) = 1$. By using the convex hull method and differential subordination, Shanmugam [178] introduced and investigated the following classes.

$$\begin{aligned}
\mathcal{ST}(g,h) &:= \left\{ f \in \mathcal{A} : \ \frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z) \right\}, \\
\mathcal{CV}(g,h) &:= \left\{ f \in \mathcal{A} : \ 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} \prec h(z) \right\}, \\
\mathcal{CCV}(g,h) &:= \left\{ f \in \mathcal{A} : \ \frac{z(g*f)'(z)}{(g*\psi)(z)} \prec h(z), \psi \in \mathcal{ST}(g,h) \right\}, \\
\mathcal{QCV}(g,h) &:= \left\{ f \in \mathcal{A} : \ \frac{z(g*f)'(z)}{(g*\phi)(z)} \prec h(z), \phi \in \mathcal{CV}(g,h) \right\}.
\end{aligned}$$

Shanmugam [178] showed that the classes of $\mathcal{ST}(g,h)$, $\mathcal{CV}(g,h)$, $\mathcal{CCV}(g,h)$, and $\mathcal{QCV}(g,h)$ are closed under convolution with convex functions. In particular, these classes reduce to various subclasses of $\mathcal{A}$ for specific choices of g and h. For instance, for $g(z) = z/(1-z)$, the classes $\mathcal{ST}(g,h)$ and $\mathcal{CV}(g,h)$ reduce to the classes $\mathcal{ST}(h)$ and $\mathcal{CV}(h)$ defined by (2.1). If $g = k_a$ where

$$k_a(z) := \frac{z}{(1-z)^a} \quad (a > 0), \tag{2.2}$$

then $\mathcal{ST}_a(h) := \mathcal{ST}(k_a,h)$, $\mathcal{CV}_a(h) := \mathcal{CV}(k_a,h)$ and $\mathcal{CCV}_a(h) := \mathcal{CCV}(k_a,h)$. Padmanabhan and Parvatham [144] introduced the classes $\mathcal{ST}_a(h)$, $\mathcal{CV}_a(h)$, and $\mathcal{CCV}_a(h)$ and investigated convolution properties of these classes. Some other related studies were also made in [3, 15, 35, 44, 120, 121, 148, 159, 198].

Let f, $g \in \mathcal{A}$ satisfy

$$\operatorname{Re} \frac{zf'(z)}{f(z)+g(z)} > 0, \quad \operatorname{Re} \frac{zg'(z)}{f(z)+g(z)} > 0.$$

By adding the two inequalities, it is seen that the function $(f(z)+g(z))/2$ is starlike and hence both f and g are close-to-convex and hence univalent. This result motivates us to consider the following classes of functions.

It is assumed in the sequel that $m \geq 1$ is a fixed integer, g a fixed function in $\mathcal{A}$, and h a convex univalent function with positive real part in $\mathcal{U}$ satisfying $h(0) = 1$.

Definition 2.1 *The class $\mathcal{ST}_m(h)$ consists of $\hat{f} := \langle f_1, f_2, \cdots, f_m \rangle$, $f_k \in \mathcal{A}$, $1 \leq k \leq m$, satisfying $\sum_{j=1}^m f_j(z)/z \neq 0$ in $\mathcal{U}$ and the subordination*

$$\frac{mzf_k'(z)}{\sum_{j=1}^m f_j(z)} \prec h(z) \quad (k = 1, \cdots, m). \tag{2.3}$$

The class $\mathcal{ST}_m(g, h)$ consists of $\hat{f}$ for which $\hat{f} * g := \langle f_1 * g, f_2 * g, \cdots, f_m * g \rangle \in \mathcal{ST}_m(h)$. The class $\mathcal{CV}_m(h)$ consists of $\hat{f}$ for which $z\hat{f}' \in \mathcal{ST}_m(h)$ where $\hat{f}' := \langle f_1', f_2', \cdots, f_m' \rangle$ and $z\hat{f}' := \langle zf_1', zf_2', \cdots, zf_m' \rangle$. Equivalently, $\hat{f} \in \mathcal{CV}_m(h)$ if $\hat{f}$ satisfies the condition $\sum_{j=1}^m f_j'(z) \neq 0$ in $\mathcal{U}$ and the subordination

$$\frac{m(zf_k')'(z)}{\sum_{j=1}^m f_j'(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

The class $\mathcal{CV}_m(g, h)$ consists of $\hat{f}$ for which $\hat{f} * g \in \mathcal{CV}_m(h)$.

Now let $\hat{f} \in \mathcal{ST}_m(h)$ and $F(z) = (1/m) \sum_{j=1}^m f_j(z)$. From (2.3), it follows that

$$\frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}) \quad (k = 1, \cdots, m).$$

The convexity of $h(\mathcal{U})$ implies that

$$\frac{1}{m} \frac{z \sum_{k=1}^m f_k'(z)}{F(z)} \in h(\mathcal{U}), \tag{2.4}$$

which shows that the function F is starlike in $\mathcal{U}$. Thus, it follows from (2.3) that the component function f_k of $\hat{f}$ is close-to-convex in $\mathcal{U}$, and hence univalent. Similarly, the component function f_k of $\hat{f} \in \mathcal{CV}_m(h)$ is also univalent.

If $m = 1$, then the classes $\mathcal{ST}_m(g,h)$ and $\mathcal{CV}_m(g,h)$ reduced respectively to $\mathcal{ST}(g,h)$ and $\mathcal{CV}(g,h)$ introduced and investigated by Shanmugam [178]. If $g = k_a$ where k_a is given by (2.2), then the class $\mathcal{ST}_m(g,h)$ coincides with the class $\mathcal{ST}_a(h)$ studied by Padmanabhan and Parvatham [144], and $\mathcal{CV}_m(g,h)$ reduces to the class $\mathcal{CV}_a(h)$ introduced by Padmanabhan and Manjini [143]. It is evident that the classes $\mathcal{ST}_m(g,h)$ and $\mathcal{CV}_m(g,h)$ extend the classical classes of starlike and convex functions respectively.

Definition 2.2 *The class $\mathcal{CCV}_m(h)$ consists of $\hat{f} := \langle f_1, f_2, \cdots, f_m\rangle$, $f_k \in \mathcal{A}$, $1 \le k \le m$, satisfying the subordination*

$$\frac{mzf_k'(z)}{\sum_{j=1}^m \psi_j(z)} \prec h(z) \quad (k = 1, \cdots, m),$$

for some $\hat{\psi} \in \mathcal{ST}_m(h)$. In this case, we say that $\hat{f} \in \mathcal{CCV}_m(h)$ with respect to $\hat{\psi} \in \mathcal{ST}_m(h)$. The class $\mathcal{CCV}_m(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g := \langle f_1 * g, f_2 * g, \cdots, f_m * g\rangle \in \mathcal{CCV}_m(h)$. The class $\mathcal{QCV}_m(h)$ consists of $\hat{f}$ for which $z\hat{f}' \in \mathcal{CCV}_m(h)$ or equivalently satisfying the subordination

$$\frac{m(zf_k')'(z)}{\sum_{j=1}^m \varphi_j'(z)} \prec h(z) \quad (k = 1, \cdots, m),$$

for some $\hat{\varphi} \in \mathcal{CV}_m(h)$ with $z\hat{\varphi}' = \hat{\psi}$, $\hat{\psi} \in \mathcal{ST}_m(h)$. In this case, we say that $\hat{f} \in \mathcal{QCV}_m(h)$ with respect to $\hat{\varphi} \in \mathcal{CV}_m(h)$. The class $\mathcal{QCV}_m(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g \in \mathcal{QCV}_m(h)$.

When $m = 1$, the class $\mathcal{CCV}_m(g,h)$ and $\mathcal{QCV}_m(g,h)$ reduces respectively to $\mathcal{C}_g(h)$ and $\mathcal{Q}_g(h)$ introduced and investigated by Shanmugam [178]. If $g = k_a$ where k_a is defined by (2.2), then the class $\mathcal{CCV}_m(g,h)$ coincides with $\mathcal{CCV}_a(h)$ studied by Padmanabhan and Parvatham [144]. The classes $\mathcal{CCV}_m(g,h)$ and $\mathcal{QCV}_m(g,h)$ extend the classical classes of close-to-convex and quasi convex functions respectively.

The class of prestarlike functions considered by Ruscheweyh [172] has seen many applications in convolution results. For $\alpha < 1$, the class $\mathcal{R}_\alpha$ of prestarlike functions of order α is defined by

$$\mathcal{R}_\alpha := \left\{ f \in \mathcal{A} : \ f * \frac{z}{(1-z)^{2-2\alpha}} \in \mathcal{S}^*(\alpha) \right\},$$

while $\mathcal{R}_1$ consists of $f \in \mathcal{A}$ satisfying $\operatorname{Re} f(z)/z > 1/2$.

The well-known result that the classes of starlike functions of order α and convex functions of order α are closed under convolution with prestarlike functions of order α follows from the following:

Theorem 2.1 (Theorem 1.12) *Let $\alpha \leq 1$, $\phi \in \mathcal{R}_\alpha$ and $f \in \mathcal{ST}(\alpha)$. Then*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\text{co}}(H(\mathcal{U}))$$

for any analytic function $H \in \mathcal{H}(\mathcal{U})$, where $\overline{\text{co}}(H(\mathcal{U}))$ denotes the closed convex hull of $H(\mathcal{U})$.

The following basic theorem on prestarlike functions is required.

Theorem 2.2 [172, Theorem 2.1, p. 49]

(1) Let $\alpha \leq 1$. If $f,\ g \in \mathcal{R}_\alpha$, then $f * g \in \mathcal{R}_\alpha$.

(2) If $\alpha < \beta \leq 1$, then $\mathcal{R}_\alpha \subset \mathcal{R}_\beta$.

Ruscheweyh [169] showed the following result for a special convex function which has an interesting convolution property.

Lemma 2.1 [169, Theorem 5, p. 113] *Let $\operatorname{Re}\gamma \geq 0$, and h_γ be defined by*

$$h_\gamma(z) = z + \sum_{n=2}^{\infty} \frac{\gamma+1}{\gamma+n} z^n.$$

If $f \in \mathcal{CV}$, then $f * h_\gamma \in \mathcal{CV}$. In particular, $h_\gamma \in \mathcal{CV}$.

In the following section, by using the methods of convex hull and differential subordination, convolution properties of functions belonging to the four classes $\mathcal{ST}_m(g,h)$, $\mathcal{CV}_m(g,h)$, $\mathcal{CCV}_m(g,h)$ and $\mathcal{QCV}_m(g,h)$ are investigated. It is seen that various earlier works, for example [3], [35], [44], [120], [148] and [159], are special instances of this study.

2.2 Convolution of Analytic Functions

The first result shows that the classes $\mathcal{ST}_m(g,h)$ and $\mathcal{CV}_m(g,h)$ are closed under convolution with prestarlike functions.

Theorem 2.3 *Let $m \geq 1$ be a fixed integer and g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$, $h(0) = 1$ *and* $\phi \in \mathcal{R}_\alpha$.

(1) If $\hat{f} \in \mathcal{ST}_m(g,h)$, then $\hat{f} * \phi \in \mathcal{ST}_m(g,h)$.

(2) If $\hat{f} \in \mathcal{CV}_m(g,h)$, then $\hat{f} * \phi \in \mathcal{CV}_m(g,h)$.

Proof. (1) It is sufficient to prove that $\hat{f} * \phi \in \mathcal{ST}_m(h)$ whenever $\hat{f} \in \mathcal{ST}_m(h)$. Once this is established, the general result for $\hat{f} \in \mathcal{ST}_m(g,h)$ follows from the fact that

$$\hat{f} \in \mathcal{ST}_m(g,h) \Leftrightarrow \hat{f} * g \in \mathcal{ST}_m(h).$$

For $k = 1, 2, \cdots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z), \quad H_k(z) = \frac{z f_k'(z)}{F(z)}.$$

It will first be proved that F belongs to $\mathcal{ST}(\alpha)$. If $\hat{f} \in \mathcal{ST}_m(h)$ and $z \in \mathcal{U}$, then by (2.4), F satisfies

$$\frac{zF'(z)}{F(z)} \prec h(z). \tag{2.5}$$

Since $\operatorname{Re} h(z) > \alpha$, the subordination (2.5) yields

$$\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) > \alpha, \tag{2.6}$$

and hence $F \in \mathcal{ST}(\alpha)$.

A computation shows that

$$\begin{aligned}
\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * f_j)(z)} &= \frac{(\phi * zf_k')(z)}{(\phi * \frac{1}{m}\sum_{j=1}^{m} f_j)(z)} \\
&= \frac{(\phi * \dfrac{zf_k'}{F}F)(z)}{(\phi * \frac{1}{m}\sum_{j=1}^{m} f_j)(z)} \\
&= \frac{(\phi * H_kF)(z)}{(\phi * F)(z)}.
\end{aligned}$$

Since $\phi \in R_\alpha$ and $F \in \mathcal{ST}(\alpha)$, Theorem 2.1 yields

$$\frac{(\phi * H_kF)(z)}{(\phi * F)(z)} \in \overline{\mathrm{co}}(H_k(\mathcal{U})),$$

and because $H_k(z) \prec h(z)$, we deduce that

$$\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * f_j)(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

Thus $\hat{f} * \phi \in \mathcal{ST}_m(h)$.

(2) The function $\hat{f}$ is in $\mathcal{CV}_m(g,h)$ if and only if $z\hat{f}'$ is in $\mathcal{ST}_m(g,h)$ and by the first part above, it follows that $\phi * z\hat{f}' = z(\phi * \hat{f})' \in \mathcal{ST}_m(g,h)$. Hence $\phi * \hat{f} \in \mathcal{CV}_m(g,h)$. ∎

Remark 2.1 *For $\phi \in \mathcal{R}_\alpha$, the above theorem can be expressed in the following* *equivalent forms:*

$$\mathcal{ST}_m(g,h) \subset \mathcal{ST}_m(\phi * g, h), \quad \mathcal{CV}_m(g,h) \subset \mathcal{CV}_m(\phi * g, h).$$

When $m = 1$, various known results are obtained as special cases of Theorem 2.3. For instance, Theorem 3.3 and Corollary 3.1 in [178, p. 336] are easily deduced from Theorem 2.3 as follows:

Corollary 2.1 *Let g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > 0$, $h(0) = 1$ *and* $\phi \in \mathcal{CV}$.

(1) If $f \in \mathcal{ST}(g,h)$, then $\phi * f \in \mathcal{ST}(g,h)$.

(2) If $f \in \mathcal{CV}(g,h)$, then $\phi * f \in \mathcal{CV}(g,h)$.

Proof. From (1.23) in page 19, it follows that $\mathcal{CV} = \mathcal{R}_0$. By Theorem 2.2 (2), for $0 \leq \alpha < 1$, $\mathcal{R}_0 \subset \mathcal{R}_\alpha$, and hence $\mathcal{CV} \subset \mathcal{R}_\alpha$. Then part (1) is deduced from Theorem 2.3 (1), while part (2) follows from Theorem 2.3 (2) when $m = 1$. ∎

For $g(z) = k_a$ where k_a is defined by (2.2), then Theorem 2.3 reduces to Theorem 4 and Corollary 4.1 in [143, pp. 110-111].

Corollary 2.2 *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > 0$, $h(0) = 1$, *and* $\phi \in \mathcal{CV}$.

(1) If $f \in \mathcal{ST}_a(h)$, then $\phi * f \in \mathcal{ST}_a(h)$.

(2) If $f \in \mathcal{CV}_a(h)$, then $\phi * f \in \mathcal{CV}_a(h)$.

Proof. If $g(z) = k_a$ where k_a is defined by (2.2), then part (1) follows from Theorem 2.3 (1), and part (2) follows from Theorem 2.3 (2) when $m = 1$. ∎

Corollary 2.3 *Let $m \geq 1$ be a fixed integer and g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$. *Define*

$$F_k(z) = \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f_k(t)dt \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1, \cdots, m). \tag{2.7}$$

If $\hat{f} \in \mathcal{ST}_m(g,h)$, then $\hat{F} = \langle F_1, \cdots, F_m\rangle \in \mathcal{ST}_m(g,h)$. Similarly, if $\hat{f} \in \mathcal{CV}_m(g,h)$, then $\hat{F} \in \mathcal{CV}_m(g,h)$.

Proof. Define the function ϕ by

$$\phi(z) = h_\gamma(z) = z + \sum_{n=2}^{\infty} \frac{\gamma+1}{\gamma+n} z^n.$$

For $\operatorname{Re}\gamma \geq 0$, by Lemma 2.1, the function ϕ is a convex function. Since $\mathcal{CV} = \mathcal{R}_0$ by (1.23), it follows that $\phi \in \mathcal{R}_0$. Theorem 2.2 (2) shows that $\mathcal{R}_0 \subset \mathcal{R}_\alpha$ for $0 \leq \alpha < 1$, and hence $\phi \in \mathcal{R}_\alpha$. It is evident from the definition of F_k that

$$\begin{aligned} F_k(z) &= \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1} f_k(t)dt \\ &= \frac{\gamma+1}{z^\gamma}\int_0^z t^{\gamma-1}\left(t + \sum_{n=2}^{\infty} a_n t^n\right)dt \\ &= z + \sum_{n=2}^{\infty} \frac{\gamma+1}{n+\gamma} a_n z^n \\ &= f_k(z) * \left(z + \sum_{n=2}^{\infty}\frac{\gamma+1}{\gamma+n} z^n\right) \\ &= (f_k * \phi)(z) \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1, \cdots, m), \end{aligned}$$

so that $\hat{F} = \hat{f} * \phi$. By Theorem 2.3 (1), it follows that $\hat{F} = \hat{f} * \phi \in \mathcal{ST}_m(g,h)$.

The second result is proved in a similar manner. ∎

Corollary 2.4 [144, Theorem 2, p. 324] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) > 0$, $h(0) = 1$. *If $f \in \mathcal{ST}_a(h)$, then $F \in \mathcal{ST}_a(h)$ where F is given by* (2.7).

Proof. With $g(z) = k_a(z)$ defined by (2.2), desired result follows from Corollary 2.3 when $m = 1$. ∎

Theorem 2.4 *Let $m \geq 1$ be a fixed integer and g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$, and $\phi \in \mathcal{R}_\alpha$.*

(1) If $\hat{f} \in \mathcal{CCV}_m(g,h)$ with respect to $\hat{\psi} \in \mathcal{ST}_m(g,h)$, then $\hat{f} * \phi \in \mathcal{CCV}_m(g,h)$ with respect to $\hat{\psi} * \phi \in \mathcal{ST}_m(g,h)$.

(2) If $\hat{f} \in \mathcal{QCV}_m(g,h)$ with respect to $\hat{\varphi} \in \mathcal{CV}_m(g,h)$, then $\hat{f} * \phi \in \mathcal{QCV}_m(g,h)$ with respect to $\hat{\varphi} * \phi \in \mathcal{CV}_m(g,h)$.

Proof. (1) In view of the fact that

$$\hat{f} \in \mathcal{CCV}_m(g,h) \Leftrightarrow \hat{f} * g \in \mathcal{CCV}_m(h),$$

we shall only prove that $\hat{f} * \phi \in \mathcal{CCV}_m(h)$ when $\hat{f} \in \mathcal{CCV}_m(h)$. Let $\hat{f} \in \mathcal{CCV}_m(h)$. For $k = 1, 2, \cdots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m} \sum_{j=1}^{m} \psi_j(z), \quad H_k(z) = \frac{z f_k'(z)}{F(z)}.$$

Since $\hat{\psi} \in \mathcal{ST}_m(h)$, it is evident from (2.6) that $F \in \mathcal{ST}(\alpha)$.

That $\hat{\psi} * \phi \in \mathcal{ST}_m(h)$ follows from Theorem 2.3 (1). Now, a computation shows that

$$\begin{aligned} \frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * \psi_j)(z)} &= \frac{(\phi * z f_k')(z)}{(\phi * \frac{1}{m}\sum_{j=1}^{m}\psi_j)(z)} \\ &= \frac{(\phi * H_k F)(z)}{(\phi * F)(z)}. \end{aligned}$$

Since $\phi \in \mathcal{R}_\alpha$ and $F \in \mathcal{ST}(\alpha)$, Theorem 2.1 yields

$$\frac{(\phi * H_k F)(z)}{(\phi * F)(z)} \in \overline{\mathrm{co}}(H_k(\mathcal{U})),$$

and because $H_k(z) \prec h(z)$, it follows that

$$\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * \psi_j)(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

Thus $\hat{f} * \phi \in \mathcal{CCV}_m(h)$.

(2) The function $\hat{f}$ is in $\mathcal{QCV}_m(g, h)$ if and only if $z\hat{f}'$ is in $\mathcal{CCV}_m(g, h)$ and by the first part, clearly $\phi * z\hat{f}' = z(\phi * \hat{f})' \in \mathcal{CCV}_m(g, h)$. Hence $\phi * \hat{f} \in \mathcal{QCV}_m(g, h)$. ∎

Again when $m = 1$, known results are obtained as special cases of Theorem 2.4. For instance, [178, Theorem 3.5, p. 337] follows from Theorem 2.4 (1), and [178, Theorem 3.9, p. 339] is a special case of Theorem 2.4 (2). Indeed, the following result was proved by Shanmugam [178].

Corollary 2.5 *Let g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) > 0$, $h(0) = 1$ and $\phi \in \mathcal{CV}$.*

(1) If $\hat{f} \in \mathcal{CCV}(g, h)$ with respect to $\hat{\psi} \in \mathcal{ST}(g, h)$, then $\hat{f} * \phi \in \mathcal{CCV}(g, h)$ with respect to $\hat{\psi} * \phi \in \mathcal{ST}g, h)$.

(2) If $\hat{f} \in \mathcal{QCV}(g, h)$ with respect to $\hat{\varphi} \in \mathcal{CV}(g, h)$, then $\hat{f} * \phi \in \mathcal{QCV}(g, h)$ with respect to $\hat{\varphi} * \phi \in \mathcal{CV}(g, h)$.

Proof. In view of the fact that $\mathcal{CV} = \mathcal{R}_0$, and by Theorem 2.2 (2), for $0 \leq \alpha < 1$, $\phi \in \mathcal{R}_0 \subset \mathcal{R}_\alpha$. When $m = 1$, the first part follows from Theorem 2.4 (1), and the second part is a special case of Theorem 2.4 (2). ∎

As an application of Theorem 2.4, the following result shows that the class $\mathcal{CCV}_m(g, h)$ is closed under Bernardi-Libera-Livingston integral transform.

Corollary 2.6 *Let $m \geq 1$ be a fixed integer and g be a fixed function in $\mathcal{A}$. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) > \alpha$, $0 \leq \alpha < 1$. Let F_k*

be the Bernardi-Libera-Livingston integral transform of f_k *defined by* (2.7)*. If* $\hat{f} \in \mathcal{CCV}_m(g,h)$*, then* $\hat{F} = \langle F_1, \cdots, F_m \rangle \in \mathcal{CCV}_m(g,h)$*.*

The proof is similar to the proof of Corollary 2.3, and is therefore omitted.

Corollary 2.7 [144, Theorem 4, p. 326] *Let* h *be a convex univalent function satisfying* $\operatorname{Re} h(z) > 0$*,* $h(0) = 1$*. Let* F *be the Bernardi-Libera-Livingston integral transform of* f *defined by* (2.7)*. If* $f \in \mathcal{CCV}_a(h)$*, then* $F \in \mathcal{CCV}_a(h)$*.*

Proof. Choose $g(z) = k_a(z)$ defined by (2.2) in Corollary 2.6. ∎

2.3 Convolution of Meromorphic Functions

Let Σ denote the class of functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n \tag{2.8}$$

that are analytic in the punctured unit disk $\mathcal{U}^* := \{z : 0 < |z| < 1\}$. The convolution of two meromorphic functions f and g, where f is given by (2.8) and $g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n$, is given by

$$(f * g)(z) := \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n.$$

In this section, several subclasses of meromorphic functions in the punctured unit disk are introduced by means of convolution with a given fixed meromorphic function. Convolution properties of these newly defined subclasses will be investigated. Simple consequences of the results obtained will include the work of Bharati and Rajagopal [35] involving the function $k_a(z) := 1/(z(1-z)^a)$, $a > 0$, as well as the work of Al-Oboudi and Al-Zkeri [3] on the modified Salagean operator.

Motivated by the investigation of Shanmugam [178], Ravichandran [159], and Ali *et al.* [15], several subclasses of meromorphic functions defined my means of the differential subordinations were introduced by Mohd *et al.* [120]. For instance, the familiar classes of meromorphic starlike and convex functions and other related subclasses of meromorphic functions can be put in the form

$$\Sigma^{st}(g,h) := \left\{ f \in \Sigma : \ -\frac{z(f*g)'(z)}{(f*g)(z)} \prec h(z) \right\},$$
$$\Sigma^{cv}(g,h) := \left\{ f \in \Sigma : \ -\frac{(z(f*g)')'(z)}{(f*g)(z)} \prec h(z) \right\},$$

where g is a fixed function in Σ and h is a suitably normalized analytic function with positive real part. The class of meromorphic starlike functions of order α, $0 \leq \alpha < 1$, defined by

$$\Sigma^{st} := \left\{ f \in \Sigma : \ -\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha \right\}$$

is a particular case of $\Sigma^{st}(g,h)$ with $g(z) = 1/z(1-z)$ and $h(z) = (1+(1-2\alpha)z)/(1-z)$.

Let

$$p_\mu(z) := \frac{1}{z(1-z)^\mu}, \quad q_{\beta,\lambda}(z) := \frac{1}{z} + \sum_{k=0}^{\infty}\left(\frac{\lambda}{k+1+\lambda}\right)^\beta z^k. \tag{2.9}$$

Now for $\beta \geq 0$, $\lambda > 0$, $\mu > 0$, define the linear operator $\mathcal{I}^\beta_{\lambda,\mu} : \Sigma \to \Sigma$ by

$$\mathcal{I}^\beta_{\lambda,\mu} f(z) = (f * p_\mu * q_{\beta,\lambda})(z),$$

where f is given by (2.8). The functions g_1, g_2, $\cdots$, g_m are in the class $\Sigma^\beta_{\lambda,\mu}(m,h)$ if they satisfy the subordination condition

$$-\frac{z(\mathcal{I}^\beta_{\lambda,\mu} g_k(z))'}{\frac{1}{m}\sum_{j=1}^m \mathcal{I}^\beta_{\lambda,\mu} g_j(z)} \prec h(z) \quad (\ z \in \mathcal{U}^*, \ k = 1, 2, \cdots, m). \tag{2.10}$$

The class $\Sigma^{\beta}_{\lambda,\mu}(m,h)$ was investigated by Piejko and Sokół [148].

Al-Oboudi generalized the known Sălăgean operator and defined the operator $D^n_\lambda f$ for meromorphic functions as follows;

$$D^0_\lambda f(z) = f(z),\ D^1_\lambda f(z) = (1-\lambda)f(z) + \lambda\frac{(z^2 f(z))'}{z}, \lambda \geq 0, \text{ and}$$
$$D^n_\lambda f(z) = D_\lambda D^{n-1}_\lambda f(z) = (1-\lambda)D^{n-1}_\lambda f(z) + \lambda\frac{(z^2 D^{n-1}_\lambda f(z))'}{z} \quad (n \in \mathbb{N}).$$

Al-Oboudi and Al-Zkeri [3] applied this operator and introduced the subclasses $\Sigma(m,n,\lambda,h)$ and $Q(m,n,\lambda,h)$ of Σ. The function $f = \{f_1, f_2, \cdots, f_m\}$, $f_i \in \Sigma$, $1 \leq i \leq m$ is in the class $\Sigma(m,n,\lambda,h)$ if

$$-\frac{z(D^n_\lambda f_i(z))'}{\frac{1}{m}\sum_{j=1}^m D^n_\lambda f_j(z)} \prec h(z) \quad (z \in \mathcal{U}^*, i = 1, \cdots, m) \tag{2.11}$$

where h is convex univalent in $\mathcal{U}^*$ with $h(0) = 1$. Let $Q(m,n,\lambda,h)$ denote the class of functions $f \in \Sigma$ such that

$$-\frac{z(D^n_\lambda f(z))'}{\frac{1}{m}\sum_{j=1}^m D^n_\lambda g_j(z)} \prec h(z) \quad (z \in \mathcal{U}^*) \tag{2.12}$$

where $g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m,n,\lambda,h)$.

The class $\Sigma(m,a,h)$ consists of functions $g = \{g_1, g_2, \cdots, g_m\}$, $g_i \in \Sigma$, $1 \leq i \leq m$ satisfying

$$-\frac{z(k_a * g_i)'(z)}{\frac{1}{m}\sum_{j=1}^m (k_a * g_j)(z)} \prec h(z) \quad (z \in \mathcal{U}^*, i = 1, 2, \cdots, m) \tag{2.13}$$

where $\sum_{j=1}^m (k_a * g_j)(z) \neq 0$ in $\mathcal{U}^*$, h is convex univalent in $\mathcal{U}^*$ with $h(0) = 1$, and k_a is given by

$$k_a(z) := \frac{1}{z(1-z)^a} \quad (a > 0). \tag{2.14}$$

Let $C^*(m,a,h)$ denote the class of functions $f \in \Sigma$ such that

$$-\frac{z(k_a * f)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(k_a * g_j)(z)} \prec h(z) \quad (z \in \mathcal{U}^*, i = 1, 2, \cdots, m) \tag{2.15}$$

where $g = \{g_1, g_2, \cdots, g_m\} \in \Sigma(m,a,h)$. The classes $\Sigma(m,a,h)$ and $C^*(m,a,h)$ were studied by Bharati and Rajagopal [35].

Here four classes $\Sigma_m^{st}(g,h)$, $\Sigma_m^{cv}(g,h)$, $\Sigma_m^{ccv}(g,h)$ and $\Sigma_m^{qcv}(g,h)$ of meromorphic functions are introduced. As before, it is assumed that $m \geq 1$ is a fixed integer, g a fixed function in Σ, and h a convex univalent function with positive real part in $\mathcal{U}$ satisfying $h(0) = 1$.

Definition 2.3 *The class $\Sigma_m^{st}(h)$ consists of $\hat{f} := \langle f_1, f_2, \cdots, f_m\rangle$, $f_k \in \Sigma$, $1 \leq k \leq m$, satisfying $\sum_{j=1}^{m} f_j(z) \neq 0$ in $\mathcal{U}^*$ and the subordination*

$$-\frac{mzf_k'(z)}{\sum_{j=1}^{m} f_j(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

The class $\Sigma_m^{st}(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g := \langle f_1 * g, f_2 * g, \cdots, f_m * g\rangle \in \Sigma_m^{st}(h)$. The class $\Sigma_m^{cv}(h)$ consists of $\hat{f}$ for which $-z\hat{f}' \in \Sigma_m^{st}(h)$ or equivalently satisfying the condition $\sum_{j=1}^{m} f_j'(z) \neq 0$ in $\mathcal{U}^*$ and the subordination

$$-\frac{m(zf_k')'(z)}{\sum_{j=1}^{m} f_j'(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

The class $\Sigma_m^{cv}(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g \in \Sigma_m^{cv}(h)$.

Various subclasses of meromorphic functions investigated in earlier works are special instances of the above defined classes. For instance, if $g(z) := 1/z(1-z)$, then $\Sigma_m^{st}(g,h)$ coincides with $\Sigma_m^{st}(h)$. By putting $g = p_\mu * q_{\beta,\lambda}$ where p_μ and $q_{\beta,\lambda}$ are given by (2.9), the class $\Sigma_m^{st}(g,h)$ reduces to the class $\Sigma_{\lambda,\mu}^{\beta}(m,h)$ given by

(2.10) investigated in [148]. If $g = k_n$ where

$$k_n(z) := \frac{1}{z} + \sum_{k=1}^{\infty}[1+\lambda(k+1)]^n z^k, \tag{2.16}$$

then the class of $\Sigma_m^{st}(g,h)$ is the class $\Sigma(m,n,\lambda,h)$ given by (2.11) studied in [3]. If $g = k_a$ is given by (2.14), then the class $\Sigma_m^{st}(g,h)$ coincides with $\Sigma(m,a,h)$ given by (2.13) investigated in [35].

Definition 2.4 *The class $\Sigma_m^{ccv}(h)$ consists of $\hat{f} := \langle f_1, f_2, \cdots, f_m\rangle$, $f_k \in \Sigma$, $1 \leq k \leq m$, satisfying the subordination*

$$-\frac{mzf_k'(z)}{\sum_{j=1}^{m}\psi_j(z)} \prec h(z) \quad (k = 1, \cdots, m),$$

for some $\hat{\psi} \in \Sigma_m^{st}(h)$. In this case, we say that $\hat{f} \in \Sigma_m^{ccv}(h)$ with respect to $\hat{\psi} \in \Sigma_m^{st}(h)$. The class $\Sigma_m^{ccv}(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g := \langle f_1 * g, f_2 * g, \cdots, f_m * g\rangle \in \Sigma_m^{ccv}(h)$. The class $\Sigma_m^{qcv}(h)$ consists of $\hat{f}$ for which $-z\hat{f}' \in \Sigma_m^{ccv}(h)$ or equivalently satisfying the subordination

$$-\frac{m(zf_k')'(z)}{\sum_{j=1}^{m}\varphi_j'(z)} \prec h(z) \quad (k = 1, \cdots, m),$$

for some $\hat{\varphi} \in \mathcal{CV}_m(h)$ with $-z\hat{\varphi}' = \hat{\psi}$ and $\hat{\psi} \in \mathcal{ST}_m(h)$. The class $\Sigma_m^{qcv}(g,h)$ consists of $\hat{f}$ for which $\hat{f} * g \in \Sigma_m^{qcv}(h)$.

If $g(z) := 1/z(1-z)$, then $\Sigma_m^{ccv}(g,h)$ coincides with $\Sigma_m^{ccv}(h)$. If $g(z) = k_n(z)$ is defined by (2.16), then $\Sigma_m^{ccv}(g,h)$ reduces to $Q(m,n,\lambda,h)$ defined by (2.12) and investigated in [3]. If $g(z) = k_a(z)$ is defined by (2.14), then the class $\Sigma_m^{ccv}(g,h)$ is the class $C^*(m,a,h)$ defined by (2.15) and studied in [35].

The following modification of Theorem 2.1 is required in the sequel.

Theorem 2.5 *Let $\alpha \leq 1$, $f, \phi \in \Sigma$, $z^2\phi \in \mathcal{R}_\alpha$ and $z^2 f \in \mathcal{S}^*(\alpha)$. Then, for any analytic function $H \in \mathcal{H}(\mathcal{U})$,*

$$\frac{\phi * (Hf)}{\phi * f}(\mathcal{U}) \subset \overline{\mathrm{co}}(H(\mathcal{U})).$$

Theorem 2.6 *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$, $h(0) = 1$, and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{R}_\alpha$.*

(1) If $\hat{f} \in \Sigma_m^{st}(g, h)$, then $\hat{f} * \phi \in \Sigma_m^{st}(g, h)$.

(2) If $\hat{f} \in \Sigma_m^{cv}(g, h)$, then $\hat{f} * \phi \in \Sigma_m^{cv}(g, h)$.

Proof. (1) Since

$$\hat{f} \in \Sigma_m^{st}(g, h) \Leftrightarrow \hat{f} * g \in \Sigma_m^{st}(h),$$

it suffices to prove the result for $g(z) = 1/z(1-z)$. For $k = 1, 2, \cdots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} f_j(z), \quad H_k(z) = -\frac{zf_k'(z)}{F(z)}.$$

We show that F satisfies the condition $z^2 F \in \mathcal{ST}(\alpha)$. For $\hat{f} \in \Sigma_m^{st}(h)$ and $z \in \mathcal{U}$, clearly

$$H_k(z) = -\frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}) \quad (k = 1, \cdots, m).$$

Since $h(\mathcal{U})$ is a convex domain, it follows that

$$-\frac{1}{m}\sum_{k=1}^{m} \frac{zf_k'(z)}{F(z)} \in h(\mathcal{U}),$$

or

$$-\frac{zF'(z)}{F(z)} \prec h(z). \tag{2.17}$$

Since $\operatorname{Re} h(z) < 2-\alpha$, the subordination (2.17) yields

$$-\operatorname{Re}\left(\frac{zF'(z)}{F(z)}\right) < 2-\alpha,$$

and thus

$$\operatorname{Re}\left(\frac{z(z^2F)'(z)}{z^2F(z)}\right) = \operatorname{Re}\frac{zF'(z)}{F(z)} + 2 > \alpha. \tag{2.18}$$

Inequality (2.18) shows that $z^2F \in \mathcal{ST}(\alpha)$.

A routine computation now gives

$$\begin{aligned} -\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * f_j)(z)} &= \frac{(\phi * (-zf_k'))(z)}{(\phi * \frac{1}{m}\sum_{j=1}^{m} f_j)(z)} \\ &= \frac{(\phi * H_kF)(z)}{(\phi * F)(z)}. \end{aligned}$$

Since $z^2\phi \in R_\alpha$ and $z^2F \in \mathcal{ST}(\alpha)$, Theorem 2.5 yields

$$\frac{(\phi * H_kF)(z)}{(\phi * F)(z)} \in \overline{\operatorname{co}}(H_k(\mathcal{U})),$$

and because $H_k(z) \prec h(z)$, it is clear that

$$-\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * f_j)(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

Thus $\hat{f} * \phi \in \Sigma_m^{st}(h)$.

(2) The function $\hat{f}$ is in $\Sigma_m^{cv}(g,h)$ if and only if $-z\hat{f}'$ is in $\Sigma_m^{st}(g,h)$ and the result of part (1) shows that $\phi * (-z\hat{f}') = -z(\phi * \hat{f})' \in \Sigma_m^{st}(g,h)$. Hence $\phi * \hat{f} \in \Sigma_m^{k}(g,h)$. ∎

Remark 2.2 *Let h be a convex univalent function satisfying $\operatorname{Re} h(z) < 2-\alpha$, $0 \le \alpha < 1$, and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{R}_\alpha$. The above theorem can be written in the* *following equivalent forms:*

$$\Sigma_m^{st}(g,h) \subset \Sigma_m^{st}(\phi * g, h), \quad \Sigma_m^{cv}(g,h) \subset \Sigma_m^{cv}(\phi * g, h).$$

Corollary 2.8 [148, Theorem 6, p. 1265] *Let h be a convex univalent function satisfying $\operatorname{Re} h(z) < 2$, and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{CV}$. If $f_1, f_2, \cdots, f_m \in \Sigma_{\lambda,\mu}^{\beta}(m,h)$, then $f_1 * \phi, f_2 * \phi, \cdots, f_m * \phi \in \Sigma_{\lambda,\mu}^{\beta}(m,h)$.*

Proof. By Theorem 2.2 (2), for $0 \leq \alpha < 1$, $z^2\phi \in \mathcal{R}_\alpha$. If $g(z) = p_\mu * q_{\beta,\lambda}$ is defined by (2.14), then the desired result follows from Theorem 2.6 (1). ∎

Corollary 2.9 *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$. Define*

$$F_k(z) = \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} f_k(t)dt \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1, \cdots, m). \qquad (2.19)$$

If $\hat{f} \in \Sigma_m^{st}(g,h)$, then $\hat{F} = \langle F_1, \cdots, F_m \rangle \in \Sigma_m^{st}(g,h)$. Similarly, if $\hat{f} \in \Sigma_m^{cv}(g,h)$, then $\hat{F} \in \Sigma_m^{cv}(g,h)$.

Proof. Define the function ϕ by

$$\phi(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\gamma+1}{\gamma+2+n} z^n.$$

For $\operatorname{Re}\gamma \geq 0$, by Lemma 2.1, the function $z^2\phi(z)$ is a convex function. Since $\mathcal{CV} = \mathcal{R}_0$, and Theorem 2.2 (2) shows that $\mathcal{R}_0 \subset \mathcal{R}_\alpha$ for $0 \leq \alpha < 1$, hence

$z^2\phi(z) \in \mathcal{R}_\alpha$. It is evident from the definition of F_k that

$$\begin{aligned}
F_k(z) &= \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} f_k(t)dt \\
&= \frac{\gamma+1}{z^{\gamma+2}} \int_0^z t^{\gamma+1} \left(\frac{1}{t} + \sum_{n=0}^{\infty} a_n t^n \right) dt \\
&= \frac{1}{z} + \sum_{n=0}^{\infty} \frac{\gamma+1}{\gamma+2+n} z^n \\
&= (f_k * \phi)(z) \quad (\gamma \in \mathbb{C},\ \operatorname{Re}\gamma \geq 0,\ k = 1, \cdots, m),
\end{aligned}$$

so that $\hat{F} = \hat{f} * \phi$. By Theorem 2.6 (1), it follows that $\hat{F} = \hat{f} * \phi \in \Sigma_m^{st}(g, h)$.

The second result is established analogously. ▮

Again we take note of how our results extend various earlier works.

Corollary 2.10 [44, Proposition 2, p. 512] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < \gamma + 1,\ \gamma > 0$. *If $f_1, f_2, \cdots, f_m \in \Sigma_{\lambda,\mu}^{\beta}(m, h)$, then $F_1, F_2, \cdots, F_m \in \Sigma_{\lambda,\mu}^{\beta}(m, h)$ where F_k is given by* (2.19).

Proof. Choose $g(z) = p_\mu * q_{\beta,\lambda}$ defined by (2.14), the result follows from Corollary 2.9. ▮

Corollary 2.11 [35, Theorem 2, p. 11] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < \operatorname{Re}\gamma + 2$. *If $f_1, f_2, \cdots, f_m \in \Sigma(m, a, h)$, then $F_1, F_2, \cdots, F_m \in \Sigma(m, a, h)$ where F_k is given by* (2.19).

Proof. Choose $g(z) = k_n(z)$ defined by (2.16), Corollary 2.9 yields the desired result. ▮

Theorem 2.7 *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2 - \alpha$, $0 \leq \alpha < 1$, *and $\phi \in \Sigma$ with $z^2\phi \in \mathcal{R}_\alpha$.*

(1) If $\hat{f} \in \Sigma_m^{ccv}(g,h)$ with respect to $\hat{\psi} \in \Sigma_m^{st}(g,h)$, then $\hat{f} * \phi \in \Sigma_m^{ccv}(g,h)$ with respect to $\hat{\psi} * \phi \in \Sigma_m^{st}(g,h)$.

(2) If $\hat{f} \in \Sigma_m^{qcv}(g,h)$ with respect to $\hat{\varphi} \in \Sigma_m^{cv}(g,h)$, then $\hat{f} * \phi \in \Sigma_m^{qcv}(g,h)$ with respect to $\hat{\varphi} * \phi \in \Sigma_m^{cv}(g,h)$.

Proof. (1) By using

$$\hat{f} \in \Sigma_m^{ccv}(g,h) \Leftrightarrow \hat{f} * g \in \Sigma_m^{ccv}(h),$$

it is sufficient to prove that $\hat{f} * \phi \in \Sigma_m^{ccv}(h)$ when $\hat{f} \in \Sigma_m^{ccv}(h)$. Let $\hat{f} \in \Sigma_m^{ccv}(h)$. For $k = 1, 2, \cdots, m$, define the functions F and H_k by

$$F(z) = \frac{1}{m}\sum_{j=1}^{m} \psi_j(z), \quad H_k(z) = -\frac{zf_k'(z)}{F(z)}.$$

Inequality (2.18) shows that $z^2F \in \mathcal{ST}(\alpha)$.

It is seen that

$$\begin{aligned} -\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * \psi_j)(z)} &= \frac{(\phi * (-zf_k'))(z)}{(\phi * \frac{1}{m}\sum_{j=1}^{m} \psi_j)(z)} \\ &= \frac{(\phi * H_kF)(z)}{(\phi * F)(z)}. \end{aligned}$$

Since $z^2\phi \in \mathcal{R}_\alpha$ and $z^2F \in \mathcal{ST}(\alpha)$, Theorem 2.5 yields

$$\frac{(\phi * H_kF)(z)}{(\phi * F)(z)} \in \overline{\text{co}}(H_k(\mathcal{U})),$$

and because $H_k(z) \prec h(z)$, it follows that

$$-\frac{z(\phi * f_k)'(z)}{\frac{1}{m}\sum_{j=1}^{m}(\phi * \psi_j)(z)} \prec h(z) \quad (k = 1, \cdots, m).$$

Thus $\hat{f} * \phi \in \Sigma_m^{ccv}(h)$.

(2) The function $\hat{f}$ is in $\Sigma_m^{qcv}(g,h)$ if and only if $-z\hat{f}'$ is in $\Sigma_m^{ccv}(g,h)$ and from the first part above, it follows that $\phi * (-z\hat{f}') = -z(\phi * \hat{f})' \in \Sigma_m^{ccv}(g,h)$. Hence $\phi * \hat{f} \in \Sigma_m^{qcv}(g,h)$. ∎

The following corollary shows that the class $\Sigma_m^{ccv}(g,h)$ is closed under the Bernardi-Libera-Livingston integral operator.

Corollary 2.12 *Assume that $m \geq 1$ is a fixed integer and g is a fixed function in Σ. Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < 2-\alpha$, $0 \leq \alpha < 1$. *Let F_k be defined by* (2.19). *If $\hat{f} \in \Sigma_m^{ccv}(g,h)$, then $\hat{F} = \langle F_1, \cdots, F_m\rangle \in \Sigma_m^{ccv}(g,h)$.*

The proof is analogous to Corollary 2.3 and is omitted.

Corollary 2.13 [3, Theorem 3.1,p. 9] *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < \operatorname{Re}\gamma + 2$. *Let F be defined by* (2.19). *If $f \in Q(m,n,\lambda,h)$, then $F \in Q(m,n,\lambda,h)$.*

Proof. Let $g(z) = k_n(z)$ be defined by (2.16). Then the desired result now follows from Corollary 2.12. ∎

Corollary 2.14 *Let h be a convex univalent function satisfying* $\operatorname{Re} h(z) < \operatorname{Re}\gamma + 2$. *Let F be defined by* (2.19). *If $f \in C^*(m,a,h)$, then $F \in C^*(m,a,h)$.*

Proof. By choosing $g(z) = k_a(z)$ defined by (2.14). Then the result follows from Corollary 2.12. ∎

CHAPTER 3

GRONWALL'S INEQUALITY AND INCLUSION CRITERIA FOR SUBCLASSES OF FUNCTIONS

3.1 Introduction

The Schwarzian derivative of analytic functions has important invariant properties. We recall that the Schwarzian derivative $S(f,z)$ of a locally univalent analytic function f is defined by

$$S(f,z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

The Schwarzian derivative is invariant under Möbius transformations. Indeed, the Schwarzian derivative of an analytic function f is identically zero if and only if it is a Möbius transformation [48, p. 259].

Nehari showed that the univalence of an analytic function in $\mathcal{U}$ can be guaranteed if its Schwarzian derivative is dominated by a suitable positive function [124, Theorem I, p. 700]. In [123], by considering two particular positive functions, a bound on the Schwarzian derivative was obtained that would ensure univalence of an analytic function in $\mathcal{A}$. In fact, the following theorem was proved.

Theorem 3.1 [123, Theorem II, p. 549] *If $f \in \mathcal{A}$ satisfies*

$$|S(f,z)| \leq \frac{\pi^2}{2} \quad (z \in \mathcal{U}),$$

then $f \in \mathcal{S}$. The result is sharp for the function f given by $f(z) = (\exp(i\pi z) - 1)/i\pi$.

The problem of finding similar bounds on the Schwarzian derivatives that would imply univalence, starlikeness or convexity of functions was investigated by a number of authors including Gabriel [55], Friedland and Nehari [54], and Ozaki and Nunokawa [139]. Corresponding results related to meromorphic functions were dealt with in [55, 68, 123, 149]. For instance, Kim and Sugawa [80] found sufficient conditions in terms of the Schwarzian derivative for locally univalent meromorphic functions in the unit disk to possess specific geometric properties such as starlikeness and convexity. The method of proof in [80] was based on comparison theorems in the theory of ordinary differential equations with real coefficients.

For $0 < \alpha \leq 1$, let $\mathcal{SST}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions f satisfying the inequality

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leq \frac{\alpha\pi}{2}.$$

Functions in $\mathcal{SST}(\alpha)$ are called strongly starlike functions of order α. Chiang [41] investigated strong-starlikeness of order α and convexity of functions f by requiring the Schwarzian derivative $S(f, z)$ and the second coefficient a_2 of f to satisfy certain inequalities. The following results were proved:

Theorem 3.2 [41, Theorem 1, pp. 108-109] *Let $f \in \mathcal{A}$, $0 < \alpha \leq 1$ and $|a_2| = \eta < \sin(\alpha\pi/2)$. Suppose*

$$\sup_{z\in\mathcal{U}} |S(f, z)| = 2\delta(\eta), \tag{3.1}$$

where $\delta(\eta)$ satisfies the inequality

$$\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\alpha\pi}{2}.$$

Then $f \in \mathcal{SST}(\alpha)$. Further, $|\arg(f(z)/z)| \leq \alpha\pi/2$.

Theorem 3.3 [41, Theorem 2, p. 109] *Let $f \in \mathcal{A}$, and $|a_2| = \eta < 1/3$. Suppose*

(3.1) *holds where* $\delta(\eta)$ *satisfies the inequality*

$$6\eta + 5(1+\eta)\delta e^{\delta/2} < 2.$$

Then

$$f \in \mathcal{CV}\left(\frac{2 - 6\eta - 5(1+\eta)\delta e^{\delta/2}}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right).$$

In particular, if $a_2 = 0$ *and* $2\delta \leq 0.6712$, *then* $f \in \mathcal{CV}$.

Chiang's proofs in [41] rely on Gronwall's inequality (see Lemma 3.1 below). In this chapter, Gronwall's inequality is used to obtain sufficient conditions for analytic functions to be univalent. Also, certain inequalities related to the Schwarzian derivative and the second coefficient will be formulated that ensure analytic functions possess certain geometric properties. The sufficient conditions for convexity obtained by Chiang [41] will be seen to be a special case of our result. A sufficient condition in terms of the Schwarzian derivative and the second coefficient for function f to be starlike will also be obtained.

3.2 Consequences of Gronwall's Inequality

Gronwall's inequality and certain relationships between the Schwarzian derivative of f and the solution of the linear second-order differential equation $y'' + A(z)y = 0$ with $A(z) := S(f;z)/2$ will be revisited in this section. We first state *Gronwall's inequality*.

Lemma 3.1 [71, p. 19] *Suppose* A *and* g *are non-negative continuous real functions for* $t \geq 0$. *Let* $k > 0$ *be a constant. Then the inequality*

$$g(t) \leq k + \int_0^t g(s)A(s)ds$$

implies

$$g(t) \leq k \exp\left(\int_0^t A(s)ds\right) \quad (t > 0).$$

For the linear second-order differential equation

$$y'' + A(z)y = 0 \tag{3.2}$$

where $A(z) := S(f;z)/2$ is an analytic function, suppose that u and v are two linearly independent solutions with initial conditions $u(0) = v'(0) = 0$ and $u'(0) = v(0) = 1$. Such solutions always exist, and $y_1(z) = au(z) + bv(z)$ and $y_2(z) = cu(z) + dv(z)$ with $ad - bc \neq 0$ and the ratio $f(z) = y_1(z)/y_2(z)$ satisfy (3.2). Indeed, the logarithmic derivative of

$$f'(z) = \frac{y_1'(z)y_2(z) - y_2'(z)y_1(z)}{y_2^2(z)} = \frac{1}{y_2^2(z)}$$

is

$$\frac{f''(z)}{f'(z)} = -\frac{2y_2'(z)}{y_2(z)}.$$

Hence,

$$S(f;z) = \left(-\frac{2y_2'(z)}{y_2(z)}\right)' - \frac{1}{2}\left(-\frac{2y_2'(z)}{y_2(z)}\right)^2 = 2A(z).$$

Thus, f can be defined by

$$f(z) = \frac{au(z) + bv(z)}{cu(z) + dv(z)}.$$

The normalization of f gives $a = 1$ and $b = 0$. Since $f'(0) = 1$, so $d = 1$. Hence the function f is represented by

$$f(z) = \frac{u(z)}{cu(z) + v(z)}. \tag{3.3}$$

It is evident that

$$
\begin{aligned}
f'(z) &= \frac{u'(z)v(z) - v'(z)u(z)}{\big(cu(z)+v(z)\big)^2} \\
&= -\frac{\mathcal{W}(u,v)}{\big(cu(z)+v(z)\big)^2} = \frac{1}{\big(cu(z)+v(z)\big)^2},
\end{aligned}
\tag{3.4}
$$

where $\mathcal{W}(u,v)$ is the Wronskian of u and v. Also, the function f'' given by

$$
f''(z) = \frac{-2\big(cu'(z)+v'(z)\big)}{\big(cu(z)+v(z)\big)^3}
$$

together with the normalization conditions for u and v show that $c = -a_2$. Estimates on bounds for various expressions related to u and v were found in [41]. Indeed, using the integral representation of the fundamental solutions

$$
\begin{aligned}
u(z) &= z + \textstyle\int_0^z (\eta - z)A(\eta)u(\eta)d\eta, \\
v(z) &= 1 + \textstyle\int_0^z (\eta - z)A(\eta)v(\eta)d\eta,
\end{aligned}
\tag{3.5}
$$

and applying Gronwall's inequality, Chiang obtained the following inequalities [41]:

$$
|u(z)| < e^{\delta/2}, \tag{3.6}
$$

$$
\left|\frac{u(z)}{z} - 1\right| < \frac{1}{2}\delta e^{\delta/2}, \tag{3.7}
$$

$$
|cu(z)+v(z)| < (1+\eta)e^{\delta/2}, \tag{3.8}
$$

$$
|cu(z)+v(z)-1| < \eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}. \tag{3.9}
$$

The proof of these inequalities follows by taking the path of integration to be $\eta(t) = te^{i\theta}$, $t \in [0,r]$, $z = re^{i\theta}$, and applying Gronwall's inequality whenever

$|A(z)| < \delta$ and $0 < r < 1$. For instance,

$$\begin{aligned}
|u(z)| &= \left| z + \int_0^z (\eta - z)A(\eta)u(\eta)d\eta \right| \\
&\le 1 + \int_0^r (r-t)|A(te^{i\theta})|\,|u(te^{i\theta})|dt \\
&\le \exp(\int_o^r (r-t)|A(te^{i\theta})|dt) \le \exp(\delta/2).
\end{aligned}$$

This proves inequality (3.6). Note that there was a typographical error in [41, inequality (8), p. 112], and that inequality (3.7) is the right form. Substituting (3.6) back into (3.5) yields

$$\begin{aligned}
|u(z) - z| &= \left| \int_0^z (\eta - z)A(\eta)u(\eta)d\eta \right| \\
&\le \int_0^r (r-t)|A(te^{i\theta})|\,|u(te^{i\theta})|dt \\
&\le \delta \exp(\delta/2) \int_o^r (r-t)dt \\
&= \delta \exp(\delta/2)\frac{r^2}{2} < \delta \exp(\delta/2)/2.
\end{aligned}$$

Hence inequality (3.7) holds. From (3.5), it follows that

$$cu(z) + v(z) = 1 + cz + \int_0^z (\zeta - z)A(\zeta)(cu(\zeta) + v(\zeta))d\zeta, \tag{3.10}$$

and therefore

$$|cu(z) + v(z)| \le 1 + |c|r + \int_0^r (r-t)|A(te^{i\theta})|\;|(cu(te^{i\theta}) + v(te^{i\theta}))|dt.$$

Gronwall's inequality shows that, whenever $|A(z)| < \delta$ and $0 < r < 1$,

$$\begin{aligned}
|cu(z) + v(z)| &< (1 + |c|)\exp\left(\int_0^r (r-t)|A(te^{i\theta})|dt\right) \\
&< (1 + |c|)\exp(\delta/2).
\end{aligned} \tag{3.11}$$

This proves inequality (3.8). If (3.11) is substituted back into (3.10), then

$$\begin{aligned}|cu(z)+v(z)-1| &\leq |c|r + \int_0^r (r-t)|A(te^{i\theta})||(cu(te^{i\theta})+v(te^{i\theta}))|dt \\ &< \eta + (1+\eta)\delta\exp(\delta/2)\int_0^r (r-t)dt \\ &< \eta + (1+\eta)\delta\frac{\exp(\delta/2)}{2}.\end{aligned}$$

Therefore, inequality (3.9) holds.

3.3 Inclusion Criteria for Subclasses of Analytic Functions

The first result leads to sufficient conditions for univalence.

Theorem 3.4 *Let $0<\alpha\leq 1$, $0\leq\beta<1$, $f\in\mathcal{A}$ and $|a_2|=\eta$, where α, β and η satisfy*

$$\sin^{-1}\left(\beta(1+\eta)^2\right) + 2\sin^{-1}\eta < \frac{\alpha\pi}{2}. \tag{3.12}$$

Suppose (3.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\sin^{-1}\left(\beta(1+\eta)^2 e^{\delta}\right) + 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\alpha\pi}{2}. \tag{3.13}$$

Then $|\arg(f'(z)-\beta)| \leq \alpha\pi/2$.

Proof. It is evident that

$$\lim_{\delta\to 0}\left(\sin^{-1}\left(\beta(1+\eta)^2 e^{\delta}\right) + 2\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)\right) = \sin^{-1}\left(\beta(1+\eta)^2\right) + 2\sin^{-1}\eta.$$

Let $\epsilon = (\alpha\pi/2) - (sin^{-1}\big(\beta(1+\eta)^2\big) + 2\sin^{-1}\eta)$. From (3.12) follows that $\epsilon > 0$.

The inequality

$$\sin^{-1}\beta(1+\eta)^2e^{\delta})+2\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) < \sin^{-1}\left(\beta(1+\eta)^2\right)+2\sin^{-1}\eta+\epsilon$$

together with condition (3.12) show that there is a real number $\delta(\eta) \geq 0$ satisfying inequality (3.13). The representation of f' in terms of the linearly independent solutions of the differential equation $y'' + A(z)y = 0$ with $A(z) := S(f;z)/2$ as given by equation (3.4) yields

$$f'(z) - \beta = \frac{1 - \beta(c\ u(z) + v(z))^2}{(c\ u(z) + v(z))^2}. \tag{3.14}$$

If follows from the fact $|\arg w| \leq \sin^{-1} r$ that inequality (3.8) implies

$$|\arg\left(1 - \beta(c\ u(z) + v(z))^2\right)| \leq \sin^{-1}\left(\beta(1+\eta)^2e^{\delta}\right). \tag{3.15}$$

Similarly, inequality (3.9) shows

$$|\arg\left(c\ u(z) + v(z)\right)| \leq \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \tag{3.16}$$

Hence, it follows from (3.14), (3.15) and (3.16) that

$$\begin{aligned} |\arg(f'(z) - \beta)| &= \left|\arg\left(\frac{1 - \beta(c\ u(z) + v(z))^2}{(c\ u(z) + v(z))^2}\right)\right| \\ &\leq |\arg\left(1 - \beta(c\ u(z) + v(z))^2\right)| + 2|\arg\left(c\ u(z) + v(z)\right)| \\ &\leq \sin^{-1}(\beta(1+\eta)^2e^{\delta}) + 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\leq \frac{\alpha\pi}{2}, \end{aligned}$$

where the last inequality follows from (3.13). This completes the proof. ∎

By taking $\beta = 0$ in Theorem 3.4, the following univalence criterion is obtained.

Corollary 3.1 *Let $f \in \mathcal{A}$, and $|a_2| = \eta < \sin(\alpha\pi/4)$, $0 < \alpha \leq 1$. Suppose that* (3.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2} \leq \sin\left(\frac{\alpha\pi}{4}\right).$$

Then $|\arg f'(z)| \leq \alpha\pi/2$, and in particular $f \in \mathcal{S}$.

Example 3.1 *Consider the univalent function g given by*

$$g(z) = \frac{z}{1+cz} \quad (|c| \leq 1,\ z \in \mathcal{U}).$$

Since the Schwarzian derivative of an analytic function is zero if and only if it is a Möbius transformation, it is evident that $S(g,z) = 0$. Therefore the condition (3.1) *is satisfied with $\delta = 0$. It is enough to take $\eta = |c|$ and to assume that η, α and β satisfy inequality* (3.12). *Now*

$$|\arg(g'(z) - \beta)| = \left|\arg\frac{1}{(1+cz)^2} - \beta\right| \leq |\arg(1-\beta(1+cz)^2)| + 2|\arg(1+cz)|$$
$$\leq \sin^{-1}(\beta(1+|c|)^2) + 2\sin^{-1}|c|.$$

In view of the latter inequality, it is necessary to assume inequality (3.12) *for g to satisfy $|\arg(g'(z)-\beta)| \leq \alpha\pi/2$.*

Let $0 \leq \rho < 1$, $0 \leq \lambda < 1$, and α be a positive integer. A function $f \in \mathcal{A}$ is called an *α-Bazilevič function of order ρ and type λ*, written $f \in \mathcal{B}(\alpha, \rho, \lambda)$, if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)^{1-\alpha}g(z)^{\alpha}}\right) > \rho \quad (z \in \mathcal{U})$$

for some function $g \in \mathcal{ST}(\lambda)$. Bazilevič [32] introduced the above class of functions and proved such functions are univalent in $\mathcal{U}$. Thomas [200] called a function satisfying the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)^{1-\alpha}g(z)^{\alpha}}\right) > 0 \quad (z \in \mathcal{U},\ g \in \mathcal{ST})$$

an α-Bazilevič function, written $f \in \mathcal{B}(\alpha)$. Certain subclasses of $\mathcal{B}(\alpha)$ have been investigated widely by Keogh and Miller [78], Merkes and Wright [106] , Zamorski [210], and Mocanu *et al.* [115].

The following subclass of α-Bazilevič functions introduced by Gao [56] is of interest. A function $f \in \mathcal{A}$ is called strongly α-Bazilevič of order β if

$$\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha} f'(z)\right)\right| < \frac{\beta\pi}{2} \quad (\alpha > 0,\ 0 < \beta \leq 1).$$

For the class of strongly α-Bazilevič functions of order β, the following sufficient condition is obtained.

Theorem 3.5 *Let $\alpha > 0$, $0 < \beta \leq 1$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η, α and β satisfy*

$$\eta < \sin\left(\frac{\beta\pi}{2(1+\alpha)}\right).$$

Suppose (3.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$|1-\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + (1+\alpha)\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \leq \frac{\beta\pi}{2}. \qquad (3.17)$$

Then f is strongly α-Bazilevič of order β.

Proof. Since

$$\lim_{\delta\to 0}\left(|1-\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + (1+\alpha)\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)\right)$$
$$= (1+\alpha)\sin^{-1}\eta,$$

then the inequality

$$|1-\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+(1+\alpha)\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)<(1+\alpha)\sin^{-1}\eta+\epsilon$$

holds for a given $\epsilon=(\beta\pi/2)-(1+\alpha)\sin^{-1}\eta$. This shows that the condition $(1+\alpha)\sin^{-1}\eta\leq\beta\pi/2$, or equivalently $\eta<\sin(\beta\pi/2(1+\alpha))$ ensures that there is a real number $\delta(\eta)$ satisfying (3.17). Using (3.3) and (3.4) lead to

$$\begin{aligned}\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha}f'(z)\right)\right| &= \left|\arg\left(\left(\frac{u(z)}{z}\right)^{\alpha-1}(cu(z)+v(z))^{-(\alpha+1)}\right)\right| \\ &\leq |1-\alpha|\left|\arg\left(\frac{u(z)}{z}\right)\right|+|\alpha+1|\;|\arg(cu(z)+v(z))|\,.\end{aligned}$$

Inequality (3.7) shows that

$$\left|\arg\left(\frac{u(z)}{z}\right)\right|\leq\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) \tag{3.18}$$

It now follows from (3.16), (3.17) and (3.18) that

$$\begin{aligned}&\left|\arg\left(\left(\frac{z}{f(z)}\right)^{1-\alpha}f'(z)\right)\right| \\ &\leq|1-\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+(1+\alpha)\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\leq\frac{\beta\pi}{2}. \quad \blacksquare\end{aligned}$$

For $\alpha\geq0$, consider the class $\mathcal{R}^\alpha$ defined by

$$\mathcal{R}^\alpha=\{f\in\mathcal{A}:\operatorname{Re}\left(f'(z)+\alpha zf''(z)\right)>0\}.$$

For this class, the following sufficient condition is obtained.

Theorem 3.6 *Let* $\alpha \geq 0$*,* $f \in \mathcal{A}$ *and* $|a_2| = \eta$*, where* η *and* α *satisfy*

$$2\sin^{-1}\eta + \sin^{-1}\left(\frac{2\eta\alpha}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.19}$$

Suppose (3.1) *holds where* $\delta(\eta)$ *satisfies the inequality*

$$2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right) \leq \frac{\pi}{2}. \tag{3.20}$$

Then $f \in \mathcal{R}(\alpha)$*.*

Proof. It is evident that

$$\begin{aligned}\lim_{\delta\to 0}\left(2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right)\right) \\ = 2\sin^{-1}\eta + \sin^{-1}\left(\frac{2\eta\alpha}{1-\eta}\right).\end{aligned}$$

Again it is easily seen from a limiting argument that condition (3.19) guarantees the existence of a real number $\delta(\eta) \geq 0$ satisfying inequality (3.20). It is sufficient to show that

$$\left|\arg\left(f'(z)\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right)\right| < \frac{\pi}{2}.$$

The equation (3.4) yields

$$\frac{zf''(z)}{f'(z)} = -2z\frac{cu'(z) + v'(z)}{cu(z) + v(z)}. \tag{3.21}$$

A simple calculation from (3.5) shows that

$$cu'(z) + v'(z) = c - \int_0^z A(\eta)\big(cu(\eta) + v(\eta)\big)d\eta,$$

and an application of (3.8) leads to

$$|cu'(z) + v'(z)| \leq |c| + \int_0^z |A(\eta)| \big|cu(\eta) + v(\eta)\big| d\eta \leq \eta + (1+\eta)\delta e^{\delta/2}. \qquad (3.22)$$

Use of (3.9) yields

$$|cu(z) + v(z)| \geq 1 - |cu(z) + v(z) - 1| \geq 1 - \eta - \frac{1}{2}(1+\eta)\delta e^{\delta/2}. \qquad (3.23)$$

The lower bound in (3.23) is non-negative from the assumption made in (3.20). From (3.21), (3.22) and (3.23) , it is evident that

$$\begin{aligned}\left|\left(1 + \alpha \frac{zf''(z)}{f'(z)}\right) - 1\right| &= \left|2z\alpha \frac{cu'(z) + v'(z)}{cu(z) + v(z)}\right| \\ &\leq \frac{2\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{1 - \eta - \frac{1}{2}(1+\eta)\delta e^{\delta/2}}.\end{aligned}$$

Hence,

$$\left|\arg\left(1 + \alpha \frac{zf''(z)}{f'(z)}\right)\right| \leq \sin^{-1}\left(\frac{4\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right). \qquad (3.24)$$

From (3.16) it follows that

$$|\arg f'(z)| = 2|\arg(cu(z) + v(z))| \leq 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \qquad (3.25)$$

Using (3.4) and (3.16), inequality (3.25) together with (3.24) and (3.20) imply that

$$\begin{aligned}&\left|\arg\left(f'(z)\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right)\right| \\ &\leq |\arg f'(z)| + \left|\arg\left(1 + \alpha\frac{zf''(z)}{f'(z)}\right)\right| \\ &\leq 2\sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\alpha\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right) \\ &\leq \frac{\pi}{2}. \qquad \blacksquare\end{aligned}$$

Generalizing the familiar starlike and convex functions, Lewandoski *et al.* [91] introduced γ-starlike functions consisting of $f \in \mathcal{A}$ satisfying

$$\operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{1-\gamma}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\gamma}\right) > 0 \quad (0 \leq \gamma \leq 1).$$

Further discussion can be found in [38, 51, 194]. More generally, let $\mathcal{M}(\alpha, \beta)$ be the class of analytic functions defined by

$$\mathcal{M}(\alpha,\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}\right) > 0, \alpha, \beta \in \mathbb{R} \right\}. \tag{3.26}$$

The following sufficient condition for the class $\mathcal{M}(\alpha, \beta)$ is obtained.

Theorem 3.7 *Let $f \in \mathcal{A}$, $|a_2| = \eta \leq 1/3$, and β, α be real numbers satisfying*

$$|\alpha| \sin^{-1} \eta + |\beta| \sin^{-1}\left(\frac{2\eta}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.27}$$

Suppose (3.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\begin{aligned} |\alpha| \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + |\alpha| \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)& \\ + |\beta| \sin^{-1}\left(\frac{4\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) &\leq \frac{\pi}{2}. \end{aligned} \tag{3.28}$$

Then $f \in \mathcal{M}(\alpha, \beta)$.

Proof. Condition (3.27) assures the existence of δ satisfying (3.28). From (3.3) and (3.4), it follows that

$$\frac{zf'(z)}{f(z)} = \frac{z}{u(z)} \frac{1}{cu(z)+v(z)} \quad (z \in \mathcal{U}). \tag{3.29}$$

By (3.18) and (3.16),

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| \leq \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \tag{3.30}$$

Using (3.24) with $\alpha = 1$, (3.30) and (3.28) lead to

$$\begin{aligned}
&\left|\arg\left(\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{\beta}\right)\right| \\
&\leq |\alpha|\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| + |\beta|\left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right| \\
&\leq |\alpha|\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + |\alpha|\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\
&\quad + |\beta|\sin^{-1}\frac{4\big(\eta+(1+\eta)\delta e^{\delta/2}\big)}{2-2\eta-(1+\eta)\delta e^{\delta/2}} \\
&\leq \frac{\pi}{2}.
\end{aligned}$$

This shows that $f \in \mathcal{M}(\alpha, \beta)$. ▮

Remark 3.1 *Theorem 3.7 yields the following interesting special cases.*

(1) If $\alpha = 0$, $\beta = 1$, a sufficient condition for convexity is obtained. This case reduces to a result in [41, Theorem 2, p. 109].

(2) For $\alpha = 1$, $\beta = 0$, a sufficient condition for starlikeness is obtained.

(3) For $\alpha = -1$ and $\beta = 1$, then the class of functions satisfying (3.26) *reduces to the class of functions*

$$\mathcal{G} := \left\{ f \in \mathcal{A} : \ \operatorname{Re}\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) > 0 \right\}.$$

This class $\mathcal{G}$ was considered by Silverman [184]. He proved that functions in the class $\mathcal{G}$ are starlike in $\mathcal{U}$. The class $\mathcal{G}$ was also studied extensively by

Tuneski [132, 201–203].

Theorem 3.8 *Let $\beta \geq 0$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η satisfies*

$$\sin^{-1}(\eta) + \sin^{-1}\left(\frac{2\beta\eta}{1-\eta}\right) < \frac{\pi}{2}. \tag{3.31}$$

Suppose (3.1) *holds where* $\delta(\eta)$ *satisfies the inequality*

$$\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) + \sin^{-1}\left(\frac{4\beta\big(\eta + (1+\eta)\delta e^{\delta/2}\big)}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}}\right) \leq \frac{\pi}{2}.$$

Then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)} + \beta\frac{z^2 f''(z)}{f(z)}\right) > 0. \tag{3.32}$$

The proof is similar to the proof of Theorem 3.7, and is therefore omitted. Condition (3.31) is equivalent to the condition

$$\eta\left(1 + \sqrt{(1-\eta)^2 - 4\beta^2\eta^2} + 2\beta\sqrt{1-\eta^2}\right) < 1.$$

For $\beta = 1$, the above equation simplifies to

$$\eta^8 - 4\eta^7 + 12\eta^6 - 12\eta^5 + 6\eta^4 + 20\eta^3 - 4\eta^2 - 4\eta + 1 = 0;$$

the value of the root η is approximately 0.321336. Lewandowski *et al.* [92] proved that analytic functions satisfying (3.32) are starlike. Functions satisfying inequality (3.32) has been extensively studied by Ramesha *et al.* [157], Obradović and Joshi [134], Nunokawa *et al.* [133], and Padmanabhan [142].

Consider the class $\mathcal{P}_\gamma$, $0 \le \gamma \le 1$, given by

$$\mathcal{P}_\gamma := \left\{ f \in \mathcal{A} : \left| \arg \left((1-\gamma)\frac{f(z)}{z} + \gamma f'(z) \right) \right| < \frac{\pi}{2}, \quad z \in \mathcal{U} \right\}.$$

The same approach using Gronwall's inequality leads to the following result on the class $\mathcal{P}_\gamma$.

Theorem 3.9 *Let $0 \le \gamma < 1$, $f \in \mathcal{A}$ and $|a_2| = \eta$, where η and γ satisfy*

$$\sin^{-1}\left(\frac{\gamma}{1-\gamma} \frac{1}{\eta - 1} \right) + \sin^{-1} \eta < \frac{\pi}{2}. \tag{3.33}$$

Suppose (3.1) *holds where $\delta(\eta)$ satisfies the inequality*

$$\begin{aligned} &\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2} \right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2} \right) \\ &+ \sin^{-1}\left(\frac{2\gamma}{1-\gamma} \frac{1}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}} \frac{1}{1 - 2e^{\delta/2}} \right) \le \frac{\pi}{2}. \end{aligned} \tag{3.34}$$

Then $f \in \mathcal{P}_\gamma$.

Proof. It is clear that

$$\begin{aligned} &\lim_{\delta \to 0} \left(\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2} \right) + \sin^{-1}\left(\eta + \frac{1}{2}(1+\eta)\delta e^{\delta/2} \right) \right. \\ &\left. + \sin^{-1}\left(\frac{2\gamma}{1-\gamma} \frac{1}{2 - 2\eta - (1+\eta)\delta e^{\delta/2}} \frac{1}{1 - 2e^{\delta/2}} \right) \right) \\ &= \sin^{-1}\left(\frac{\gamma}{1-\gamma} \frac{1}{\eta - 1} \right) + \sin^{-1} \eta. \end{aligned}$$

Thus, the inequality

$$\begin{aligned}
&\sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right)\\
&+\sin^{-1}\left(\frac{2\gamma}{1-\gamma}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\frac{1}{1-2e^{\delta/2}}\right)\\
&<\sin^{-1}\left(\frac{\gamma}{1-\gamma}\frac{1}{\eta-1}\right)+\sin^{-1}\eta+\epsilon.
\end{aligned}$$

holds for $\epsilon = (\pi/2) - \sin^{-1}(\gamma/(1-\gamma)(\eta-1)) - \sin^{-1}\eta$. Therefore, condition (3.33) assures the existence of a small enough real number $\delta(\eta) \geq 0$ satisfying inequality (3.34).

A simple calculation from (3.5) and Lemma 3.1 shows that

$$\begin{aligned}
|u(z)-1| &\leq |z-1| + \left|\int_0^z (\zeta - z)A(\zeta)u(\zeta)d\zeta\right|\\
&\leq (z-1)\exp\left(\int_0^z (\zeta-z)|A(\zeta)|d\zeta\right)\\
&\leq 2e^{\delta/2}.
\end{aligned}$$

The above inequality gives

$$\left|\frac{z}{u(z)}\right| \leq \frac{1}{|u(z)|} \leq \frac{1}{1-|u(z)-1|} \leq \frac{1}{1-2e^{\delta/2}}. \tag{3.35}$$

Therefore, for some $0 < \beta \leq \gamma/(1-\gamma)$, (3.29), (3.35) and (3.23) lead to

$$\begin{aligned}
\left|1+\beta\frac{zf'(z)}{f(z)}-1\right| &= \beta\left|\frac{z}{u(z)}\right|\frac{1}{|cu(z)+v(z)|}\\
&\leq \frac{\beta}{1-2e^{\delta/2}}\frac{1}{1-\eta-\frac{1}{2}(1+\eta)\delta e^{\delta/2}}\\
&= \frac{2\beta}{1-2e^{\delta/2}}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}.
\end{aligned}$$

Hence

$$\left|\arg\left(1+\beta\frac{zf'(z)}{f(z)}\right)\right| \le \sin^{-1}\left(\frac{2\beta}{1-2e^{\delta/2}}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right). \tag{3.36}$$

Also, (3.7) and (3.16) yield

$$\begin{aligned}\left|\arg\frac{f(z)}{z}\right| &= \left|\arg\frac{u(z)}{z(cu(z)+v(z))}\right| \\ &\le \left|\arg\frac{u(z)}{z}\right| + |\arg(cu(z)+v(z))| \\ &\le \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right). \end{aligned}\tag{3.37}$$

Replacing β by $\gamma/(1-\gamma)$ in inequality (7.14), and using (3.37) and (3.34) yield

$$\begin{aligned}\left|\arg\left((1-\gamma)\frac{f(z)}{z}+\gamma f'(z)\right)\right| &\le \left|\arg\frac{f(z)}{z}\right| + \left|\arg\left(1+\frac{\gamma}{1-\gamma}\frac{zf'(z)}{f(z)}\right)\right| \\ &\le \sin^{-1}\left(\frac{1}{2}\delta e^{\delta/2}\right)+\sin^{-1}\left(\eta+\frac{1}{2}(1+\eta)\delta e^{\delta/2}\right) \\ &\quad + \sin^{-1}\left(\frac{2\gamma}{1-\gamma}\frac{1}{1-2e^{\delta/2}}\frac{1}{2-2\eta-(1+\eta)\delta e^{\delta/2}}\right) \\ &\le \frac{\pi}{2}, \end{aligned}$$

and hence $f \in P_\gamma$. ∎

CHAPTER 4

CONVEXITY OF INTEGRAL TRANSFORMS AND DUALITY

4.1 Duality Technique

For $f \in \mathcal{A}$, Fournier and Ruscheweyh [53] investigated starlikeness properties of the operator

$$F(z) = V_\lambda(f)(z) := \int_0^1 \lambda(t) \frac{f(tz)}{t} dt \tag{4.1}$$

over functions f in the class

$$\mathcal{P}(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi} \big(f'(z) - \beta \big) > 0, \quad z \in \mathcal{U} \right\}.$$

Here λ is a non-negative real-valued integrable function satisfying the condition $\int_0^1 \lambda(t)dt = 1$. Ali and Singh [21] used the Duality Principle [172] to find a sharp estimate of the parameter β that ensures $V_\lambda(f)$ is convex over $\mathcal{P}(\beta)$. In 2002, Choi *et al.* [45] investigated convexity property of the integral transform (4.1) over functions f in the class

$$\mathcal{P}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi} \left((1-\alpha)\frac{f(z)}{z} + \alpha f'(z) - \beta \right) > 0, \ z \in \mathcal{U} \right\},$$

$\alpha \in \mathbb{R}$. The class $\mathcal{P}_\alpha(\beta)$ is closely related to the class $\mathcal{R}_\alpha(\beta)$ defined by

$$\mathcal{R}_\alpha(\beta) := \left\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with } \operatorname{Re} e^{i\phi} \big(f'(z) + \alpha z f''(z) - \beta \big) > 0, \quad z \in \mathcal{U} \right\}.$$

It is evident that $f \in \mathcal{R}_\alpha(\beta)$ if and only if zf' belongs to $\mathcal{P}_\alpha(\beta)$.

Consider now the following class of functions that includes both classes $\mathcal{P}(\beta)$

and $\mathcal{P}_\alpha(\beta)$. For $\alpha \geq 0$, $\gamma \geq 0$ and $\beta < 1$, define the class

$$\mathcal{W}_\beta(\alpha,\gamma) := \Big\{ f \in \mathcal{A} : \exists \phi \in \mathbb{R} \text{ with} \\ \operatorname{Re} e^{i\phi} \left((1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) - \beta \right) > 0, \quad z \in \mathcal{U} \Big\}. \tag{4.2}$$

Thus $\mathcal{P}(\beta) \equiv \mathcal{W}_\beta(1,0)$, $\mathcal{P}_\alpha(\beta) \equiv \mathcal{W}_\beta(\alpha,0)$, and $\mathcal{R}_\gamma(\beta) \equiv \mathcal{W}_\beta(1+2\gamma,\gamma)$. The class $\mathcal{W}_\beta(\alpha,\gamma)$ is closely related to the class $R(\alpha,\gamma,h)$ consisting of all solutions $f \in \mathcal{A}$ satisfying

$$f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z) \prec h(z) \quad (z \in \mathcal{U}),$$

with $h(z) := h_\beta(z) = (1+(1-2\beta)z)/(1-z)$. Here $g(z) \prec h(z)$ indicates the function g is subordinate to h, or in other words, there is an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = h(w(z))$, $z \in \mathcal{U}$. When $\phi = 0$ in (4.2), it is clear that $f \in R(\alpha,\gamma,h_\beta)$ if and only if zf' belongs to $\mathcal{W}_\beta(\alpha,\gamma)$. Every function $f \in R(\alpha,\gamma,h)$ for a suitably normalized convex function h has a double integral representation, which was recently investigated by Ali *et al.* [12].

In a recent paper, Ali *et al.* [7] investigated starlikeness properties of the integral transform (4.1) over the class $\mathcal{W}_\beta(\alpha,\gamma)$. The present chapter investigates convexity of the integral transform V_λ over the class $\mathcal{W}_\beta(\alpha,\gamma)$ by applying the Duality Principle. Specifically, in Section 4.2, the best value $\beta < 1$ is determined that ensures V_λ maps $\mathcal{W}_\beta(\alpha,\gamma)$ into the class of convex functions $\mathcal{CV}$. Necessary and sufficient conditions are also derived that ensure $V_\lambda(f)$ is convex univalent. In Section 4.3, simpler sufficient conditions for $V_\lambda(f)$ to be convex are derived. These are used in Section 4.4 in the discussion of several interesting applications for specific choices of the admissible function λ. As a consequence, the smallest value $\beta < 1$ is obtained that ensures a function f satisfying $\operatorname{Re}\left(f'(z) + \alpha z f''(z) + \gamma z^2 f'''(z)\right) > \beta$

is convex in the unit disk. The results obtained in this section extend and improve earlier works by several authors. The final section is devoted to extending the main convexity result to the generalized integral operator of the form $\rho z+(1-\rho)V_\lambda(f)$, $\rho<1$. The best value $\beta<1$ is obtained that ensures the latter operator maps $\mathcal{W}_\beta(\alpha,\gamma)$ into the class $\mathcal{CV}$.

4.2 Convexity of Integral Operators

The following notations introduced in [7] are used. Let $\mu\geq 0$ and $\nu\geq 0$ satisfy

$$\mu+\nu=\alpha-\gamma \quad \text{and} \quad \mu\nu=\gamma. \tag{4.3}$$

When $\gamma=0$, then μ is chosen to be 0, in which case, $\nu=\alpha\geq 0$. When $\alpha=1+2\gamma$, (4.3) yields $\mu+\nu=1+\gamma=1+\mu\nu$, or $(\mu-1)(1-\nu)=0$.

(i) For $\gamma>0$, choosing $\mu=1$ gives $\nu=\gamma$.

(ii) For $\gamma=0$, then $\mu=0$ and $\nu=\alpha=1$.

In the sequel, whenever the particular case $\alpha=1+2\gamma$ is considered, the values of μ and ν for $\gamma>0$ will be taken as $\mu=1$ and $\nu=\gamma$ respectively, while $\mu=0$ and $\nu=1=\alpha$ in the case $\gamma=0$.

Next we introduce two auxiliary functions. Let

$$\phi_{\mu,\nu}(z)=1+\sum_{n=1}^{\infty}\frac{(n\nu+1)(n\mu+1)}{n+1}z^n, \tag{4.4}$$

and

$$\begin{aligned}\psi_{\mu,\nu}(z)=\phi_{\mu,\nu}^{-1}(z) &= 1+\sum_{n=1}^{\infty}\frac{n+1}{(n\nu+1)(n\mu+1)}z^n \\ &= \int_0^1\int_0^1\frac{ds\,dt}{(1-t^\nu s^\mu z)^2}. \end{aligned}\tag{4.5}$$

Here $\phi_{\mu,\nu}^{-1}$ denotes the convolution inverse of $\phi_{\mu,\nu}$ such that $\phi_{\mu,\nu} * \phi_{\mu,\nu}^{-1} = 1/(1-z)$. If $\gamma = 0$, then $\mu = 0$, $\nu = \alpha$, and it is clear that

$$\psi_{0,\alpha}(z) = 1 + \sum_{n=1}^{\infty} \frac{n+1}{n\alpha+1} z^n = \int_0^1 \frac{dt}{(1-t^\alpha z)^2}.$$

If $\gamma > 0$, then $\nu > 0$, $\mu > 0$, and making the change of variables $u = t^\nu$, $v = s^\mu$ result in

$$\psi_{\mu,\nu}(z) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv.$$

Thus the function $\psi_{\mu,\nu}$ can be written as

$$\psi_{\mu,\nu}(z) = \begin{cases} \dfrac{1}{\mu\nu} \displaystyle\int_0^1 \int_0^1 \frac{u^{1/\nu-1} v^{1/\mu-1}}{(1-uvz)^2} dudv, & \gamma > 0, \\[2ex] \displaystyle\int_0^1 \frac{dt}{(1-t^\alpha z)^2}, & \gamma = 0, \ \alpha \geq 0. \end{cases}$$

Now let q be the solution of the initial-value problem

$$\frac{d}{dt} t^{1/\nu} q(t) = \begin{cases} \dfrac{1}{\mu\nu} t^{1/\nu-1} \displaystyle\int_0^1 s^{1/\mu-1} \frac{1-st}{(1+st)^3} ds, & \gamma > 0, \\[2ex] \dfrac{1}{\alpha} t^{1/\alpha-1} \dfrac{1-t}{(1+t)^3}, & \gamma = 0, \ \alpha > 0, \end{cases} \tag{4.6}$$

satisfying $q(0) = 0$. It is easily seen that the solution is given by

$$q(t) = \frac{1}{\mu\nu} \int_0^1 \int_0^1 s^{1/\mu-1} w^{1/\nu-1} \frac{1-swt}{(1+swt)^3} dsdw = \sum_{n=0}^{\infty} \frac{(n+1)^2(-1)^n t^n}{(1+\mu n)(1+\nu n)}. \tag{4.7}$$

In particular,

$$
\begin{aligned}
q_\gamma(t) &= \frac{1}{\gamma}\int_0^1 s^{1/\gamma-1}\frac{1}{(1+st)^2}ds, & \gamma>0,\ \alpha=1+2\gamma,\\
q_\alpha(t) &= \frac{1}{\alpha}t^{-1/\alpha}\int_0^t \tau^{1/\alpha-1}\frac{1-\tau}{(1+\tau)^3}d\tau, & \gamma=0,\ \alpha>0. \qquad (4.8)
\end{aligned}
$$

A well-known result [172, p. 94] states that

$$
f\in\mathcal{ST} \Longleftrightarrow \frac{1}{z}(f*h)(z)\neq 0 \quad (z\in\mathcal{U}), \tag{4.9}
$$

where h is given by

$$
h(z)=\frac{z(1+\frac{\epsilon-1}{2}z)}{(1-z)^2} \quad (|\epsilon|=1). \tag{4.10}
$$

This can be verified from Hadamard product in (4.9). Let $\epsilon=(1-iT)/(1+iT)$, and

$$
h(z)=h_T(z)=\frac{1}{1+iT}\left(\frac{z}{(1-z)^2}+\frac{iTz}{1-z}\right). \tag{4.11}
$$

Now, $f\in\mathcal{ST}$ implies $\mathrm{Re}(zf'/f)>0$, or equivalently $zf'/f\neq -iT$, thus

$$
\begin{aligned}
\frac{zf'+iTf}{z(1+iT)}\neq 0 \Leftrightarrow & \frac{1}{z(1+iT)}\left(f*\frac{z}{(1-z)^2}+iTf*\frac{z}{1-z}\right)\neq 0\\
\Leftrightarrow & \frac{1}{z}\big(h_T*f\big)(z)\neq 0,
\end{aligned}
$$

where h_T given by (4.11).

Functions in the class $\mathcal{W}_\beta(\alpha,\gamma)$ generally are not convex. The following is the main result that gives conditions for convexity.

Theorem 4.1 *Let $\mu\geq 0$, $\nu\geq 0$ satisfy* (4.3), *and let $\beta<1$ satisfy*

$$
\frac{\beta-1/2}{1-\beta}=-\int_0^1\lambda(t)q(t)dt, \tag{4.12}
$$

where q is the solution of the initial-value problem (4.6)*. Further let*

$$\Lambda_\nu(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\nu}} dx \quad (\nu > 0), \tag{4.13}$$

$$\Pi_{\mu,\nu}(t) = \begin{cases} \displaystyle\int_t^1 \Lambda_\nu(x) x^{1/\nu - 1 - 1/\mu} dx, & \gamma > 0 \ (\mu > 0,\ \nu > 0), \\ \\ \Lambda_\alpha(t), & \gamma = 0 \ (\mu = 0,\ \nu = \alpha > 0), \end{cases} \tag{4.14}$$

and assume that $t^{1/\nu}\Lambda_\nu(t) \to 0$, and $t^{1/\mu}\Pi_{\mu,\nu}(t) \to 0$ as $t \to 0^+$. Let V_λ be given by (4.1) *and h given by* (4.10)*. Then*

$$\begin{cases} \operatorname{Re} \displaystyle\int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0, \ \gamma > 0, \\ \\ \operatorname{Re} \displaystyle\int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0, \ \gamma = 0, \end{cases} \tag{4.15}$$

if and only if $F(z) = V_\lambda(f)(z)$ is in $\mathcal{CV}$ for $f \in W_\beta(\alpha,\gamma)$. This conclusion does not hold for smaller values of β.

Proof. Since the case $\gamma = 0$ ($\mu = 0$ and $\nu = \alpha$) corresponds to Lemma 3(ii) [45, p. 121], it is sufficient to consider only the case $\gamma > 0$.

Let

$$H(z) = (1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma) f'(z) + \gamma z f''(z).$$

Since $\mu + \nu = \alpha - \gamma$ and $\mu\nu = \gamma$, then

$$\begin{aligned} H(z) &= (1 + \gamma - (\alpha - \gamma)) \frac{f(z)}{z} + (\alpha - \gamma - \gamma) f'(z) + \gamma z f''(z) \\ &= (1 + \mu\nu - \mu - \nu) \frac{f(z)}{z} + (\mu + \nu - \mu\nu) f'(z) + \mu\nu z f''(z). \end{aligned}$$

With $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, it follows from (4.4) that

$$H(z) = 1 + \sum_{n=1}^{\infty} a_{n+1}(n\nu + 1)(n\mu + 1)z^n = f'(z) * \phi_{\mu,\nu}(z), \tag{4.16}$$

and (4.5) yields

$$f'(z) = H(z) * \psi_{\mu,\nu}(z). \tag{4.17}$$

Let g be given by

$$g(z) = \frac{H(z) - \beta}{1 - \beta}.$$

Since $\operatorname{Re} e^{i\phi} g(z) > 0$, the Duality Principle allows us to assume that

$$g(z) = \frac{1 + xz}{1 + yz} \quad (|x| = 1,\ |y| = 1). \tag{4.18}$$

Now, (4.17) implies that $f'(z) = [(1-\beta)g(z) + \beta] * \psi_{\mu,\nu}(z)$, and (4.18) readily gives

$$\frac{f(z)}{z} = \frac{1}{z}\int_0^z \left((1-\beta)\frac{1 + xw}{1 + yw} + \beta\right) dw * \psi(z), \tag{4.19}$$

where for convenience, we write $\psi := \psi_{\mu,\nu}$.

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then (4.9) states that

$$F \in \mathcal{ST} \Longleftrightarrow \frac{1}{z}(F * h)(z) \neq 0 \quad (z \in \mathcal{U}),$$

where h is given by (4.10). Now $F \in \mathcal{CV}$ if and only if $zF' \in \mathcal{ST}$, and so

$$\begin{aligned}
0 &\neq \frac{1}{z}(zF'(z) * h(z)) \\
&= \frac{1}{z}(F(z) * zh'(z)) = \frac{1}{z}\left[\int_0^1 \lambda(t)\frac{f(tz)}{t}dt * zh'(z)\right] \\
&= \int_0^1 \frac{\lambda(t)}{1 - tz}dt * \frac{f(z)}{z} * h'(z).
\end{aligned}$$

From (4.19), it follows that

$$\begin{aligned}
0 &\neq \int_0^1 \frac{\lambda(t)}{1-tz}dt * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw * \psi(z)\right] * h'(z) \\
&= \int_0^1 \frac{\lambda(t)}{1-tz}dt * h'(z) * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw\right] * \psi(z) \\
&= \int_0^1 \lambda(t)h'(tz)dt * (1-\beta)\left[\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw + \frac{\beta}{1-\beta}\right] * \psi(z) \\
&= (1-\beta)\left[\int_0^1 \lambda(t)h'(tz)dt + \frac{\beta}{1-\beta}\right] * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw * \psi(z).
\end{aligned}$$

Theorem 1.14 (p. 21) states that the dual set of functions g given by (4.18) consists of analytic functions p satisfying $p(0) = 1$ and $\operatorname{Re} p(z) > 1/2$ in $\mathcal{U}$. Hence

$$\begin{aligned}
&0 \neq (1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt + \frac{\beta}{1-\beta}\right] * \frac{1+xz}{1+yz} * \psi(z) \\
\Longleftrightarrow &\operatorname{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt + \frac{\beta}{1-\beta}\right] * \psi(z) > \frac{1}{2} \\
\Longleftrightarrow &\operatorname{Re}(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt + \frac{\beta}{1-\beta} - \frac{1}{2(1-\beta)}\right] * \psi(z) > 0.
\end{aligned}$$

Using (4.12), the latter condition is equivalent to

$$\operatorname{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw - q(t)\right)dt\right] * \psi(z) > 0. \tag{4.20}$$

From (4.5), the above inequality is equivalent to

$$\operatorname{Re}\left[\int_0^1 \lambda(t)\left(1+\sum_{n=1}^{\infty} B_{n+1}(tz)^n - q(t)\right)dt\right] * \left(1+\sum_{n=1}^{\infty}\frac{n+1}{(n\nu+1)(n\mu+1)}z^n\right) > 0$$

where $h(z) = z + \sum_{n=2}^{\infty} B_n z^n$. Thus

$$\begin{aligned}
0 &< \operatorname{Re} \int_0^1 \lambda(t) \left(1 + \sum_{n=1}^{\infty} \frac{(n+1)B_{n+1}}{(n\nu+1)(n\mu+1)} (tz)^n - q(t) \right) dt \\
&= \operatorname{Re} \int_0^1 \lambda(t) \left(\sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * h'(tz) - q(t) \right) dt \\
&= \operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 - z\eta^{\nu}\zeta^{\mu}} * h'(tz) - q(t) \right) dt \\
&= \operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 h'(tz\eta^{\nu}\zeta^{\mu}) d\eta d\zeta - q(t) \right) dt,
\end{aligned}$$

which reduces to

$$\operatorname{Re} \int_0^1 \lambda(t) \left[\int_0^1 \int_0^1 \frac{1}{\mu\nu} h'(tzuv) u^{1/\nu-1} v^{1/\mu-1} dv du - q(t) \right] dt > 0.$$

A change of variable $w = tu$ leads to

$$\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left[\int_0^t \int_0^1 h'(wzv) w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} q(t) \right] dt > 0.$$

Integrating-by-parts with respect to t and using (4.6) gives the equivalent form

$$\operatorname{Re} \int_0^1 \Lambda_{\nu}(t) \left[\int_0^1 h'(tzv) t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 s^{1/\mu-1} \frac{1-st}{(1+st)^3} ds \right] dt > 0.$$

Making the variable change $w = vt$ and $\eta = st$ reduces the above inequality to

$$\operatorname{Re} \int_0^1 \Lambda_{\nu}(t) t^{1/\nu-1/\mu-1} \left[\int_0^t h'(wz) w^{1/\mu-1} dw - \int_0^t \eta^{1/\mu-1} \frac{1-\eta}{(1+\eta)^3} d\eta \right] dt > 0,$$

which after integrating-by-parts with respect to t yields

$$\operatorname{Re} \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0.$$

Thus $F \in \mathcal{CV}$ if and only if condition (4.15) holds.

To verify sharpness, let β_0 satisfy

$$\frac{\beta_0 - 1/2}{1-\beta_0} = -\int_0^1 \lambda(t)q(t)dt.$$

Assume that $\beta < \beta_0$ and let $f \in \mathcal{W}_\beta(\alpha,\gamma)$ be the solution of the differential equation

$$(1-\alpha+2\gamma)\frac{f(z)}{z} + (\alpha-2\gamma)f'(z) + \gamma z f''(z) = \beta + (1-\beta)\frac{1+z}{1-z}.$$

From (4.16), it follows that

$$f(z) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{(n\nu+1)(n\mu+1)} z^{n+1},$$

and

$$G(z) = \int_0^1 \lambda(t)\frac{f(tz)}{t}dt = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)}{(n\nu+1)(n\mu+1)}\left(\int_0^1 \lambda(t)t^n dt\right) z^{n+1}.$$

Thus

$$G(z) = V_\lambda(f)(z) = z + \sum_{n=1}^{\infty} \frac{2(1-\beta)\tau_n}{(n\nu+1)(n\mu+1)} z^{n+1},$$

where $\tau_n = \int_0^1 \lambda(t)t^n dt$. Now (4.7) implies that

$$\frac{\beta_0 - 1/2}{1-\beta_0} = -\int_0^1 \lambda(t)q(t)dt = -\sum_{n=1}^{\infty} \frac{(n+1)^2(-1)^n\tau_n}{(1+\mu n)(1+\nu n)}.$$

This means that

$$(zG')'\Big|_{z=-1} = 1 + 2(1-\beta)\sum_{n=1}^{\infty} \frac{(n+1)^2(-1)^n\tau_n}{(1+\mu n)(1+\nu n)} = 1 - \frac{1-\beta}{1-\beta_0} < 0.$$

Hence $(zG')'(z) = 0$ for some $z \in \mathcal{U}$, and so zG' is not even locally univalent in $\mathcal{U}$.

Therefore the value of β in (4.12) is sharp. ∎

Theorem 4.1 yields several known results. The case $\gamma = 0$ ($\mu = 0$ and $\nu = \alpha$) corresponds to the following result proved by Choi *et al.* [45].

Corollary 4.1 [45, Lemma 3(ii), p. 121] *Let* $\Lambda_\alpha(t)$ *be an integrable function on* $[0,1]$ *defined by*

$$\Lambda_\alpha(t) = \int_t^1 \frac{\lambda(x)}{x^{1/\alpha}}dx,$$

satisfying $\lim_{t\to 0}\Lambda_\alpha(t) = 0$. *Further, let* $1/2 \le \alpha \le 1$ *and* q_α *be defined by* (4.8). *Defined* $\beta < 1$ *by*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)q_\alpha(t)dt.$$

Then $V_\lambda(\mathcal{P}_\alpha(\beta)) \subset \mathcal{CV}$ *if and only if*

$$\mathrm{Re}\int_0^1 \Lambda_\alpha(t)t^{1/\alpha-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)dt > 0$$

holds.

Remark 4.1 *When* $\gamma = 0$, *then* $\mu = 0$, $\nu = \alpha$, *and in this particular instance, Theorem 4.1 gives Lemma 3(ii) in Choi* et al. *[45, p. 121]. There the range of* α *lies in* $[1/2, 1]$, *whereas the range of* α *in Theorem 4.1 for this particular case is* $\alpha > 0$.

The special case $\alpha = 1$ above yields a result of Ali and Singh [21, Theorem 1(ii), p. 301].

Corollary 4.2 [21, Theorem 1, p. 301] *Let* $f \in \mathcal{W}_\beta(1,0) = \mathcal{P}_\beta$ *and* $\beta < 1,$ *with*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)\frac{1}{(1+t)^2}dt.$$

and

$$\Lambda(t) = \int_t^1 \frac{\lambda(s)}{s} ds, \tag{4.21}$$

satisfies $t\Lambda \to 0$ *as* $t \to 0^+$. *Then*

$$\mathrm{Re} \int_0^1 \Lambda(t) \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt \geq 0, \tag{4.22}$$

where h *as defined in* (4.10) *if and only if* $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{CV}$. *The conclusion does not hold for smaller values of* β.

If $\alpha = 1 + 2\gamma$, then $\mu = 1$ and $\nu = \gamma$ for $\gamma > 0$, while $\mu = 0$ and $\nu = \alpha = 1$ for $\gamma = 0$. In this instance, Theorem 4.1 gives the following result.

Corollary 4.3 *Let* $f \in \mathcal{W}_\beta(1 + 2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, $\gamma \geq 0$, *and let* $\beta < 1$ *satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 \lambda(t) q_\gamma(t) dt,$$

where q_γ *is given by* (4.8). *Further let* Λ_γ *be defined by* (4.13),

$$\Pi_\gamma(t) = \begin{cases} \displaystyle\int_t^1 \Lambda_\gamma(s) s^{1/\gamma - 2} ds, & \gamma > 0, \\ \\ \displaystyle\int_t^1 \frac{\lambda(s)}{s} ds, & \gamma = 0, \end{cases} \tag{4.23}$$

and h *be given by* (4.10). *Then*

$$\mathrm{Re} \int_0^1 \Pi_\gamma(t) \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0$$

if and only if $F(z) = V_\lambda(f)(z)$ *is in* $\mathcal{CV}$. *The conclusion does not hold for smaller values of* β.

4.3 Sufficient Conditions for Convexity of Integral Transforms

The conditions stipulated in Theorem 4.1 can be cumbersome to use. A simpler sufficient condition for convexity of the integral operator (4.1) is now given in the following theorem. The following lemma of Fournier and Ruscheweyh [53] is required.

Lemma 4.1 [53, Theorem 1, p. 530] *Let* Λ *be integrable on* $[0,1]$, *and positive on* $(0,1)$. *If* $\Lambda(t)/(1-t^2)$ *is decreasing on (0,1), then* $L_\Lambda(\mathcal{CCV})=0$, *where*

$$L_\Lambda(f)=\inf_{z\in D}\int_0^1 \Lambda(t)\left(Re\ \frac{f(tz)}{tz}-\frac{1}{(1+t)^2}\right)dt \quad (f\in\mathcal{S})$$

and

$$L_\Lambda(\mathcal{CCV})=\inf_{f\in\mathcal{CCV}}L_\Lambda(f).$$

Theorem 4.2 *Let* $\Pi_{\mu,\nu}$ *and* Λ_ν *be given as in Theorem* 4.1. *Assume that both* $\Pi_{\mu,\nu}$ *and* Λ_ν *are integrable on* $[0,1]$, *and positive on* $(0,1)$. *Assume further that* $\mu\geq 1$ *and*

$$\frac{\Lambda_\nu(t)t^{1/\nu-1/\mu}+(1-1/\mu)\Pi_{\mu,\nu}(t)}{1-t^2} \qquad \textit{is decreasing on } (0,1). \tag{4.24}$$

If β *satisfies* (4.12), *and* $f\in\mathcal{W}_\beta(\alpha,\gamma)$, *then* $V_\lambda(f)\in\mathcal{CV}$.

Proof. Integrating-by-parts with respect to t yields

$$\begin{aligned}
&\text{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)dt\\
&=\text{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\frac{\partial}{\partial t}\left(\frac{h(tz)}{z}-\frac{t}{(1+t)^2}\right)dt\\
&=\text{Re}\int_0^1 t^{1/\mu-1}\left(\Lambda_\nu(t)t^{1/\nu-1/\mu}+\left(1-\frac{1}{\mu}\right)\Pi_{\mu,\nu}(t)\right)\left(\frac{h(tz)}{tz}-\frac{1}{(1+t)^2}\right)dt.
\end{aligned}$$

The function $t^{1/\mu-1}$ is decreasing on $(0,1)$ when $\mu\geq 1$. Thus, condition (4.24) along with Lemma 4.1 yield

$$\mathrm{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz)-\frac{1-t}{(1+t)^3}\right)dt>0.$$

The desired conclusion now follows from Theorem 4.1. ∎

Let us scrutinize Theorem 4.2 for helpful conditions to ensure convexity of $V_\lambda(f)$. Now for $\gamma>0$,

$$\Pi_{\mu,\nu}(t)=\int_t^1 \Lambda_\nu(y)y^{1/\nu-1-1/\mu}dy \quad \text{and} \quad \Lambda_\nu(t)=\int_t^1 \frac{\lambda(x)}{x^{1/\nu}}dx.$$

To apply Theorem 4.2, it is sufficient to show that the function

$$k(t)=\frac{\Lambda_\nu(t)t^{1/\nu-1/\mu}+(1-1/\mu)\Pi_{\mu,\nu}(t)}{1-t^2}:=\frac{p(t)}{1-t^2}$$

is decreasing in the interval $(0,1)$. Note that $k(t)>0$ and decreasing in the interval $(0,1)$ provided

$$q(t):=p(t)+\frac{1-t^2}{2}t^{-1}p'(t)\leq 0.$$

Since $q(1)=0$, this will certainly hold if q is increasing in $(0,1)$. Now

$$q'(t)=\frac{1-t^2}{2}t^{-2}[tp''(t)-p'(t)],$$

and

$$\begin{aligned} tp''(t)-p'(t)=&-\lambda(t)t^{-1/\mu}\left(\frac{1}{\nu}-\frac{1}{\mu}-2+\frac{t\lambda'(t)}{\lambda(t)}\right)\\ &+\left(\frac{1}{\nu}-\frac{1}{\mu}-2\right)\left(\frac{1}{\nu}-1\right)t^{1/\nu-1/\mu-1}\Lambda_\nu(t). \end{aligned}$$

Thus $tp''(t) - p'(t)$ is nonnegative if

$$\frac{1}{\nu} - \frac{1}{\mu} - 2 + \frac{t\lambda'(t)}{\lambda(t)} \leq 0 \text{ and } \left(\frac{1}{\nu} - \frac{1}{\mu} - 2\right)\left(\frac{1}{\nu} - 1\right) \geq 0. \tag{4.25}$$

For $\mu \geq 1$, the condition (4.3) implies $\nu \geq \mu \geq 1$. Thus condition (4.25) is equivalent to

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1).$$

These observations result in the following theorem.

Theorem 4.3 *Let λ be a nonnegative real-valued integrable function on $[0,1]$. Assume that Λ_ν and $\Pi_{\mu,\nu}$ given by (4.13) and (4.14) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 4.1, if λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1), \tag{4.26}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

Remark 4.2 *The condition $\mu \geq 1$ is equivalent to $0 < \gamma \leq \alpha \leq 2\gamma + 1$.*

Taking $\alpha = 1 + 2\gamma$, $\gamma > 0$ and $\mu = 1$ in Theorem 4.14 yields the following result.

Corollary 4.4 *Let λ be a nonnegative real-valued integrable function on $[0,1]$. Let $f \in \mathcal{W}_\beta(1+2\gamma,\gamma) = \mathcal{R}_\beta(\gamma),\ \gamma \in [1,\infty)$, and let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t) q_\gamma(t) dt,$$

where q_γ is given by (4.8). Assume further that $\Pi_{1,\gamma}$ and Λ_γ are integrable on $[0,1]$ and positive on $(0,1)$. If λ satisfies

$$\frac{t\lambda'(t)}{\lambda(t)} \leq 3 - \frac{1}{\gamma},$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

In the case $\gamma = 0$ and $\alpha \geq 1$ ($\mu = 0, \nu = \alpha$), an easier sufficient condition for convexity of the integral operator (4.1) is obtained in the following theorem.

Theorem 4.4 *Let λ be a nonnegative real-valued integrable function on $[0,1]$. Assume that Λ_α and $\Pi_{0,\alpha}$ given by (4.13) and (4.14) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 4.1, if $\lambda(1) = 0$ and λ satisfies*

$$t\lambda''(t) - \frac{1}{\alpha}\lambda'(t) > 0 \quad (\alpha \geq 1), \tag{4.27}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{CV}$. The conclusion does not hold for smaller values of β.

Proof. From Theorem 4.1, it suffices to show that

$$\mathrm{Re} \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0 \quad (\gamma = 0).$$

Integrating-by-parts with respect to t yields

$$\begin{aligned} &\mathrm{Re} \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt \\ &= \mathrm{Re} \int_0^1 t^{1/\alpha - 1} \left(t^{1-1/\alpha}\lambda(t) + \left(1 - \frac{1}{\alpha}\right) \Lambda_\alpha(t) \right) \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2} \right) dt. \end{aligned}$$

The function $t^{1/\alpha - 1}$ is decreasing on $(0,1)$ when $\alpha \geq 1$. Thus, the condition

$$\frac{t^{1-1/\alpha}\lambda(t) + \left(1 - \frac{1}{\alpha}\right) \Lambda_\alpha(t)}{1 - t^2} \qquad \text{is decreasing on } (0,1)$$

along with Lemma 4.1 will yield

$$\mathrm{Re} \int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0.$$

Let $p(t) = k(t)/(1-t^2)$, where $k(t) = t^{1-1/\alpha}\lambda(t) + (1-1/\alpha)\Lambda_\alpha(t)$. Taking the logarithmic derivative of p and using the fact that $p(t) > 0$ for $\alpha \geq 1$, the condition $p'(t) \leq 0$ in $(0,1)$ is equivalent to the inequality

$$q(t) = k(t) + t^{-1}k'(t)\frac{1-t^2}{2} \leq 0.$$

Clearly $q(1) = 0$ and if q is increasing in $(0,1)$, then p will be decreasing in $(0,1)$. Direct computations show that $q'(t) \geq 0$ provided $(t^{-1}k'(t))' \geq 0$. Since

$$(t^{-1}k'(t))' = t^{-1/\alpha-1}\left(t\lambda''(t) - \frac{1}{\alpha}\lambda'(t)\right),$$

the desired result follows from (4.27). ∎

4.4 Applications to Integral Transforms

The integral operators has been investigated extensively in Section 1.9. In this section, various well-known integral operators are considered, and conditions for convexity for $f \in \mathcal{W}_\beta(\alpha,\gamma)$ under these integral operators are obtained. First let λ be defined by

$$\lambda(t) = (1+c)t^c \quad (c > -1).$$

Then the integral transform

$$F_c(z) = V_\lambda(f)(z) = (1+c)\int_0^1 t^{c-1}f(tz)dt \quad (c > -1), \tag{4.28}$$

is the Bernardi integral operator. The classical Alexander and Libera transforms are special cases of (4.28) with $c = 0$ and $c = 1$ respectively. For this special case of λ, the following result holds.

Theorem 4.5 *Let $c > -1$, $0 < \gamma \leq \alpha \leq 1 + 2\gamma$, and $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -(c+1) \int_0^1 t^c q(t) dt,$$

where q is given by (4.7). *If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function*

$$V_\lambda(f)(z) = (1+c) \int_0^1 t^{c-1} f(tz) dt$$

belongs to $\mathcal{CV}$ provided

$$c \leq \; 2 + \frac{1}{\mu} - \frac{1}{\nu} \qquad (\nu \geq \mu \geq 1).$$

The value of β is sharp.

Proof. With $\lambda(t) = (1+c)t^c$, then $t\lambda'(t)/\lambda(t) = c$, and the result readily follows from Theorem 4.14. ∎

When $\alpha = 1 + 2\gamma$, $\gamma > 0$, and $\mu = 1$, Theorem 4.28 yields the following result.

Corollary 4.5 *Let $-1 < c \leq 3 - 1/\gamma$, $\gamma \in [1, \infty)$, and $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1 - \beta} = -(c+1) \int_0^1 t^c q_\gamma(t) dt,$$

where q_γ is given by (4.8). *If $f \in \mathcal{W}_\beta(1 + 2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, then the function*

$$V_\lambda(f)(z) = (1+c) \int_0^1 t^{c-1} f(tz) dt$$

belongs to $\mathcal{CV}$. The value of β is sharp.

The case $c = 0$ in Theorem 4.5 yields the following interesting result, which we state as a theorem.

Theorem 4.6 *Let $0 < \gamma \leq \alpha \leq 1 + 2\gamma$. If $F \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left(F'(z) + \alpha z F''(z) + \gamma z^2 F'''(z)\right) > \beta$$

in $\mathcal{U}$, and $\beta < 1$ satisfies

$$\frac{\beta - 1/2}{1 - \beta} = -\int_0^1 q(t)dt,$$

where q is given by (4.7)*, then F is convex. The value of β is sharp.*

Proof. It is evident that the function $f = zF'$ belongs to the class

$$\mathcal{W}_{\beta,0}(\alpha,\gamma) = \Bigg\{ f \in \mathcal{A} : \\ \mathrm{Re}\left((1 - \alpha + 2\gamma)\frac{f(z)}{z} + (\alpha - 2\gamma)f'(z) + \gamma z f''(z)\right) > \beta, \quad z \in \mathcal{U} \Bigg\}.$$

Thus

$$F(z) = \int_0^1 \frac{f(tz)}{t} dt.$$

The conditions on α and γ imply that $1 \leq \mu \leq \nu$. Thus the result now follows from Theorem 4.5 with $c = 0$. It is also evident from the proof of sharpness in Theorem 4.1 that the extremal function in $\mathcal{W}_\beta(\alpha,\gamma)$ indeed also belongs to the class $\mathcal{W}_{\beta,0}(\alpha,\gamma)$. ∎

Example 4.1 *If $\gamma = 1$, $\alpha = 3$, then $\mu = 1 = \nu$. In this case,* (4.7) *yields* $\beta = (1 - 2\ln 2)/2(1 - \ln 2) = -0.629445$. *Thus*

$$\mathrm{Re}\left(f'(z) + 3zf''(z) + z^2 f'''(z)\right) > \beta \Rightarrow f \in \mathcal{CV}.$$

Theorem 4.7 *Let $b > -1,\ a > -1,$ and $0 < \gamma \leq \alpha \leq 2\gamma + 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)q(t)dt,$$

where q is given by (4.7) and

$$\lambda(t) = \begin{cases} (a+1)(b+1)\frac{t^a(1-t^{b-a})}{b-a}, & b \neq a, \\ (a+1)^2 t^a \log(1/t), & b = a. \end{cases} \tag{4.29}$$

If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then

$$\mathcal{G}_f(a,b;z) = \begin{cases} \frac{(a+1)(b+1)}{b-a}\int_0^1 t^{a-1}(1-t^{b-a})f(tz)dt, & b \neq a, \\ \\ (a+1)^2 \int_0^1 t^{a-1}\log(1/t)f(tz)dt, & b = a, \end{cases} \tag{4.30}$$

belongs to $\mathcal{CV}$ provided

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1). \tag{4.31}$$

The value of β is sharp.

Proof. It is seen that $\int_0^1 \lambda(t)dt = 1$. There are two cases to consider. When $b \neq a$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(b-a)t^{b-a}}{1-t^{b-a}}.$$

The function λ satisfies (4.43) if

$$a - \frac{(b-a)t^{b-a}}{1-t^{b-a}} \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1). \tag{4.32}$$

Since $t \in (0,1)$, the condition $b > a$ implies $(b-a)t^{b-a}/(1-t^{b-a}) > 0$, and so inequality (4.32) holds true whenever a satisfies (4.31). When $b < a$, then $(a-b)/(t^{a-b}-1) < b-a$, and hence $a-(b-a)t^{b-a}/(1-t^{b-a}) < b < a$, and thus (4.32) holds if a satisfies (4.31).

For the case $b = a$, then

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{1}{\log(1/t)}.$$

Since $t < 1$ implies $1/\log(1/t) \geq 0$, condition (4.43) is satisfied provided a satisfies (4.31). This completes the proof. ∎

The simpler condition (4.43) can also be applied to the choice

$$\lambda(t) = \frac{(1+a)^p}{\Gamma(p)} t^a \left(\log\left(\frac{1}{t}\right)\right)^{p-1} \qquad (a > -1,\ p \geq 0).$$

The integral transform V_λ in this case takes the form

$$V_\lambda(f)(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)dt \quad (a > -1,\ p \geq 0).$$

This is the Komatu operator, which reduces to the Bernardi integral operator when $p = 1$. For this λ, the following result holds.

Theorem 4.8 *Let $a > p-2 \geq -1$, and $0 < \gamma \leq \alpha \leq 2\gamma + 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \left(\log\left(\frac{1}{t}\right)\right)^{p-1} q(t)dt,$$

where q is given by (4.7). For $f \in \mathcal{W}_\beta(\alpha,\gamma)$, the function

$$\Phi_p(a;z) * f(z) = \frac{(1+a)^p}{\Gamma(p)} \int_0^1 \left(\log\left(\frac{1}{t}\right)\right)^{p-1} t^{a-1} f(tz)dt \tag{4.33}$$

belongs to $\mathcal{CV}$ provided

$$a \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1). \tag{4.34}$$

The value of β is sharp.

Proof. Brief computations show that

$$\frac{t\lambda'(t)}{\lambda(t)} = a - \frac{(p-1)}{\log(1/t)}.$$

Since $\log(1/t) > 0$ for $t \in (0,1)$, and $p \geq 1$, condition (4.43) is satisfied whenever a satisfies (4.34). ∎

We next apply Theorem 4.4 to the case $\lambda(1) = 0$ as shown by the following two theorems.

Theorem 4.9 *Suppose $a > -1,\ b > -1,\ \alpha \geq 1$ are related by*

(1) $-1 < a \leq 0$ and $a = b$, or

(2) $-1 < a \leq 0$ and $-1 < a < b \leq 1 + 1/\alpha$.

Let $\beta < 1$ satisfy

$$\frac{\beta - 1/2}{1-\beta} = -\int_0^1 \lambda(t)q(t)dt,$$

where q and λ are given by (4.8) *and* (4.29) *respectively. If $f \in \mathcal{W}_\beta(\alpha, 0) = \mathcal{P}_\beta(\alpha)$, then $\mathcal{G}_f(a,b;z)$ defined by* (4.30) *belongs to $\mathcal{CV}$. The value of β is sharp.*

Proof. To apply Theorem 4.4, it suffices to verify inequality (4.27) for λ defined by (4.29). It is seen that

$$\lambda'(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a} t^{a-1}(a - bt^{b-a}), & b > a, \\ (a+1)^2 \left(-1 + a\log(\frac{1}{t})\right) t^{a-1}, & b = a, \end{cases}$$

and

$$t\lambda''(t) = \begin{cases} \frac{(a+1)(b+1)}{b-a} t^{a-1}(a(a-1) - b(b-1)t^{b-a}), & b > a, \\ (a+1)^2 \left(1 - 2a + a(a-1)\log(\frac{1}{t})\right) t^{a-1}, & b = a. \end{cases}$$

Case (i). Let $b = a > -1$. Substituting the expression for λ' and $t\lambda''$ in (4.27) yields the equivalent condition

$$a \log\left(\frac{1}{t}\right)\left(a - \frac{1}{\alpha} - 1\right) + \left(-2a + \frac{1}{\alpha} + 1\right) \geq 0.$$

This clearly holds for $t \in (0,1)$ whenever $-1 < a \leq \min\{0, 1+1/\alpha, (1+1/\alpha)/2\} = 0$.

Case (ii). Let $b > a > -1$ with $a \in (-1,0]$ and $-1 < b \leq 1 + 1/\alpha$. In this case, condition (4.27) is equivalent to $\psi_t(a) \geq \psi_t(b)$, where

$$\psi_t(a) = a(a-1)t^a - \frac{1}{\alpha} a t^a.$$

For a fixed t,

$$\psi_t'(a) = t^a \left(2a - 1 - \frac{1}{\alpha} - a^2 \log\left(\frac{1}{t}\right) + a\left(1 + \frac{1}{\alpha}\right)\log\left(\frac{1}{t}\right)\right),$$

that is, $\psi_t'(a) \leq 0$ for $a \in (-1,0)$. Thus, $\psi_t(a)$ is a decreasing function of a for each fixed $t \in (0,1)$. In particular, for $b > a$ with $b \in (-1,0)$ and $a \in (-1,0)$, inequality (4.27) holds. When $b > a$ with $0 < b \leq 1 + 1/\alpha$, then $\psi_t(a) \geq \psi_t(0) = 0$ for each fixed $t \in (0,1)$. For $0 < b \leq 1 + 1/\alpha$,

$$\psi_t(b) = bt^b \left(b - 1 - \frac{1}{\alpha}\right) \leq 0.$$

It follows then that $\psi_t(a) \geq 0 \geq \psi_t(b)$ holds for each fixed $t \in (0,1)$. Thus, inequality (4.27) holds for $b > a > -1$ with $a \in (-1,0]$ and $0 < b \leq 1 + 1/\alpha$. ∎

Remark 4.3 *The conditions $b > -1$ and $a > -1$ in Theorem 4.9 yield several improvements of known results.*

(1) Taking $\gamma = 0$ and $\alpha \geq 1$ in Theorem 4.9 leads to a result extending Theorem 3.4 obtained in [28, p. 12] for the case $\alpha \in [1/2, 1]$. When $\alpha = 1$, the range of b there lies in $(-1, 0]$, whereas the range of b in Theorem 4.9 lies in the larger interval $(-1, 2]$.

(2) With $\alpha = 1$ above leads to improvement of a result obtained by Ponnusamy and Rønning [152, Corollary 3.2]. There the parameters $b > a > -1$ must satisfy a fairly complicated equation to deduce $\mathcal{G}_f(a, b; z)$ is starlike of order 1/2. In our present situation, the conditions on the parameters a and b are simpler to infer convexity.

For $\alpha = 1$, $\gamma = 0$, $a = -\eta$ and $b = -\eta + 2$, Theorem 4.9 reduces to Corollary 1 [21, p. 302] and Corollary 1 in [47, pp. 915-916, $(\gamma = 0)$].

Corollary 4.6 [21, Corollary 1, p. 302] *Let $1 \geq \eta \geq 0$ and $\beta < 1$ be given by*

$$\frac{\beta - 1/2}{1 - \beta} = -\frac{(1-\eta)(3-\eta)}{2} \int_0^1 t^{-\eta} \frac{1-t}{1+t} dt. \tag{4.35}$$

Then for $f \in \mathcal{W}_\beta(1, 0) = \mathcal{P}_\beta$ the function

$$V_\lambda(f)(z) = \frac{(1-\eta)(3-\eta)}{2} \int_0^1 t^{-\eta-1}(1-t^2) f(tz) dt \tag{4.36}$$

belongs to $\mathcal{CV}$. The value of β is sharp.

Now, let Φ be defined by $\Phi(1-t) = 1 + \Sigma_{n=1}^{\infty} b_n (1-t)^n$, $b_n \geq 0$ for $n \geq 1$, and

$$\lambda(t) = K t^{b-1} (1-t)^{c-a-b} \Phi(1-t), \tag{4.37}$$

where K is a constant chosen such that $\int_0^1 \lambda(t) dt = 1$.

Theorem 4.10 *Let $a, b, c > 0$ and $\alpha \geq 1$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -K \int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t)q(t)dt,$$

where q is given by (4.8), and K is a constant such that $K\int_0^1 t^{b-1}(1-t)^{c-a-b}\Phi(1-t) = 1$. If $f \in \mathcal{W}_\beta(\alpha, 0)$, then the function

$$V_\lambda(f)(z) = K \int_0^1 t^{b-2}(1-t)^{c-a-b}\Phi(1-t)f(tz)dt$$

belongs to $\mathcal{CV}$ provided

$$c \geq a+b+1 \quad \textit{and} \quad 0 < b \leq 1.$$

The value of β is sharp.

Proof. As in the earlier proof, it suffices to verify inequality (4.27). Consider λ given by (4.37). Direct computations show that

$$\begin{aligned}\lambda'(t) = Kt^{b-2}(1-t)^{c-a-b-1}\\ &\Big(\big((b-1)(1-t)-(c-a-b)t\big)\Phi(1-t) - t(1-t)\Phi'(1-t)\Big),\end{aligned}$$

and

$$\begin{aligned}t\lambda''(t) =& Kt^{b-2}(1-t)^{c-a-b-2}\Big(\big((b-1)(b-2)(1-t)^2\\ &-2(b-1)(c-a-b)t(1-t)+(c-a-b)(c-a-b-1)t^2\big)\Phi(1-t)\\ &+\big(2(c-a-b)t-2(b-1)(1-t)\big)t(1-t)\Phi'(1-t)+t^2(1-t)^2\Phi''(1-t)\Big).\end{aligned}$$

Thus, (4.27) is satisfied provided

$$\psi(t) = \Phi(1-t)X(t) + t(1-t)\Phi'(1-t)Y(t) + t^2(1-t)^2\Phi''(1-t) \geq 0,$$

where

$$\begin{aligned}X(t) &= (1-t)^2(b-1)\left(-\frac{1}{\alpha}+b-2\right)-(c-a-b)t(1-t)\left(-\frac{1}{\alpha}+2b-2\right)\\&\quad+(c-a-b)(c-a-b-1)t^2,\\Y(t) &= 2(c-a-b)t+(1-t)\left(\frac{1}{\alpha}-2b+2\right).\end{aligned}$$

Since $\Phi(1-t) = 1+\Sigma_{n=1}^{\infty} b_n(1-t)^n$, $b_n \geq 0$ for $n \geq 1$, the functions $\Phi(1-t)$, $\Phi'(1-t)$ and $\Phi''(1-t)$ are nonnegative for $t \in (0,1)$. Therefore, it suffices to show

$$X(t) \geq 0 \quad \text{and} \quad Y(t) \geq 0,$$

and these evidently hold provided $c \geq a+b+1$ and $0 < b \leq \min\{1, 2+1/\alpha, (2+1/\alpha)/2\} = 1$. ∎

Remark 4.4 *For* $\gamma = 0$ *and* $\alpha \geq 1$, *Theorem* 4.10 *extends Theorem 3.1 in [28, p. 9, (*$\mu = 0$*)] for* $\alpha \in [1/2, 1]$. *When* $\alpha = 1$, *the range of* b *obtained in [28] lies in the interval* $(0, 1/2]$, *whereas the range of* b *obtained in Theorem 4.10 for this particular case lies in* $(0, 1]$.

Remark 4.5 *As shown in [28], choosing*

$$\Phi(1-t) = F\left(c-a, 1-a, c-a-b+1; 1-t\right)$$

gives

$$K = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}.$$

In this case, $V_\lambda(f)$ reduces to the Hohlov operator given by

$$\begin{aligned} V_\lambda(f)(z) &= H_{a,b,c}(f)(z) = zF(a,b;c;z) * f(z) \\ &= K\int_0^1 t^{b-2}(1-t)^{c-a-b} F\left(c-a, 1-a, c-a-b+1; 1-t\right) f(tz) dt, \end{aligned}$$

where $a > 0$, $b > 0$, and $c-a-b+1 > 0$. In the case $\gamma = 0$ and $\alpha \geq 1$, Theorem 4.10 extends Theorem 1 in [45, p. 122] and Theorem 3.2 in [28, p. 11]. When $\alpha = 1$, the range of b obtained in [28] lies in the interval $(0, 1/2]$, whereas the range of b in Theorem 4.10 lies in $(0, 1]$. This result improves as well Theorem 1 obtained by Choi et al. *[45] for the particular case $\alpha = 1$.*

In particular, for $a = 1$, $b = a$ and $c = a + b$, Theorem 4.10 yields Corollary 2 in [21, p. 302] and Corollary 2 in [47, p. 916, ($\gamma = 0$)].

Corollary 4.7 [21, Corollary 2, p. 302] *For $0 < a < 1$, $b > 2$ and $f \in \mathcal{P}_\beta$, let*

$$F(z) = zF(1, a; a+b; z) * f(z),$$

and $\beta < 1$ defined by

$$\frac{\beta - \frac{1}{2}}{1-\beta} = -\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1} \frac{dt}{(1+t)^2}.$$

Then F is convex. The result does not holds for smaller β.

Choosing now (see [28, Theorem 3.3, p.12])

$$\Phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1}$$

in Theorem 4.10 yields the following interesting result, which we state as a theorem.

Theorem 4.11 *Let $-1 < a \le 0$, $\alpha \ge 1$, and $p \ge 2$. Let $\beta < 1$ satisfy*

$$\frac{\beta - 1/2}{1-\beta} = -\frac{(1+a)^p}{\Gamma(p)} \int_0^1 t^a \left(\log\frac{1}{t}\right)^{p-1} q(t)dt,$$

where q is given by (4.8). If $f \in \mathcal{W}_\beta(\alpha, 0)$, then the function $\Phi_p(a; z) * f(z)$ defined by (4.33) belongs to $\mathcal{CV}$. The value of β is sharp.

Proof. Choose

$$\Phi(1-t) = \left(\frac{\log(1/t)}{1-t}\right)^{p-1},$$

$c - a - b = p - 1$ and $b = a + 1$ so that λ defined by equation (4.37) takes the form

$$\lambda(t) = Kt^a(1-t)^{p-1}\Phi(1-t), \quad K = \frac{(1+a)^p}{\Gamma(p)}.$$

The desired result now follows from Theorem 4.10. ∎

Remark 4.6 *For the particular case $\alpha = 1$, this result improves Theorem 3.3 by Balasubramanian* et al. *[28], where the range of a obtained there has been improved from $(-1, -1/2]$ to $(-1, 0]$.*

4.5 A Generalized Integral Operator

In [21], Ali and Singh generalized the operator (4.1) and considered

$$\mathcal{V}_\lambda(f)(z) := \rho z + (1-\rho)V_\lambda(f) = z \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * f(z) \quad (\rho < 1). \tag{4.38}$$

In this final section, Theorem 4.1 is generalized to obtain conditions on λ such that $\mathcal{V}_\lambda(f)$ is starlike or convex for $f \in \mathcal{W}_\beta(\alpha, \gamma)$. As an application, the results obtained will be applied for $\lambda(t) = (1+c)t^c$. Specifically, Starlikeness and convexity of a generalization of the Bernardi transform are investigated.

Theorem 4.12 *Let $\Pi_{\mu,\nu}$ and Λ_ν be given as in Theorem 4.1. Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on $[0,1]$, and positive on $(0,1)$. Assume further that $\rho < 1$ and $f \in \mathcal{A}$. Let $\beta < 1$ satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\,(1-q(t))\,dt, \tag{4.39}$$

where q is the solution of the initial-value problem (4.6). Then

$$\begin{cases} \operatorname{Re} \displaystyle\int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0, \ \gamma > 0, \\ \\ \operatorname{Re} \displaystyle\int_0^1 \Pi_{0,\alpha}(t) t^{1/\alpha - 1} \left(h'(tz) - \frac{1-t}{(1+t)^3} \right) dt > 0, \ \gamma = 0, \end{cases}$$

if and only if $\mathcal{V}_\lambda(f)$ is in $\mathcal{CV}$ for $f \in \mathcal{W}_\beta(\alpha,\gamma)$. The conclusion does not hold for smaller values of β.

Proof. Let $f \in \mathcal{W}_\beta(\alpha,\gamma)$. In view of the fact stated in (4.9) that

$$\mathcal{V}_\lambda(f) \in \mathcal{ST} \Longleftrightarrow \frac{1}{z}(\mathcal{V}_\lambda(f) * h)(z) \neq 0 \quad (z \in \mathcal{U}),$$

where h is given by (4.10), and $\mathcal{V}_\lambda(f) \in \mathcal{CV}$ if and only if $z\mathcal{V}_\lambda'(f) \in \mathcal{ST}$, the condition becomes

$$\begin{aligned} 0 &\neq \frac{1}{z}(z\mathcal{V}_\lambda'(f) * h(z)) \\ &= \frac{1}{z}(\mathcal{V}_\lambda(f) * zh'(z)) = \frac{1}{z}\left[z \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * f(z) dt * zh'(z) \right] \\ &= \int_0^1 \lambda(t) \frac{1-\rho tz}{1-tz} dt * \frac{f(z)}{z} * h'(z). \end{aligned}$$

From (4.19), it follows that

$$
\begin{aligned}
0 &\neq \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw * \psi(z)\right] * h'(z)\\
&= \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * h'(z) * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw\right] * \psi(z)\\
&= \int_0^1 \lambda(t)(1-\rho)\left[h'(tz)+\frac{\rho}{1-\rho}\right]dt * (1-\beta)\left[\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw+\frac{\beta}{1-\beta}\right] * \psi(z)\\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)h'(tz)dt+\frac{\rho}{(1-\rho)(1-\beta)}+\frac{\beta}{1-\beta}\right]\\
&\qquad * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw * \psi(z)\\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt+\frac{\rho}{(1-\rho)(1-\beta)}+\frac{\beta}{1-\beta}\right]\\
&\qquad * \frac{1+xz}{1+yz} * \psi(z)\\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt+\frac{1}{(1-\rho)(1-\beta)}-1\right]\\
&\qquad * \frac{1+xz}{1+yz} * \psi(z).
\end{aligned}
$$

Theorem 1.14 (p. 21) shows that the set of functions g given by (4.18) has a dual set which includes the analytic functions p satisfying $p(0)=1$ and $\operatorname{Re} p(z) > 1/2$

in $\mathcal{U}$. Hence

$$\begin{aligned}
0 \neq &(1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt + \frac{1}{(1-\rho)(1-\beta)} - 1\right] \\
&* \frac{1+xz}{1+yz} * \psi(z) \\
\Longleftrightarrow &\mathrm{Re}\,(1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt + \frac{1}{(1-\rho)(1-\beta)} - 1\right] \\
&* \psi(z) > 1/2 \\
\Longleftrightarrow &\mathrm{Re}\,(1-\rho)(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt - (1 - \frac{1}{2(1-\rho)(1-\beta)})\right] \\
&* \psi(z) > 0 \\
\Longleftrightarrow &\mathrm{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw\right)dt - (1 - \frac{1}{2(1-\rho)(1-\beta)})\right] * \psi(z) > 0.
\end{aligned}$$

Using (4.39), this condition is equivalent to

$$\mathrm{Re}\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z h'(tw)dw - q(t)\right)dt\right] * \psi(z) > 0,$$

which as proved in Theorem 4.1, after changing variable and integrating-by-parts with respect to t yields

$$\mathrm{Re}\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)dt > 0.$$

∎

Taking $\gamma = 0$ and $\alpha > 0$, Theorem 4.12 leads to Theorem 2.5 obtained by Balasubramanian *et al.* in [28] for the case $\alpha \in [1/2, 1]$, and it reduces to Theorem 3 obtained by Ali and Singh in [21] when $\gamma = 0$, $\alpha = 1$.

Corollary 4.8 [21, Theorem3, p. 303] *Let* λ, Λ *be given as in Corollary* 4.2. *Assume that* Λ *is integrable on* $[0,1]$, *and positive on* $(0,1)$. *Assume further that*

$\rho < 1$ and $f \in \mathcal{A}$. Let $\beta < 1$ satisfy

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\frac{t(2+t)}{(1+t)^2}dt.$$

Then (4.22) holds if and only if $\mathcal{V}_\lambda(f)$ is in $\mathcal{CV}$ for $f \in \mathcal{W}_\beta(1,0) = \mathcal{P}_\beta$. The conclusion does not hold for smaller values of β.

Theorem 4.13 *Let $\Pi_{\mu,\nu}$ and Λ_ν be given as in Theorem 4.1. Assume that both $\Pi_{\mu,\nu}$ and Λ_ν are integrable on $[0,1]$, and positive on $(0,1)$. Assume further that $\rho < 1$ and $f \in \mathcal{A}$. Let β satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\left(\frac{1-g(t)}{2}\right)dt, \tag{4.40}$$

where g is the solution of the initial-value problem

$$\frac{d}{dt}t^{1/\nu}(1+g(t)) = \begin{cases} \dfrac{2}{\mu\nu}t^{1/\nu-1}\displaystyle\int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2}ds, & \gamma > 0, \\ \dfrac{2}{\alpha}\dfrac{t^{1/\alpha-1}}{(1+t)^2}, & \gamma = 0,\ \alpha > 0, \end{cases} \tag{4.41}$$

with $g(0) = 1$. Then

$$\begin{cases} \operatorname{Re}\displaystyle\int_0^1 \Pi_{\mu,\nu}(t)t^{1/\mu-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt > 0,\ \gamma > 0, \\ \\ \operatorname{Re}\displaystyle\int_0^1 \Pi_{0,\alpha}(t)t^{1/\alpha-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt > 0,\ \gamma = 0, \end{cases} \tag{4.42}$$

if and only if $\mathcal{V}_\lambda(f)$ is in $\mathcal{ST}$ for $f \in \mathcal{W}_\beta(\alpha,\gamma)$. The conclusion does not hold for smaller values of β.

Proof. Let $f \in \mathcal{W}_\beta(\alpha,\gamma)$. From (4.9) follows that $\mathcal{V}_\lambda(f) \in \mathcal{ST}$ if and only if

$(\mathcal{V}_\lambda(f)(z) * h(z))/z \neq 0$ where h is given by (4.10), hence the condition becomes

$$0 \neq \frac{1}{z}(\mathcal{V}_\lambda(f) * h(z)) = \frac{1}{z}\left[z\int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * f(z)dt * h(z)\right]$$
$$= \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * \frac{f(z)}{z} * \frac{h(z)}{z}.$$

From (4.19), it is evident that

$$\begin{aligned}
0 &\neq \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw * \psi(z)\right] * \frac{h(z)}{z} \\
&= \int_0^1 \lambda(t)\frac{1-\rho tz}{1-tz}dt * \frac{h(z)}{z} * \left[\frac{1}{z}\int_0^z \left((1-\beta)\frac{1+xw}{1+yw}+\beta\right)dw\right] * \psi(z) \\
&= \int_0^1 \lambda(t)(1-\rho)\left[\frac{h(tz)}{tz}+\frac{\rho}{1-\rho}\right]dt * (1-\beta)\left[\frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw + \frac{\beta}{1-\beta}\right] * \psi(z) \\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\frac{h(tz)}{tz}dt + \frac{\rho}{(1-\rho)(1-\beta)} + \frac{\beta}{1-\beta}\right] \\
&\qquad * \frac{1}{z}\int_0^z \frac{1+xw}{1+yw}dw * \psi(z) \\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{\rho}{(1-\rho)(1-\beta)} + \frac{\beta}{1-\beta}\right] \\
&\qquad * \frac{1+xz}{1+yz} * \psi(z) \\
&= (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{1}{(1-\rho)(1-\beta)} - 1\right] \\
&\qquad * \frac{1+xz}{1+yz} * \psi(z).
\end{aligned}$$

By Theorem 1.14 (p. 21), it follows that

$$\begin{aligned}
0 \neq & (1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{1}{(1-\rho)(1-\beta)} - 1\right] \\
& * \frac{1+xz}{1+yz} * \psi(z) \\
\Longleftrightarrow & \mathrm{Re}\,(1-\beta)(1-\rho)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt + \frac{1}{(1-\rho)(1-\beta)} - 1\right] \\
& * \psi(z) > 1/2 \\
\Longleftrightarrow & \mathrm{Re}\,(1-\rho)(1-\beta)\left[\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt - (1 - \frac{1}{2(1-\rho)(1-\beta)})\right] \\
& * \psi(z) > 0
\end{aligned}$$

To complete the proof, it suffices to verify

$$\mathrm{Re}\int_0^1 \lambda(t)\left[\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw\right)dt - \left(1 - \frac{1}{2(1-\rho)(1-\beta)}\right)\right] * \psi(z) > 0.$$

Substituting the value of β given by (4.40), the latter condition is equivalent to

$$\mathrm{Re}\left(\int_0^1 \lambda(t)\left(\frac{1}{z}\int_0^z \frac{h(tw)}{tw}dw - \frac{1+g(t)}{2}\right)dt\right) * \psi(z) > 0.$$

From (4.5), the above inequality is equivalent to

$$\mathrm{Re}\left[\int_0^1 \lambda(t)\left(1 + \sum_{n=1}^{\infty} B_{n+1}\frac{(tz)^n}{n+1} - q(t)\right)dt\right] * \left(1 + \sum_{n=1}^{\infty} \frac{n+1}{(n\nu+1)(n\mu+1)} z^n\right) > 0$$

where $h(z) = z + \sum_{n=2}^{\infty} B_n z^n$. Thus

$$
\begin{aligned}
0 < \operatorname{Re} \int_0^1 \lambda(t) \left(1 + \sum_{n=1}^{\infty} \frac{B_{n+1}}{(n\nu+1)(n\mu+1)} (tz)^n - \frac{1+g(t)}{2}\right) dt \\
= \operatorname{Re} \int_0^1 \lambda(t) \left(\sum_{n=0}^{\infty} \frac{z^n}{(n\nu+1)(n\mu+1)} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2}\right) dt \\
= \operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{d\eta d\zeta}{1 - z\eta^\nu \zeta^\mu} * \frac{h(tz)}{tz} - \frac{1+g(t)}{2}\right) dt \\
= \operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{h(tz\eta^\nu\zeta^\mu)}{tz\eta^\nu\zeta^\mu} d\eta d\zeta - \frac{1+g(t)}{2}\right) dt,
\end{aligned}
$$

which reduces to

$$
\operatorname{Re} \int_0^1 \lambda(t) \left(\int_0^1 \int_0^1 \frac{1}{\mu\nu} \frac{h(tzuv)}{tzuv} u^{1/\nu-1} v^{1/\mu-1} dv du - \frac{1+g(t)}{2}\right) dt > 0.
$$

A change of variable $w = tu$ leads to

$$
\operatorname{Re} \int_0^1 \frac{\lambda(t)}{t^{1/\nu}} \left(\int_0^t \int_0^1 \frac{h(wzv)}{wzv} w^{1/\nu-1} v^{1/\mu-1} dv dw - \mu\nu t^{1/\nu} \frac{1+g(t)}{2}\right) dt > 0.
$$

Integrating by parts with respect to t and using (4.41) gives the equivalent form

$$
\operatorname{Re} \int_0^1 \Lambda_\nu(t) \left(\int_0^1 \frac{h(tzv)}{tzv} t^{1/\nu-1} v^{1/\mu-1} dv - t^{1/\nu-1} \int_0^1 \frac{s^{1/\mu-1}}{(1+st)^2} ds\right) dt > 0.
$$

Making the variable change $w = vt$ and $\eta = st$ reduces the above inequality to

$$
\operatorname{Re} \int_0^1 \Lambda_\nu(t) t^{1/\nu-1/\mu-1} \left(\int_0^t \frac{h(wz)}{wz} w^{1/\mu-1} dw - \int_0^t \frac{\eta^{1/\mu-1}}{(1+\eta)^2} d\eta\right) dt > 0,
$$

which after integrating by parts with respect to t yields

$$
\operatorname{Re} \int_0^1 \Pi_{\mu,\nu}(t) t^{1/\mu-1} \left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right) dt > 0.
$$

Thus $F \in \mathcal{ST}$ if and only if condition (4.42) holds. ∎

Taking $\gamma = 0$ and $\alpha > 0$, Theorem 4.13 leads to Theorem 2.4 obtained by Balasubramanian *et al.* in [28] for the case $\alpha \in [1/2, 1]$, and to Theorem 2 obtained by Ali and Singh in [21] and Theorem 3 in [47, p. 916] for the case $\gamma = 0$, $\alpha = 1$.

Corollary 4.9 [21, Theorem 3, p. 303] *Let λ, Λ be given as in Corollary 4.2. Assume that Λ is integrable on $[0,1]$, and positive on $(0,1)$. Assume further that $\rho < 1$ and $f \in \mathcal{A}$. Let $\beta < 1$ satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = \int_0^1 \lambda(t)\frac{t}{1+t}dt.$$

Then

$$\mathrm{Re}\int_0^1 \Lambda(t)t^{1/\alpha-1}\left(\frac{h(tz)}{tz} - \frac{1}{(1+t)^2}\right)dt > 0$$

holds if and only if $\mathcal{V}_\lambda(f)$ is in $\mathcal{ST}$ for $f \in \mathcal{W}_\beta(1,0) = \mathcal{P}_\beta$. The conclusion does not hold for smaller values of β.

For the case $\alpha = 1 + 2\gamma$, Theorem 4.12 reduces to the following result.

Corollary 4.10 *Let $f \in \mathcal{W}_\beta(1+2\gamma, \gamma) = \mathcal{R}_\gamma(\beta)$, $\gamma \geq 0$, and let $\beta < 1$ satisfy* (4.39)*. Assume Λ_γ is defined by* (4.13) *and Π_γ is defined by* (4.23)*. Then*

$$\mathrm{Re}\int_0^1 \Pi_\gamma(t)\left(h'(tz) - \frac{1-t}{(1+t)^3}\right)dt > 0,$$

where h is given by (4.10)*, if and only if $F(z) = \mathcal{V}_\lambda(f)(z)$ is in $\mathcal{CV}$. The conclusion does not hold for smaller values of β.*

Now the results obtained are applied to the generalization of the Bernadri transform. Choosing $\lambda(t) = (1+c)t^c$, Theorem 4.12 leads to the following corollary.

Corollary 4.11 *Let $c > -1$, $\gamma > 0$, $\rho < 1$, and $\beta < 1$ satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = (1+c)\int_0^1 t^c\,(1-q(t))\,dt,$$

where q is given by (4.6). If $f \in \mathcal{W}_\beta(\alpha, \gamma)$, then the function

$$\mathcal{V}_\lambda(f)(z) = \rho z + (1-\rho)(1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{CV}$ provided

$$c \leq 2 + \frac{1}{\mu} - \frac{1}{\nu} \quad (\nu \geq \mu \geq 1).$$

The value of β is sharp.

The following sufficient condition for $f \in \mathcal{W}_\beta(\alpha, \gamma)$ to be starlike was obtained by Ali *et al.* [7, p. 816].

Theorem 4.14 [7, Theorem 4.2, p. 816] *Let λ be a non-negative real-valued integrable function on $[0,1]$. Assume that Λ_ν and $\Pi_{\mu,\nu}$ given respectively by (4.13) and (4.14) are both integrable on $[0,1]$, and positive on $(0,1)$. Under the assumptions stated in Theorem 4.1, if λ satisfies*

$$\frac{t\lambda'(t)}{\lambda(t)} \leq \begin{cases} 1 + \dfrac{1}{\mu}, & \mu \geq 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (0, 1/3] \cup [1, \infty), \end{cases} \tag{4.43}$$

then $F(z) = V_\lambda(f)(z) \in \mathcal{ST}$. The conclusion does not hold for smaller values of β.

Using the sufficient condition (4.43) obtained by Ali *et al.* [7] and Choosing $\lambda(t) = (1+c)t^c$, Theorem 4.13 reduces to the following result.

Corollary 4.12 *Let $c > -1$, $\rho < 1$, and $\beta < 1$ satisfy*

$$\frac{1}{2(1-\beta)(1-\rho)} = (1+c)\int_0^1 t^c \left(\frac{1-g(t)}{2}\right) dt,$$

where g is given by (4.41). If $f \in \mathcal{W}_\beta(\alpha,\gamma)$, then the function

$$\mathcal{V}_\lambda(f)(z) = \rho z + (1-\rho)(1+c)\int_0^1 t^{c-1} f(tz)dt$$

belongs to $\mathcal{ST}$ provided

$$c \leq \begin{cases} 1 + \frac{1}{\mu}, & \mu \geq 1 \ (\gamma > 0), \\ 3 - \frac{1}{\alpha}, & \gamma = 0, \ \alpha \in (0, 1/3] \cap [1, \infty). \end{cases}$$

The value of β is sharp.

CHAPTER 5

COEFFICIENT CONDITION FOR STARLIKENESS AND CONVEXITY

5.1 Introduction

For $0 \le \alpha < 1$, let $\mathcal{ST}(\alpha)$ and $\mathcal{CV}(\alpha)$ be the class of starlike and convex functions of order α. Analytically,

$$\mathcal{ST}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \right\},$$

and

$$\mathcal{CV}(\alpha) := \left\{ f \in \mathcal{A} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \right\}.$$

Closely related are the classes of functions

$$\mathcal{ST}_\alpha := \left\{ f \in \mathcal{A} : \left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \alpha \right\},$$

and

$$\mathcal{CV}_\alpha := \left\{ f \in \mathcal{A} : \left|\frac{zf''(z)}{f'(z)}\right| < 1 - \alpha \right\},$$

introduced by Ruscheweyh [171]. Note that $\mathcal{ST}_\alpha \subseteq \mathcal{ST}(\alpha)$ and $\mathcal{CV}_\alpha \subseteq \mathcal{CV}(\alpha)$. For $\alpha \ge 0$, and $\beta < 1$, let

$$\mathcal{R}_\alpha(\beta) = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(f'(z) + \alpha z f''(z)\right) > \beta,\ z \in \mathcal{U} \right\}, \tag{5.1}$$

and $\mathcal{R}(\beta) := \mathcal{R}_1(\beta)$. In 1962, Krzyż [86] by a counter example showed that functions $f \in \mathcal{R}(0)$ are not necessarily convex. In 1977, Chichra [42] proved that $\mathcal{R}(0) \subset \mathcal{S}$, while in 1981, Singh and Singh [185] showed that $\mathcal{R}(0) \subset \mathcal{ST}$. Singh and Singh [186] proved that for $\beta \ge -1/4$, $\mathcal{R}(\beta) \subset \mathcal{ST}$. Ali [5] conjectured the best value of β is $\beta = (1 - 2\log 2)/(2 - 2\log 2) = -0.626$. In 1994, Fournier and

Ruscheweyh [53] applied the Duality Principle and verified Ali's conjecture [5] for β such that $\mathcal{R}(\beta) \subset \mathcal{ST}$. Ali and Singh [21] also investigated the sharp bound for β that ensures functions $f \in \mathcal{R}(\beta)$ are starlike of order γ.

Mocanu [113, 114] proved that

$$\begin{aligned}
&\operatorname{Re}\left(f'(z) + \frac{1}{2}zf''(z)\right) > 0 \Rightarrow \operatorname{Re}\frac{zf'(z)}{f(z)} > 0,\\
&\operatorname{Re}\left(f'(z) + zf''(z)\right) > 0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{3},\\
&\operatorname{Re}\left(f'(z) + \frac{1}{2}zf''(z)\right) > 0 \Rightarrow \left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{4\pi}{9}.
\end{aligned}$$

Sălăgean [176] improved the above mentioned results for the class $\mathcal{T}$ of all analytic functions with negative coefficients of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0). \tag{5.2}$$

We recall here the notations for subclasses of $\mathcal{T}$ as follows: $\mathcal{TST}(\alpha) = \mathcal{T} \cap \mathcal{ST}(\alpha)$, $\mathcal{TST}_\alpha = \mathcal{T} \cap \mathcal{ST}_\alpha$, $\mathcal{TCV}(\alpha) = \mathcal{T} \cap \mathcal{CV}(\alpha)$, and $\mathcal{TCV}_\alpha = \mathcal{T} \cap \mathcal{CV}_\alpha$. Several interesting results for functions $f \in \mathcal{T}$ were determined by Sălăgean [176]. For instance, Sălăgean [176] proved the following result.

Theorem 5.1 [176, Corollary 2.2] *If $f \in \mathcal{T}$, then*

$$\operatorname{Re}\left(f'(z) + \gamma z f''(z)\right) > \beta \Rightarrow \operatorname{Re}\frac{zf'(z)}{f(z)} > \delta,$$

where

$$\delta = \begin{cases} \frac{2(\beta+\gamma)}{2\gamma+\beta+1}, & \beta \in [-1, 0], \gamma > 1,\\ \frac{2\gamma}{2\gamma-\beta+1}, & \beta \in [0, 1), \gamma > 0. \end{cases}$$

Lewandowski *et al.* [92] investigated the class of analytic functions satisfying

$$\operatorname{Re}\left(\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}) \tag{5.3}$$

and proved that a function f satisfying (5.3) is starlike. For $\beta < 1$, $\alpha \geq 0$, a function $f \in \mathcal{A}$ belongs to the class $\mathcal{L}(\alpha, \beta)$ if it satisfies the inequality

$$\operatorname{Re}\left(\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \beta. \tag{5.4}$$

Clearly, $\mathcal{L}(0, \beta) = \mathcal{ST}(\beta)$. For $\beta \geq -\alpha/2$, Li and Owa [93] proved that $\mathcal{L}(\alpha, \beta) \subset \mathcal{ST}$. The class $\mathcal{L}(\alpha, \beta)$ has been extensively studied by Ramesha *et al.* [157], Obradović and Joshi [134], Nunokawa *et al.* [133], Padmanabhan [142], Liu *et al.* [97], and Ravichandran [162]. Ravichandran [160] also investigated functions f satisfying (5.4) using differential subordination.

The following sufficient condition for functions $f \in \mathcal{A}$ to belong to the class $\mathcal{L}(\alpha, \beta)$ is needed (Lemma 1.1).

Lemma 5.1 [97] *Let $\beta < 1$, and $\alpha \geq 0$. If $f \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| \leq 1 - \beta, \tag{5.5}$$

then $f \in \mathcal{L}(\alpha, \beta)$.

In Section 5.2, the largest bound for β is determined so that analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfying the inequality $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta$ is either starlike or convex of some positive order. In Section 5.3, a similar problem is investigated for functions f satisfying the coefficient inequality $\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| \leq 1 - \beta$. In fact, the sharp bound for the order of starlikeness and convexity of functions f satisfying the coefficient inequality $\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n-\beta\big)|a_n| \leq 1-\beta$ are obtained. In Section 5.4, the largest value is obtained that bounds each coefficient inequality of the form $\sum na_n$, $\sum n(n-1)a_n$, $\sum(n-1)a_n$ and $\sum n^2 a_n$ so that the function is starlike or convex of positive order. In the final section, the results obtained are applied to determine conditions on the parameters so that the hypergeometric functions belong to the class $\mathcal{L}(\alpha,\beta)$.

5.2 Sufficient Coefficient Estimates for Starlikeness and Convexity

In 1962, necessary and sufficient condition for a function to be starlike was proved by Merkes *et al.* [105, Theorem 2, p. 961] (Theorem 1.11). In 1975, Silverman [182, Theorem 1, p. 110] independently also obtained the same result and used it to investigate several problems on functions with negative coefficients. Necessary and sufficient conditions for convex functions follow by an application of Alexander's result, and it was proved in [182, Corollary 1, p. 110]. These results are stated in the following theorem.

Theorem 5.2 *Let* $0 \leq \alpha < 1$.

(1) If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfies the inequality*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| \leq 1-\alpha, \tag{5.6}$$

then $f \in \mathcal{ST}_\alpha$. *If* $a_n \leq 0$, *then* (5.6) *is also necessary for* $f \in \mathcal{ST}(\alpha)$.

(2) Similarly, if f *satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq 1-\alpha, \tag{5.7}$$

then $f \in \mathcal{CV}_\alpha$. *If* $a_n \leq 0$, *then* (5.7) *is necessary for* $f \in \mathcal{CV}(\alpha)$.

The following theorem provides a sufficient coefficient inequality for functions to be in the classes $\mathcal{CV}_\alpha$ or $\mathcal{ST}_\alpha$.

Theorem 5.3 *Let $\alpha \in [0, 1)$, and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfy the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \beta < 1. \tag{5.8}$$

(1) The function f belongs to the class $\mathcal{CV}_\alpha$ if $\beta \leq (1-\alpha)/(2-\alpha)$. The bound $(1-\alpha)/(2-\alpha)$ is sharp.

(2) The function f belongs to the class $\mathcal{ST}_\alpha$ if $\beta \leq 2(1-\alpha)/(2-\alpha)$. The bound $2(1-\alpha)/(2-\alpha)$ is sharp.

Proof. (1) Let f satisfy inequality (5.8) with $\beta \leq (1-\alpha)/(2-\alpha)$. Since

$$n - \alpha \leq (2-\alpha)(n-1) \tag{5.9}$$

for $n \geq 2$, inequality (5.8) leads to

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq (2-\alpha)\sum_{n=2}^{\infty} n(n-1)|a_n| \leq (2-\alpha)\beta \leq 1-\alpha.$$

Thus, it follows from Theorem 5.2 (2) that $f \in \mathcal{CV}_\alpha$. The function $f_0 : \mathcal{U} \to \mathbb{C}$ defined by

$$f_0(z) = z - \frac{1-\alpha}{2(2-\alpha)} z^2$$

satisfies the hypothesis of Theorem 5.2 and therefore $f_0 \in \mathcal{CV}_\alpha$. This function f_0 shows that the bound for β is sharp.

(2) Now, let f satisfy inequality (5.8) with $\beta \leq 2(1-\alpha)/(2-\alpha)$. When $n \geq 2$, inequality (5.9) leads to

$$(n-\alpha) \leq \frac{n(n-\alpha)}{2} \leq \frac{(2-\alpha)n(n-1)}{2},$$

and hence

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq \frac{(2-\alpha)}{2}\sum_{n=2}^{\infty}n(n-1)|a_n| \leq (1-\alpha).$$

By Theorem 5.2 (1), $f \in \mathcal{ST}_\alpha$. The function

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha}z^2 \in \mathcal{ST}_\alpha$$

shows that the result is sharp. ∎

Theorem 5.3 (1) will be applied to a certain class of uniformly convex functions. A function $f \in \mathcal{S}$ is *uniformly convex*, if f maps every circular arc γ contained in $\mathcal{U}$ with center $\zeta \in \mathcal{U}$ onto a convex arc. Goodman [62] introduced the class $\mathcal{UCV}$ and gave a two-variable analytic characterization for the class $\mathcal{UCV}$, while Rønning [167], and Ma and Minda [103] independently gave a one-variable characterization for $f \in \mathcal{UCV}$ by using the minimum principle for harmonic functions:

$$f \in \mathcal{UCV} \Leftrightarrow \left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (z \in \mathcal{U}).$$

In 1999, Kanas and Wisniowska [75,76] extended the class $\mathcal{UCV}$ by introducing the class $k-\mathcal{UCV}$ of k-uniformly convex functions. A function $f \in \mathcal{S}$ is *k-uniformly convex* ($k \geq 0$), if f maps every circular arc γ contained in $\mathcal{U}$ with center ζ, $|\zeta| \leq k$, onto a convex arc. Kanas and Wisniowska [75,76] showed that $f \in k-\mathcal{UCV}$ if and only if f satisfies the inequality

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \quad (0 \leq k < \infty,\ z \in \mathcal{U}).$$

This analytic characterization was used by Kanas and Wisniowska [75] to obtain the following sufficient condition for a function to be k-uniformly convex.

Corollary 5.1 [75, Theorem 3.3, p. 334] *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *satisfies the inequality*

$$\sum_{n=2}^{\infty} n(n-1)|a_n| \leq \frac{1}{k+2},$$

then $f \in k-\mathcal{UCV}$. *Further, the bound* $1/(k+2)$ *is sharp.*

Proof. From Theorem 5.3 (1), it follows that $f \in \mathcal{CV}_{k/(k+1)}$, and hence

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{1}{k+1}. \tag{5.10}$$

Inequality (5.10) yields

$$k\left|\frac{zf''(z)}{f'(z)}\right| < \frac{k}{k+1} = 1 - \frac{1}{k+1} < 1 - \left|\frac{zf''(z)}{f'(z)}\right| < 1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right),$$

and hence $f \in k-\mathcal{UCV}$. The result is sharp for the function $f_0 \in k-\mathcal{UCV}$ given by

$$f_0(z) = z - \frac{1}{2(k+2)}z^2.$$

∎

Remark 5.1 *The above result extended Goodman's [62, Theorem 6] case of* $k = 1$ *for functions to be k-uniformly convex. In the special case* $k = 0$, *Corollary 5.1 shows that the bound is 1/2 for functions to be convex.*

Alexander's relation shows that $f \in \mathcal{CV}_\alpha$ if and only if $zf' \in \mathcal{ST}_\alpha$, and Theorem 5.3 (1) now readily yields the following result.

Corollary 5.2 *Let* $\alpha \in [0, 1)$. *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ *and*

$$\sum_{n=2}^{\infty} (n-1)|a_n| \leq \frac{1-\alpha}{2-\alpha},$$

then $f \in \mathcal{ST}_\alpha$. Further, the bound $(1-\alpha)/(2-\alpha)$ is sharp.

The corollary above can also be deduced from Theorem 5.2 (1) and the inequality $n - \alpha \leq (2-\alpha)(n-1)$, $n \geq 2$.

The class $\mathcal{PST}$ of parabolic starlike functions of order $\alpha \in [0,1)$ consists of functions f satisfying the inequality

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\alpha + \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right).$$

A sufficient coefficient inequality condition for functions to be parabolic starlike is given in the following result.

Corollary 5.3 [6, Theorem 3.1, p. 564] *Let $\alpha \in [0,1)$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies the inequality $\sum_{n=2}^{\infty}(n-1)|a_n| \leq (1-\alpha)/(2-\alpha)$, then f is parabolic starlike of order α. The bound $(1-\alpha)/(2-\alpha)$ cannot be replaced by a larger number.*

Remark 5.2 *Corollary 5.3 for the class of parabolic starlike functions of order ρ was obtained by Ali [6, Theorem 3.1, p. 564] by using a two-variable characterization of a corresponding class of uniformly convex functions.*

Theorem 5.4 *Let $\alpha \in [0,1)$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$.*

(1) If $\sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha$, then $f \in \mathcal{ST}_\alpha$.

(2) If $\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha$, then $f \in \mathcal{CV}_\alpha$.

(3) If $\sum_{n=2}^{\infty} n^2|a_n| \leq 4(1-\alpha)/(2-\alpha)$, then $f \in \mathcal{ST}_\alpha$ and the bound $4(1-\alpha)/(2-\alpha)$ is sharp.

Proof. The first two parts follow from Theorem 5.2 and the simple inequality

$n - \alpha < n$. Indeed,

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq \sum_{n=2}^{\infty} n|a_n| \leq 1-\alpha \Rightarrow f \in \mathcal{ST}_\alpha,$$
$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| \leq \sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha \Rightarrow f \in \mathcal{CV}_\alpha.$$

The third follows from Theorem 5.2 (1) and use of the identity $(n-\alpha) \leq n^2(2-\alpha)/4 \quad (n \geq 2)$. In view of this, it follows that

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \leq \frac{2-\alpha}{4}\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\alpha \Rightarrow f \in \mathcal{ST}_\alpha.$$

The result is sharp as demonstrated by the function f_0 given by

$$f_0(z) = z - \frac{1-\alpha}{2-\alpha}z^2.$$ ∎

5.3 The Subclass $\mathcal{L}(\alpha, \beta)$

The following theorem provides sufficient coefficient conditions for functions to belong to either $\mathcal{L}(\alpha,\beta) \cap \mathcal{ST}_\eta$ or $\mathcal{L}(\alpha,\beta) \cap \mathcal{CV}_\eta$, $\beta < 1$, for an appropriate value η.

Theorem 5.5 *Let $\beta < 1$, and $f \in \mathcal{A}$ satisfy inequality* (5.5).

(1) The function f is in the class $\mathcal{ST}_\eta$ if $\eta \leq (2\alpha+\beta)/(2\alpha+1)$, $\alpha \geq 0$. The bound $(2\alpha+\beta)/(2\alpha+1)$ is sharp.

(2) The function f is in the class $\mathcal{CV}_\eta$ if $\eta \leq (\alpha - 1 + \beta)/\alpha$, $\alpha > 0$, $\beta \geq 0$.

Proof. (1) If $\eta \leq \eta_0 := (2\alpha+\beta)/(2\alpha+1)$, then $\mathcal{ST}_{\eta_0} \subset \mathcal{ST}_\eta$. Hence it is enough to prove that $f \in \mathcal{ST}_{\eta_0}$. The inequality

$$(2\alpha+1)n - 2\alpha \leq \alpha n^2 + (1-\alpha)n \quad (n \geq 2,\ \alpha \geq 0)$$

together with inequality (5.5) show that

$$\begin{aligned}
\sum_{n=2}^{\infty}(n-\eta_0)|a_n| &= \sum_{n=2}^{\infty}\frac{(2\alpha+1)n-2\alpha-\beta}{2\alpha+1}|a_n| \\
&\leq \sum_{n=2}^{\infty}\frac{\alpha n^2+(1-\alpha)n-\beta}{2\alpha+1}|a_n| \\
&\leq \frac{1-\beta}{2\alpha+1} = 1-\eta_0.
\end{aligned}$$

It is now evident from Theorem 5.2 (1) that $f \in \mathcal{ST}_{\eta_0}$. The result is sharp for the function $f_0 \in \mathcal{ST}_{\eta_0}$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta}z^2.$$

(2) If $\eta \leq \eta_0 := (\alpha-1+\beta)/\alpha$, then $\mathcal{CV}_{\eta_0} \subset \mathcal{CV}_{\eta}$. Hence it suffices to show $f \in \mathcal{CV}_{\eta_0}$. The inequality

$$\alpha n^2+(1-\alpha)n-n\beta \leq \alpha n^2+(1-\alpha)n-\beta \quad (n \geq 2,\ \beta \geq 0)$$

together with inequality (5.5) yield

$$\begin{aligned}
\sum_{n=2}^{\infty}n(n-\eta_0)|a_n| &= \frac{1}{\alpha}\sum_{n=2}^{\infty}\big(\alpha n^2+(1-\alpha)n-n\beta\big)|a_n| \\
&\leq \frac{1}{\alpha}\sum_{n=2}^{\infty}\Big(\alpha n^2+(1-\alpha)n-\beta\Big)|a_n| \\
&\leq \frac{1-\beta}{\alpha} = 1-\eta_0.
\end{aligned}$$

It follows now from Theorem 5.2(2) that $f \in \mathcal{CV}_{\eta_0}$. ∎

Along similar lines with Theorem 5.3, the following result provides a sufficient coefficient inequality for functions to belong to the class $\mathcal{L}(\alpha,\beta)$.

Theorem 5.6 *Let $\beta < 1$, and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$.*

(1) If f satisfies $\sum_{n=2}^{\infty} n(n-1)|a_n| \leq 2(1-\beta)/(2\alpha+2-\beta)$, $\alpha \geq 0$, then $f \in \mathcal{L}(\alpha,\beta)$. The bound $2(1-\beta)/(2\alpha+2-\beta)$ is sharp.

(2) Let $0 \leq \alpha \leq 1$ and $\eta \in \mathbb{R}$ be given by

$$\eta = \begin{cases} 4(1-\beta)/(3\alpha+1), & \alpha+\beta \geq 1, \\ 4(1-\beta)/(2\alpha+2-\beta), & \alpha+\beta \leq 1. \end{cases}$$

If f satisfies $\sum_{n=2}^{\infty} n^2|a_n| \leq \eta$, then $f \in \mathcal{L}(\alpha,\beta)$. The result is sharp for $\alpha+\beta \leq 1$.

Proof. (1) Since

$$2\alpha n^2 + 2(1-\alpha)n - 2\beta \leq (2\alpha+2-\beta)n(n-1) \quad (n \geq 2),$$

it follows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq \frac{1}{2}\sum_{n=2}^{\infty} n(n-1)(2\alpha+2-\beta)|a_n| \leq 1-\beta.$$

Lemma 5.1 now yields $f \in \mathcal{L}(\alpha,\beta)$. The result is sharp for the function $f_0 \in \mathcal{L}(\alpha,\beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha+2-\beta}z^2.$$

(2) Let $\alpha+\beta \geq 1$, and $\sum_{n=2}^{\infty} n^2|a_n| \leq 4(1-\beta)/(3\alpha+1)$. In this case, since

$$4\left(\alpha n^2 + (1-\alpha)n - \beta\right) \leq (3\alpha+1)n^2 \quad (n \geq 2),$$

it readily follows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq \frac{3\alpha+1}{4}\sum_{n=2}^{\infty} n^2|a_n| \leq 1-\beta.$$

Lemma 5.1 shows that $f \in \mathcal{L}(\alpha, \beta)$.

Now, let $\alpha + \beta \leq 1$ and $\sum_{n=2}^{\infty} n^2 |a_n| \leq 4(1-\beta)/(2\alpha + 2 - \beta)$. In this case, the inequality

$$4\left(\alpha n^2 + (1-\alpha)n - \beta\right) \leq n^2(2\alpha + 2 - \beta) \quad (n \geq 2)$$

shows that

$$\sum_{n=2}^{\infty} \left(\alpha n^2 + (1-\alpha)n - \beta\right)|a_n| \leq \frac{1}{4} \sum_{n=2}^{\infty} n^2(2\alpha + 2 - \beta)|a_n| \leq 1 - \beta,$$

and hence, Lemma 5.1 implies that $f \in \mathcal{L}(\alpha, \beta)$. The function $f_0 \in \mathcal{L}(\alpha, \beta)$ given by

$$f_0(z) = z - \frac{1-\beta}{2\alpha + 2 - \beta} z^2$$

demonstrates sharpness of the result. ∎

5.4 Functions with Negative Coefficients

For functions with negative coefficients, the next theorem proves the equivalence between the inequalities $\sum_{n=2}^{\infty} n(n-1)a_n \leq \beta$ and $|f''(z)| < \beta$.

Theorem 5.7 *Let $\beta > 0$, and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$. Then*

$$|f''(z)| \leq \beta \Longleftrightarrow \sum_{n=2}^{\infty} n(n-1)a_n \leq \beta.$$

Proof. The necessary condition follows by allowing $z \to 1^-$ in

$$|f''(z)| = \left|\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right| \leq \beta.$$

If f satisfies the coefficient inequality $\sum_{n=2}^{\infty} n(n-1)a_n \leq \beta$, then

$$|f''(z)| \leq \sum_{n=2}^{\infty} n(n-1)a_n|z|^{n-2} \leq \sum_{n=2}^{\infty} n(n-1)a_n \leq \beta.$$

∎

Remark 5.3 *It is known that functions $f \in \mathcal{A}$ satisfying the inequality $|f''(z)| \leq \beta$ for $0 < \beta \leq 1$are starlike, and if $|f''(z)| \leq \beta$ for $0 < \beta \leq 1/2$, then $f \in \mathcal{CV}$ [188, Theorem 1, p.1861].*

Theorem 5.8 *Let $0 \leq \alpha < 1$, and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$.*

(1) If $f \in \mathcal{TCV}(\alpha)$, then $\sum_{n=2}^{\infty} na_n \leq (1-\alpha)/(2-\alpha)$. The bound $(1-\alpha)/(2-\alpha)$ is sharp.

(2) If $f \in \mathcal{TCV}(\alpha)$, then $\sum_{n=2}^{\infty} n(n-1)a_n \leq 1-\alpha$.

(3) If $f \in \mathcal{TCV}(\alpha)$, then $\sum_{n=2}^{\infty}(n-1)a_n \leq (1-\alpha)/2(2-\alpha)$. The bound $(1-\alpha)/2(2-\alpha)$ is sharp.

(4) If $f \in \mathcal{TCV}(\alpha)$, then $\sum_{n=2}^{\infty} n^2 a_n \leq 2(1-\alpha)/(2-\alpha)$. The bound $2(1-\alpha)/(2-\alpha)$ is sharp.

Proof. The results follow from Theorem 5.2 (2).

(1) Since $2-\alpha \leq n-\alpha$ $(n \geq 2)$, it follows that

$$\sum_{n=2}^{\infty} na_n \leq \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{2-\alpha} a_n \leq \frac{1-\alpha}{2-\alpha}.$$

(2) From $n-1 \leq n-\alpha$ readily follows that

$$\sum_{n=2}^{\infty} n(n-1)a_n \leq \sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha.$$

(3) The inequality $2(2-\alpha)(n-1) \leq n(n-\alpha)$ shows that

$$\sum_{n=2}^{\infty}(n-1)a_n \leq \sum_{n=2}^{\infty}\frac{n(n-\alpha)}{2(2-\alpha)}a_n \leq \frac{1-\alpha}{2(2-\alpha)}.$$

(4) By using $n^2(2-\alpha) \leq 2n(n-\alpha)$, it follows that

$$\sum_{n=2}^{\infty} n^2 a_n \leq \sum_{n=2}^{\infty}\frac{2n(n-\alpha)}{2-\alpha}a_n \leq \frac{2(1-\alpha)}{2-\alpha}.$$

The results are sharp for the function f_0 given by

$$f_0(z) = z - \frac{1-\alpha}{2(2-\alpha)}z^2. \qquad \blacksquare$$

The Alexander's relation between $\mathcal{TCV}(\alpha)$ and $\mathcal{TST}(\alpha)$ readily yields the following corollary.

Corollary 5.4 *Let $0 \leq \alpha < 1$, and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{T}$.*

(1) If $f \in \mathcal{TST}(\alpha)$, then $\sum_{n=2}^{\infty} a_n \leq (1-\alpha)/(2-\alpha)$. The bound $(1-\alpha)/(2-\alpha)$ is sharp.

(2) If $f \in \mathcal{TST}(\alpha)$, then $\sum_{n=2}^{\infty}(n-1)a_n \leq 1-\alpha$.

(3) If $f \in \mathcal{TST}(\alpha)$, then $\sum_{n=2}^{\infty} n a_n \leq 2(1-\alpha)/(2-\alpha)$. The bound $2(1-\alpha)/(2-\alpha)$ is sharp.

Let $\mathcal{TL}(\alpha, \beta)$ be the subclass of $\mathcal{L}(\alpha, \beta)$ consisting of functions with negative coefficients. For functions in $\mathcal{TL}(\alpha, \beta)$, the following lemma holds.

Lemma 5.2 [97, Theorem 8, p.414] *Let $\beta < 1$, $\alpha \geq 0$, and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$. Then*

$$f \in \mathcal{TL}(\alpha, \beta) \Longleftrightarrow \sum_{n=2}^{\infty}\big(\alpha n^2 + (1-\alpha)n - \beta\big)a_n \leq 1-\beta.$$

Corollary 5.5 *Let $\beta < 1$, $\alpha > 0$ and $f \in \mathcal{TL}(\alpha, \beta)$.*

(1) The function $f \in \mathcal{TST}_\eta$ provided $\eta \leq (2\alpha+\beta)/(2\alpha+1)$, and the bound $(2\alpha+\beta)/(2\alpha+1)$ is sharp.

(2) The function $f \in \mathcal{TCV}_\eta$ provided $\eta \leq (\alpha - 1 + \beta)/\alpha$, $\beta \geq 0$.

Proof. The result follows from Lemma 5.2 and Theorem 5.5. ∎

The next result shows that $\mathcal{TCV}\big((2\alpha+3\beta-2)/(2\alpha+\beta)\big) \subset \mathcal{TL}(\alpha, \beta)$ for $0 \leq \beta < 1$, $\alpha \geq 0$.

Theorem 5.9 *Let $0 \leq \beta < 1$, and $\alpha \geq 0$. If $\eta \geq (2\alpha+3\beta-2)/(2\alpha+\beta)$, then $\mathcal{TCV}(\eta) \subseteq \mathcal{TL}(\alpha, \beta)$.*

Proof. For $\eta_0 \leq \eta$, $\mathcal{TCV}(\eta) \subset \mathcal{TCV}(\eta_0)$ and therefore it is sufficient to prove $\mathcal{TCV}(\eta_0) \subseteq \mathcal{TL}(\alpha, \beta)$ where $\eta_0 = (2\alpha+3\beta-2)/(2\alpha+\beta)$. For $n \geq 2$, the inequality

$$2\alpha n^2 + 2(1-\alpha)n - 2\beta \leq n\big((2\alpha+\beta)n - (2\alpha+3\beta-2)\big)$$

holds. Theorem 5.2 (2) for $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ $(a_n \geq 0)$ yields

$$\begin{aligned}
\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)a_n &\leq \frac{1}{2}\sum_{n=2}^{\infty} n\big((2\alpha+\beta)n - (2\alpha+3\beta-2)\big)a_n \\
&= \frac{2\alpha+\beta}{2}\sum_{n=2}^{\infty} n(n-\eta_0)a_n \\
&\leq \frac{2\alpha+\beta}{2}(1-\eta_0) \\
&= 1-\beta.
\end{aligned}$$

It is now evident from Lemma 5.2 that $f \in \mathcal{TL}(\alpha, \beta)$. ∎

Theorem 5.10 *Let $\beta < 1$, and $f(z) = z - \sum_{n=2}^{\infty} a_n z^n \in \mathcal{TL}(\alpha, \beta)$. Then*

(1) $\sum_{n=2}^{\infty} n(n-1)a_n < (1-\beta)/\alpha$ when $\alpha > 0$;

(2) $\sum_{n=2}^{\infty}(n-1)a_n \le \eta$ *where* $\eta = (1-\beta)/(1-\alpha),\ \ \beta \le 3\alpha + 1,\ \ 0 \le \alpha < 1.$

(3) For $0 \le \alpha \le 1$*, and*

$$\eta = \begin{cases} (1-\beta)/\alpha, & \beta \le 2(1-\alpha), \alpha > 0 \\ 4(1-\beta)/(2\alpha+2-\beta), & \beta \ge 2(1-\alpha), \beta \ge 0, \alpha > 1/2 \end{cases}$$

then $\sum_{n=2}^{\infty} n^2 a_n \le \eta$*. The result for* $\beta > 2(1-\alpha)$ *is sharp.*

(4) $\sum_{n=2}^{\infty} n a_n \le 2(1-\beta)/(2\alpha+2-\beta),\ \alpha, \beta \ge 0$*. The result is sharp.*

Proof. The equivalence in Lemma 5.2 between $f \in \mathcal{TL}(\alpha, \beta)$ and

$$\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big) a_n \le 1 - \beta$$

is used throughout the proof of this theorem.

(1) Since

$$\alpha n(n-1) \le \alpha n^2 + (1-\alpha)n - \beta \quad (n \ge 2),$$

it readily follows that

$$\sum_{n=2}^{\infty} n(n-1)a_n < \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{\alpha} a_n \le \frac{1-\beta}{\alpha}.$$

(2) If $\beta \le 3\alpha + 1$, then

$$(n-1)(1-\alpha) \le \alpha n(n-1) + n - \beta \quad (n \ge 2),$$

and use of this inequality shows that

$$\sum_{n=2}^{\infty}(n-1)a_n \le \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{1-\alpha} a_n \le \frac{1-\beta}{1-\alpha}.$$

(3) If $\beta \le 2(1-\alpha)$, the inequality

$$\alpha n^2 \le \alpha n^2 + 2(1-\alpha) - \beta \le \alpha n^2 + n(1-\alpha) - \beta$$

shows that

$$\sum_{n=2}^{\infty} n^2 a_n \le \sum_{n=2}^{\infty} \frac{\alpha n^2 + (1-\alpha)n - \beta}{\alpha} a_n \le \frac{1-\beta}{\alpha}.$$

In the case $\beta \ge 2(1-\alpha)$, the inequality

$$n^2(2\alpha + 2 - \beta) \le 4(\alpha n^2 + (1-\alpha)n - \beta) \quad (n \ge 2),$$

readily gives

$$\sum_{n=2}^{\infty} n^2 a_n \le \sum_{n=2}^{\infty} \frac{4(\alpha n^2 + (1-\alpha)n - \beta)}{2\alpha + 2 - \beta} a_n \le \frac{4(1-\beta)}{2\alpha + 2 - \beta}.$$

(4) For $\alpha, \beta \ge 0$, the inequality

$$(2\alpha + 2 - \beta)n \le 2\big(\alpha n^2 + (1-\alpha)n - \beta\big)$$

shows that

$$\sum_{n=2}^{\infty} n a_n \le \sum_{n=2}^{\infty} \frac{2\big(\alpha n^2 + (1-\alpha)n - \beta\big)}{2\alpha + 2 - \beta} a_n \le \frac{2(1-\beta)}{2\alpha + 2 - \beta}.$$

The sharpness can be seen by considering the function f_0 given by

$$f(z) = z - \frac{1-\beta}{2\alpha + 2 - \beta} z^2 \in \mathcal{TL}(\alpha, \beta).$$

$\blacksquare$

5.5 Applications to Gaussian Hypergeometric Functions

In this section, appropriate theorems from the previous sections are applied to the Gaussian hypergeometric functions. For $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$, the *Gaussian hypergeometric function* is defined by

$$F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \cdots ,$$

where $(\lambda)_n$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (n = 0, 1, 2, \ldots).$$

This is equivalent to $(\lambda)_n = \lambda(\lambda+1)_{n-1}, \ (\lambda)_0 = 1$ defined on page 29. The series converges absolutely in $\mathcal{U}$. It also converges on $|z| = 1$ when $\mathrm{Re}(c-a-b) > 0$. For $\mathrm{Re}(c - a - b) > 0$, the value of the hypergeometric function $F(a, b; c; z)$ at $z = 1$ is related to the Gamma function by the Gauss summation formula

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \quad (c \neq 0, -1, -2, \cdots). \tag{5.11}$$

By making use of Theorem 5.2, Silverman [183] determined conditions on a, b, c so that the function $zF(a, b; c; z)$ belongs to certain subclasses of starlike and convex functions. In this section, conditions on the parameters a, b, c are determined so that the function $zF(a, b; c; z)$ belongs to the class $\mathcal{L}(\alpha, \beta)$. Similar results holds for other classes of functions investigated in this chapter. The proof follows by applying appropriate theorems from the previous sections, and the Gauss summation formula for the Gaussian hypergeometric functions. The method of proof is similar to those of Silverman [183], and Kim and Ponnusamy [82]. The following Gauss summation formula for the Gaussian hypergeometric functions is required.

Lemma 5.3 [2, Lemma 10, p.169] *Let a, b, $c > 0$.*

(1) For $c > a + b + 1$,

$$\sum_{n=1}^{\infty} n \frac{(a)_n (b)_n}{(c)_n (1)_n} = \frac{ab}{c - a - b - 1} F(a, b; c; 1).$$

(2) For $c > a + b + 2$,

$$\sum_{n=1}^{\infty} n^2 \frac{(a)_n (b)_n}{(c)_n (1)_n} = \left(\frac{(a)_2 (b)_2}{(c - a - b - 2)_2} + \frac{ab}{c - a - b - 1} \right) F(a, b; c; 1).$$

Theorem 5.11 *Let $a, b \in \mathbb{C}$ and $c \in \mathbb{R}$ satisfy $c > |a| + |b| + 2$. If either*

(1) for $\alpha \geq 0$, $\beta < 1$,

$$F(|a|, |b|; c; 1) \left(\frac{(|a|)_2 (|b|)_2}{(c - |a| - |b| - 2)_2} + \frac{2|ab|}{c - |a| - |b| - 1} \right) \leq \frac{2(1 - \beta)}{2\alpha + 2 - \beta}, \tag{5.12}$$

or

(2) for $1 - \alpha \geq \beta$, $\alpha \in [0, 1]$,

$$F(|a|, |b|; c; 1) \left(\frac{(|a|)_2 (|b|)_2}{(c - |a| - |b| - 2)_2} + \frac{3|ab|}{c - |a| - |b| - 1} + 1 \right) \leq \frac{6 - 5\beta + 2\alpha}{2\alpha + 2 - \beta}, \tag{5.13}$$

then the function $zF(a, b; c; z) \in \mathcal{L}(\alpha, \beta)$. In the case $b = \overline{a}$, the range of c in either case can be improved to $c > \max\{0, 2(1 + \operatorname{Re} a)\}$.

Proof. **Case 1.** For $\alpha \geq 0$, $\beta < 1$, it follows from the fact $|(a)_n| \leq (|a|)_n$ and

Lemma 5.3 that

$$\begin{aligned}
\sum_{n=2}^{\infty} n(n-1)\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| &\leq \sum_{n=2}^{\infty} n(n-1)\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= F(|a|,|b|;c;1) \\
&\times \left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{2|ab|}{c-|a|-|b|-1}\right) \\
&\leq \frac{2(1-\beta)}{2\alpha+2-\beta},
\end{aligned}$$

and Theorem 5.6 (1) shows that $zF(a,b;c;z) \in \mathcal{R}(\alpha,\beta)$.

Case 2. Let $1-\alpha \geq \beta$, $\alpha \in [0,1]$. From Lemma 5.3 follows that

$$\begin{aligned}
\sum_{n=2}^{\infty} n^2\left|\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\right| &\leq \sum_{n=2}^{\infty} n^2\frac{(|a|)_{n-1}(|b|)_{n-1}}{(c)_{n-1}(1)_{n-1}} \\
&= F(|a|,|b|;c;1) \\
&\times \left(\frac{(|a|)_2(|b|)_2}{(c-|a|-|b|-2)_2} + \frac{3|ab|}{c-|a|-|b|-1} + 1\right) - 1 \\
&\leq \frac{4(1-\beta)}{2\alpha+2-\beta}.
\end{aligned}$$

The result follows from Theorem 5.6 (2). For $b = \overline{a}$, the proof is similar to the previous proof, and is therefore omitted. ∎

CHAPTER 6

SUBORDINATION OF LINEAR OPERATORS SATISFYING A RECURRENCE RELATION

6.1 Introduction

This chapter deals with the class $\mathcal{A}_m$ of all analytic functions f of the form

$$f(z) = z^m + \sum_{k=m+1}^{\infty} a_k z^k \quad (z \in \mathcal{U}), \tag{6.1}$$

with $\mathcal{A} := \mathcal{A}_1$. Linear operators on $\mathcal{A}_m$ continued to be of considerable interest, and widely studied. Many of these investigations rely on the differential recurrence relation inherited by the operators. For example, the Carlson-Shaffer operator [40] $L_m(a,c)$ on $\mathcal{A}_m$ defined by

$$L_m(a,c)f(z) := z^m + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} a_{k+m} z^{k+m}$$

satisfies the first-order differential recurrence relation

$$z(L_m(a,c)f(z))' = aL_m(a+1,c)f(z) - (a-m)L_m(a,c)f(z).$$

Here $(a)_k$ is the Pochhammer symbol given by $(a)_k = a(a+1)_{k-1}$, $(a)_0 = 1$ (p. 29). The multiplier transformation operator and the Dziok-Srivastava are other examples with a similar first-order differential recurrence relation [18, 49].

Consider now the class of all linear operators satisfying a certain first-order differential recurrence relation. Specifically, let $\mathcal{O}_m$ be the class of linear operators $L_a : \mathcal{A}_m \to \mathcal{A}_m$ $(a \in \mathbb{C})$ satisfying the first-order differential recurrence relation

$$z[L_a f(z)]' = c_a L_{a+1} f(z) - (c_a - m) L_a f(z) \tag{6.2}$$

for some $c_a \in \mathbb{C}$.

This chapter aims to show that the class of functions defined through each linear operator L_a in $\mathcal{O}_m$ can be given a unified treatment. The class $\mathcal{O}_m$ contains various operators. These include the Bernardi-Libera-Livingston operator, the Ruscheweyh derivative operator, the Carlson-Shaffer operator, the Hohlov operator, the multiplier transformation as well as several other operators introduced by various authors [17–19].

The Dziok-Srivastava operator [49, 190] $H_m^{l,j}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_j; z) : \mathcal{A}_m \to \mathcal{A}_m$ is defined by

$$H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)f(z) := z^m + \sum_{k=m+1}^{\infty} \frac{(\alpha_1)_{k-m} \cdots (\alpha_l)_{k-m}}{(\beta_1)_{k-m} \cdots (\beta_j)_{k-m}} \frac{a_k z^k}{(k-m)!}. \tag{6.3}$$

It is known [49] that

$$\begin{aligned} z[H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)f(z)]' &= \alpha_1 H_m^{l,j}(\alpha_1 + 1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)f(z) \\ &\quad -(\alpha_1 - m)H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)f(z). \end{aligned} \tag{6.4}$$

Clearly, if $a = c_a = \alpha_1$, then $L_a f(z) = H_m^{l,j}(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_j; z)$ is the Dziok-Srivastava operator defined in (6.3). Special cases of the Dziok-Srivastava linear operator are the Hohlov linear operator [72], the Carlson-Shaffer linear operator [40], the Ruscheweyh derivative operator [169], the generalized Bernardi-Libera-Livingston linear integral operator [34, 94, 98], and the Srivastava-Owa fractional derivative operator [135, 137].

The multiplier transformation $I_m(n, \lambda)$ on $\mathcal{A}_m$ given by

$$I_m(n, \lambda)f(z) := z^m + \sum_{k=m+1}^{\infty} \left(\frac{k+\lambda}{m+\lambda}\right)^n a_k z^k, \tag{6.5}$$

satisfies

$$z[I_m(n,\lambda)f(z)]' = (m+\lambda)I_m(n+1,\lambda)f(z) - \lambda I_m(n,\lambda)f(z). \tag{6.6}$$

The multiplier transformation $I_m(n,\lambda)f(z)$ defined in (6.5) is a special case of $L_a f$ when $a = n$ and $c_a = m + \lambda$. Note that in this case a and c_a are independent.

For α real, a function $f \in \mathcal{A}$ is *α-convex* if

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > 0.$$

Miller and Mocanu [111, p. 10] have shown that α-convex functions are starlike for $0 \leq \alpha \leq 1$. This result is in fact a particular case of the general subordination implication

$$\phi\left(\frac{zf'(z)}{f(z)}, 1 + \frac{zf''(z)}{f'(z)}, S(f,z)\right) \prec h(z) \Rightarrow \frac{zf'(z)}{f(z)} \prec q(z),$$

investigated by Miller and Mocanu [107], and Ali *et al.* [17]. Here $S(f,z)$ denotes the Schwarzian derivative of f.

In the sequel, we shall consider second-order differential subordination satisfied by the class of functions defined through a linear operator in $\mathcal{O}_m$. This chapter investigates three differential subordination implications for functions associated with the linear operator $L_a \in \mathcal{O}_m$:

$$\{\phi\left(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\right): \ z \in \mathcal{U}\} \subset \Omega \Rightarrow L_a f(z) \prec q(z), \tag{6.7}$$

$$\{\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right): \ z \in \mathcal{U}\} \subset \Omega \Rightarrow \frac{L_a f(z)}{z^{m-1}} \prec q(z), \tag{6.8}$$

$$\left\{\phi\left(\frac{L_{a+1} f(z)}{L_a f(z)}, \frac{L_{a+2} f(z)}{L_{a+1} f(z)}, \frac{L_{a+3} f(z)}{L_{a+2} f(z)}; z\right): \ z \in \mathcal{U}\right\} \subset \Omega \Rightarrow \frac{L_{a+1} f(z)}{L_a f(z)} \prec q(z). \tag{6.9}$$

Several differential inequalities and subordinations are obtained as applications of these results. In Section 6.3, the corresponding differential superordination implications are also obtained. These results are applied to prove sandwich-type results for the linear operator $L_a \in \mathcal{O}_m$. In Section 6.4, various interesting examples for different choices of admissible functions ϕ are investigated. The obtained results unify various earlier results, for example, those of [17–19].

6.2 Subordination Implications of Linear Operators

The following terminology introduced by Miller and Mocanu [111] will be required. Denote by $\mathcal{Q}$ the set of functions q that are analytic and injective on $\bar{\mathcal{U}} \backslash E(q)$ where

$$E(q) = \{\zeta \in \partial\mathcal{U} : \lim_{z \to \zeta} q(z) = \infty\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \backslash E(q)$. Let $\mathcal{Q}(a)$ be the subclass of $\mathcal{Q}$ for which $q(0) = a$. Further, let $\mathcal{Q}_0 := \mathcal{Q}(0)$ and $\mathcal{Q}_1 := \mathcal{Q}(1)$.

Definition 6.1 (Definition 1.1) *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and m be a positive integer. The class of admissible functions $\Psi_m[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; z) \notin \Omega$ whenever*

$r = q(\zeta), s = k\zeta q'(\zeta)$ and

$$\mathrm{Re}\left(\frac{t}{s} + 1\right) \geq k\,\mathrm{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$z \in \mathcal{U}, \zeta \in \partial\mathcal{U} \backslash E(q)$ and $k \geq m$. Denote by $\Psi[\Omega, q] := \Psi_1[\Omega, q]$.

The next theorem is the foundation result in the theory of first and second-order differential subordinations.

Theorem 6.1 (Theorem 1.15) *Let $\psi \in \Psi_m[\Omega, q]$ with $q(z) = a + a_1 z + a_2 z^2 + \cdots$. If $p(z) = a + a_m z^m + a_{m+1} z^{m+1} + \cdots$ satisfies*

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) \in \Omega,$$

then $p(z) \prec q(z)$.

In the sequel, L_a is a linear operator in $\mathcal{O}_m$. First, the differential subordination implication given by (6.7) is investigated. An appropriate class of admissible functions is required to apply the subordination methodology. The class of admissible functions $\Phi_H[\Omega, q]$ associated with the Dziok-Srivastava linear operator given by (6.3) was studied by Ali *et al.* [19]. Ali *et al.* [18] investigated the class of admissible functions $\Phi_I[\Omega, q]$ related to the multiplier transformation defined by (6.5). Now the following family of admissible functions is defined and theorems analogous to those of Miller and Mocanu [111] are obtained.

Definition 6.2 *Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{A}$, c_a, $c_{a+1} \in \mathbb{C} \setminus \{0\}$, and m be a positive integer. The class of admissible functions $\Phi_L[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times$*

$\mathcal{U} \to \mathbb{C}$ *satisfying the admissibility condition* $\phi(u, v, w; z) \notin \Omega$ *whenever*

$$u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (c_a - m)q(\zeta)}{c_a}$$
$$\operatorname{Re}\left(\frac{c_a c_{a+1} w - (m - c_a)(m - c_{a+1})u}{c_a v + (m - c_a)u} - (c_a + c_{a+1} - 2m)\right)$$
$$\geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$
$$z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \backslash E(q) \quad \text{and} \quad k \geq m.$$

If $a = c_a = \alpha_1$, then the class $\Phi_L[\Omega, q]$ reduces to $\Phi_H[\Omega, q]$ introduced and investigated in [19, Definition 2.1, p. 1794]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_L[\Omega, q]$ coincides with the class $\Phi_I[\Omega, q]$ studied in [18, Definition 2.1, p. 125]. It is seen that the class $\Phi_L[\Omega, q]$ extends the classical class $\Psi_m[\Omega, q]$ introduced by Miller and Mocanu [111].

Theorem 6.2 *Let* $f \in \mathcal{A}_m$*, and* $\phi \in \Phi_L[\Omega, q]$*. If*

$$\{\phi\left(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\right) : z \in \mathcal{U}\} \subset \Omega, \tag{6.10}$$

then $L_a f(z) \prec q(z)$*.*

Proof. The function $p : \mathcal{U} \to \mathbb{C}$ defined by

$$p(z) := L_a f(z) \tag{6.11}$$

is analytic in $\mathcal{U}$ as $L_a f$ is well-defined and (6.10) holds. Thus the first-order differential recurrence relation (6.2) yields

$$L_{a+1} f(z) = \frac{zp'(z) + (c_a - m)p(z)}{c_a} \tag{6.12}$$

and

$$L_{a+2}f(z) = \frac{z^2p''(z) + (c_a + c_{a+1} - 2m + 1)zp'(z) + (c_a - m)(c_{a+1} - m)p(z)}{c_a c_{a+1}}. \tag{6.13}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r, \quad v = \frac{s + (c_a - m)r}{c_a}, \\ w &= \frac{t + (c_a + c_{a+1} - 2m + 1)s + (c_a - m)(c_{a+1} - m)r}{c_a c_{a+1}}, \end{aligned} \tag{6.14}$$

and the function $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ by

$$\begin{aligned} \psi(r, s, t; z) := \phi(u, v, w; z) = \phi\Bigg(& r, \frac{s + (c_a - m)r}{c_a}, \\ & \frac{t + (c_a + c_{a+1} - 2m + 1)s + (c_a - m)(c_{a+1} - m)r}{c_a c_{a+1}}; z \Bigg). \end{aligned} \tag{6.15}$$

It follows from (6.11), (6.12), (6.13), and (6.15) that

$$\psi\big(p(z), zp'(z), z^2p''(z); z\big) = \phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big). \tag{6.16}$$

The theorem follows from Theorem 6.1 provided $\psi \in \Psi_m[\Omega, q]$ where $q \in \mathcal{Q}(a)$. Since

$$\frac{t}{s} + 1 = \frac{c_a c_{a+1} w + (m - c_a)(c_{a+1} - m)u}{c_a v + (m - c_a)u} - (c_a + c_{a+1} - 2m),$$

condition (6.10) shows that $\psi \in \Psi_m[\Omega, q]$. Thus, the admissibility condition for $\psi \in \Psi_m[\Omega, q]$ as per Definition 6.1 is equivalent to the admissibility condition for $\phi \in \Phi_L[\Omega, q]$ as given in Definition 6.2. Hence from Theorem 6.1, $p(z) = L_a f(z) \prec q(z)$.

∎

Remark 6.1 *Various known results are special case of Theorem 6.2 for suitable choices of the parameters a and c_a. For instance, Theorem 2.1 in [19, p. 1765] is deduced from Theorem 6.2 where $a = c_a = \alpha_1$. If $a = n$ and $c_a = m + \lambda$, then Theorem 2.1 in [18, p. 125] follows from Theorem 6.2.*

For a simply connected domain $\Omega \neq \mathbb{C}$, there is a conformal mapping h of $\mathcal{U}$ onto Ω (Theorem 1.1). In this case, the class $\Phi_L[\Omega, q]$ is written as $\Phi_L[h, q]$. The following result is an immediate consequence of Theorem 6.2.

Corollary 6.1 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_L[h, q]$. If $\phi\,(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z)$ is analytic in $\mathcal{U}$, and*

$$\phi\,(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z) \prec h(z), \tag{6.17}$$

then $L_a f(z) \prec q(z)$.

Remark 6.2 *If $a = c_a = \alpha_1$, then Corollary 6.1 reduces to Theorem 2.2 in [19, p. 1766], and Theorem 2.2 in [18, p. 126] follows from Corollary 6.1 where $a = n$ and $c_a = m + \lambda$.*

The next result extends Theorem 6.1 to the case where the behavior of q on $\partial\mathcal{U}$ is unknown.

Corollary 6.2 *Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{A}$. Let $\phi \in \Phi_L[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}_m$, and*

$$\left\{\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big) : z \in \mathcal{U}\right\} \subset \Omega,$$

then $L_a f(z) \prec q(z)$.

Proof. Theorem 6.2 yields $L_a f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$. ∎

Remark 6.3 *If $a = c_a = \alpha_1$, then Corollary 6.2 reduces to Corollary 2.1 in [19, p. 1766], and Corollary 2.1 in [18, p. 126] follows from Corollary 6.2 where $a = n$ and $c_a = m + \lambda$.*

The following result is similar to Theorem 2.3d in [111, pp. 30-31].

Theorem 6.3 *Let h and q be univalent in $\mathcal{U}$ with $q(0) = 0$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfy one of the following conditions:*

(1) $\phi \in \Phi_L[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(2) there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_L[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}_m$, $\phi\left(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\right)$ is analytic in $\mathcal{U}$, and $L_a f$ satisfies (6.17)*, then $L_a f(z) \prec q(z)$.*

Proof. **Case 1.** Let condition (1) be satisfied. From Corollary 6.1, it follows that $L_a f(z) \prec q_\rho(z)$, and since $q_\rho(z) \prec q(z)$, it is deduced that $L_a f(z) \prec q(z)$.

Case 2. Let $p_\rho(z) = p(\rho z) = L_a f(\rho z)$. If the condition (2) is satisfied, then from (6.16) follows that

$$\begin{aligned}\psi\big(p_\rho(z), zp_\rho'(z), z^2 p_\rho''(z); \rho z\big) &= \psi\big(p(\rho z), zp'(\rho z), z^2 p''(\rho z); \rho z\big) \\ &= \phi\big(L_a f(\rho z), L_{a+1} f(\rho z), L_{a+2} f(\rho z); \rho z\big) \in h_\rho(\mathcal{U}).\end{aligned}$$

Using Corollary 6.1 shows that $L_a f(\rho z) \prec q_\rho(z)$ for all $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, it is seen that $L_a f(z) \prec q(z)$. ∎

The next theorem yields the best dominant of differential subordination (6.17).

Theorem 6.4 *Let h be univalent in $\mathcal{U}$, and $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\psi\big(q(z), zq'(z), z^2 q''(z); z\big) = h(z) \tag{6.18}$$

has a solution q with $q(0) = 0$ where ψ is given by (6.15) and satisfies one of the following conditions:

(1) $q \in \mathcal{Q}_0$ and $\phi \in \Phi_L[h, q]$,

(2) q is univalent in $\mathcal{U}$ and $\phi \in \Phi_L[h, q_\rho]$ for some $\rho \in (0, 1)$, or

(3) q is univalent in $\mathcal{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_L[h_\rho, q_\rho]$, for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}_m$, $\phi(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z)$ is analytic in $\mathcal{U}$, and $L_a f$ satisfies (6.17), then $L_a f(z) \prec q(z)$, and q is the best dominant.

Proof. From Corollary 6.1 and Theorem 6.3, it can be deduced that q is a dominant of (6.17). Since q satisfies (6.18), it is also a solution of (6.17) and therefore q will be dominated by all dominants of (6.17). Hence q is the best dominant of (6.17). ∎

Remark 6.4 *If $a = c_a = \alpha_1$, then Theorem 6.4 reduces to Theorem 2.4 in [19, p. 1767], and Corollary 2.4 in [18, p. 127] follows from Theorem 6.4 where $a = n$ and $c_a = m + \lambda$.*

As an application, it is of interest to investigate the differential subordination implication (6.7) for the case of dominant by the disk of radius M. The class of admissible functions $\Phi_L[\Omega, q]$ reduces to the class $\Phi_L[\Omega, M]$ defined below where $q(z) = Mz$ $(M > 0)$ in Definition 6.2.

Definition 6.3 *Let $\Omega \subset \mathbb{C}$, $M > 0$, and c_a, $c_{a+1} \in \mathbb{C} \setminus \{0\}$. The class of admissible functions $\Phi_L[\Omega, M]$ consists of functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ such that*

$$\phi\Bigg(Me^{i\theta}, \frac{k + c_a - m}{c_a} Me^{i\theta}, \\ \frac{L + (c_a + c_{a+1} - 2m + 1)kMe^{i\theta} + (c_a - m)(c_{a+1} - m)Me^{i\theta}}{c_a c_{a+1}}; z\Bigg) \notin \Omega \tag{6.19}$$

whenever $z \in \mathcal{U}$*,* $\theta \in \mathbb{R}$*,* $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ *for all real* θ*, and* $k \geq m$*.*

When $a = c_a = \alpha_1$, the class $\Phi_L[\Omega, M]$ reduces to $\Phi_H[\Omega, M]$ introduced and investigated in [19]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_L[\Omega, M]$ coincides with $\Phi_I[\Omega, M]$ studied in [18].

Corollary 6.3 *Let* $f \in \mathcal{A}_m$*, and* $\phi \in \Phi_L[\Omega, M]$*. If*

$$\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big) \in \Omega,$$

then $|L_a f(z)| < M$*.*

Remark 6.5 *When* $\Omega = \mathcal{U}$ *and* $M = 1$*, various known results are obtained as special case of Corollary 6.3. For instance, Theorem 1 in [1, p. 269] is deduced from Corollary 6.3 when* $a = c_a = \alpha_1$*. If* $a = n$ *and* $c_a = m + \lambda$*, then Theorem 2 in [1, p. 271] follows from Corollary 6.3. If* $c_a = a$*, Corollary 6.3 is reduced to Theorem [25, p. 767]. Also, if* $a = c_a = \alpha_1$*, Corollary 2.2 in [19, p. 1767] follows from Corollary 6.3. If* $a = n$ *and* $c_a = m + \lambda$*, then Corollary 2.2 in [18, p. 128] follows from Corollary 6.3.*

In the special case $\Omega = q(\mathcal{U}) = \{w : |w| < M\}$, the class $\Phi_L[\Omega, M]$ is simply denoted by $\Phi_L[M]$. Corollary 6.3 can be rewritten in the following form:

Corollary 6.4 *Let* $f \in \mathcal{A}_m$*, and* $\phi \in \Phi_L[M]$*. If*

$$|\phi(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z)| < M,$$

then $|L_a f(z)| < M$*.*

Remark 6.6 *If* $a = c_a = \alpha_1$*, then Corollary 2.3 in [19, p. 1767] follows from Corollary 6.4. If* $a = n$ *and* $c_a = m + \lambda$*, then Corollary 6.4 yields Corollary 2.3 in [18, p. 4].*

Now the differential subordination implication given in (6.8) is investigated. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ associated with the Dziok-Srivastava linear operator given by (6.3) was studied by Ali *et al.* [19]. Ali *et al.* [18] investigated the class of admissible functions $\Phi_{I,1}[\Omega, q]$ related to multiplier transform defined by (6.5). The differential subordination implication given in (6.8) holds when the function ϕ belongs to the class of admissible functions introduced in the following definition.

Definition 6.4 *Let $\Omega \subset \mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{A}$, and c_a, $c_{a+1} \in \mathbb{C} \setminus \{0\}$. The class of admissible functions $\Phi_{L,1}[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever*

$$\begin{gathered} u = q(\zeta), \quad v = \frac{k\zeta q'(\zeta) + (c_a - 1)q(\zeta)}{c_a}, \\ \operatorname{Re}\left(\frac{c_a c_{a+1} w + (1 - c_a)(c_{a+1} - 1)u}{c_a v + (1 - c_a)u} - c_a - c_{a+1} + 2\right) \\ \geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right), \\ z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \backslash E(q) \text{ and } \quad k \geq 1. \end{gathered}$$

When $a = c_a = \alpha_1$, the class $\Phi_{L,1}[\Omega, q]$ reduces to $\Phi_{H,1}[\Omega, q]$ introduced and investigated in [19]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_{L,1}[\Omega, q]$ coincides with $\Phi_{I,1}[\Omega, q]$ studied in [18].

Theorem 6.5 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,1}[\Omega, q]$. If*

$$\left\{\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right) : z \in \mathcal{U}\right\} \subset \Omega, \tag{6.20}$$

then

$$\frac{L_a f(z)}{z^{m-1}} \prec q(z).$$

Proof. The function $p : \mathcal{U} \to \mathbb{C}$ defined by

$$p(z) := \frac{L_a f(z)}{z^{m-1}}. \tag{6.21}$$

is well-defined and analytic in $\mathcal{U}$ by (6.20). Thus the recurrence relation (6.2) yields

$$\frac{L_{a+1} f(z)}{z^{m-1}} = \frac{zp'(z) + (c_a - 1)p(z)}{c_a}, \tag{6.22}$$

and

$$\frac{L_{a+2} f(z)}{z^{m-1}} = \frac{z^2 p''(z) + (c_a + c_{a+1} - 1)zp'(z) + (c_a - 1)(c_{a+1} - 1)p(z)}{c_a c_{a+1}}. \tag{6.23}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r, \quad v = \frac{s + (c_a - 1)r}{c_a}, \\ w &= \frac{t + (c_a + c_{a+1} - 1)s + (c_a - 1)(c_{a+1} - 1)r}{c_a c_{a+1}}, \end{aligned}$$

and the function $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ by

$$\begin{aligned} \psi(r, s, t; z) := \phi(u, v, w; z) = \phi\Bigg(& r, \frac{s + (c_a - 1)r}{c_a}, \\ & \frac{t + (c_a + c_{a+1} - 1)s + (c_a - 1)(c_{a+1} - 1)r}{c_a c_{a+1}}; z \Bigg). \end{aligned} \tag{6.24}$$

It follows from (6.21), (6.22), (6.23) and (6.24) that

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) = \phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z \right). \tag{6.25}$$

In view of Theorem 6.1, it suffices to show that $\psi \in \Psi_m[\Omega, q]$. Since

$$\frac{t}{s}+1=\frac{c_a c_{a+1} w+(1-c_a)(c_{a+1}-1)u}{c_a v+(1-c_a)u}-c_a-c_{a+1}+2,$$

condition (6.20) shows that $\psi \in \Psi_m[\Omega, q]$. Thus, the admissibility condition for ψ as given in Definition 6.1 and the admissibility condition for $\phi \in \Phi_{L,1}[\Omega, q]$ are equivalent. Hence by Theorem 6.1, $p(z) = L_a f(z)/z^{m-1} \prec q(z)$. ∎

Remark 6.7 *Theorem 2.5 in [19, p. 1769] is deduced from Theorem 6.5 where* $a = c_a = \alpha_1$. *If* $a = n$ *and* $c_a = m + \lambda$, *then Theorem 2.5 in [18, p. 129] follows from Theorem 6.5.*

For a conformal mapping h from $\mathcal{U}$ onto a simply connected domain $\Omega \neq \mathbb{C}$, the class $\Phi_{L,1}[\Omega, q]$ is written as $\Phi_{L,1}[h, q]$, and the following corollary is deduced from Theorem 6.5.

Corollary 6.5 *Let* $f \in \mathcal{A}_m$, *and* $\phi \in \Phi_{L,1}[h, q]$. *If*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)$$

is analytic in $\mathcal{U}$, *and*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right) \prec h(z),$$

then

$$\frac{L_a f(z)}{z^{m-1}} \prec q(z).$$

A special case of differential subordination implication (6.8) is of interest. Let $q(z) = Mz$ with $M > 0$. Then $q(\mathcal{U})$ is the disk of radius M centered at origin. In this case, the class of admissible functions $\Phi_{L,1}[\Omega, M]$ will be defined as follows.

Definition 6.5 *Let $\Omega \subset \mathbb{C}$, $M > 0$, and $c_a, c_{a+1} \in \mathbb{C} \setminus \{0\}$. The class of admissible function $\Phi_{L,1}[\Omega, M]$ consists of functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\phi(r, s, t; z) \notin \Omega$ where*

$$\begin{aligned} r =& Me^{i\theta}, \quad s = \frac{k + c_a - 1}{c_a} Me^{i\theta}, \\ t =& \frac{L + (c_a + c_{a+1} - 1)kMe^{i\theta} + (c_a - 1)(c_{a+1} - 1)Me^{i\theta}}{c_a c_{a+1}}, \end{aligned}$$

whenever $z \in \mathcal{U}$, $\theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ for all real θ, and $k \geq 1$.

When $a = c_a = \alpha_1$, the class $\Phi_{L,1}[\Omega, M]$ reduces to $\Phi_{H,1}[\Omega, M]$ introduced and investigated in [19]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_{L,1}[\Omega, M]$ coincides with $\Phi_{I,1}[\Omega, M]$ studied in [18].

Corollary 6.6 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,1}[\Omega, M]$. If*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right) \in \Omega,$$

then

$$\left|\frac{L_a f(z)}{z^{m-1}}\right| < M.$$

In the special case $\Omega = q(\mathcal{U}) = \{w : |w| < M\}$, the class $\Phi_{L,1}[\Omega, M]$ is simply denoted by $\Phi_{L,1}[M]$.

Corollary 6.7 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,1}[M]$. If*

$$\left|\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)\right| < M,$$

then

$$\left|\frac{L_a f(z)}{z^{m-1}}\right| < M.$$

Ali *et al.* [19] introduced the class of admissible functions $\Phi_{H,2}[\Omega, q]$ so that the differential subordination (6.9) holds when $L_a f$ is the Dziok-Srivastava linear operator given by (6.3). Ali *et al.* [18] determined the corresponding class $\Phi_{I,2}[\Omega, q]$ of admissible functions where $L_a f$ is the multiplier transformation defined by (6.5). Now, appropriate class of admissible functions is determined so that the differential subordination implication (6.9) holds.

Definition 6.6 *Let $\Omega \subset \mathbb{C}$, $c_a \in \mathbb{C}, c_{a+2}c_{a+1}c_a \neq 0$ and $q \in \mathcal{Q}_1 \cap \mathcal{A}_0$. The class of admissible functions $\Phi_{L,2}[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; z) \notin \Omega$ whenever*

$$u = q(\zeta), \quad v = 1 - \frac{c_a}{c_{a+1}} + \frac{1}{c_{a+1}}\left(c_a q(\zeta) + \frac{k\zeta q'(\zeta)}{q(\zeta)}\right),$$

$$\operatorname{Re}\left(\frac{vc_{a+1}\big((w-1)c_{a+2} + (1-v)c_{a+1}\big)}{c_{a+1}v - c_a u - c_{a+1} + c_a} + c_{a+1}(v-1) + c_a(1-2u)\right)$$

$$\geq k \operatorname{Re}\left(\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right),$$

$$z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \backslash E(q) \text{ and } \quad k \geq 1.$$

When $a = c_a = \alpha_1$, the class $\Phi_{L,2}[\Omega, q]$ reduces to $\Phi_{H,2}[\Omega, q]$ introduced and investigated in [19]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_{L,2}[\Omega, q]$ coincides with $\Phi_{I,2}[\Omega, q]$ studied in [18].

Theorem 6.6 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,2}[\Omega, q]$. If*

$$\left\{\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) : z \in \mathcal{U}\right\} \subset \Omega \tag{6.26}$$

then

$$\frac{L_{a+1}f(z)}{L_a f(z)} \prec q(z).$$

Proof. Define the function $p : \mathcal{U} \to \mathbb{C}$ by

$$p(z) := \frac{L_{a+1}f(z)}{L_a f(z)}. \tag{6.27}$$

From (6.26), the function p is well-defined and analytic in $\mathcal{U}$. By (6.2) and (6.27), it follows that

$$\frac{L_{a+2}f(z)}{L_{a+1}f(z)} = \frac{1}{c_{a+1}} \left(c_a p(z) + c_{a+1} - c_a + \frac{zp'(z)}{p(z)} \right), \tag{6.28}$$

and

$$\begin{aligned} \frac{L_{a+3}f(z)}{L_{a+2}f(z)} = \frac{1}{c_{a+2}} \Bigg(& c_a p(z) - c_a + \frac{zp'(z)}{p(z)} + c_{a+2} + \\ & \frac{zc_a p'(z) + \frac{zp'(z)}{p(z)} + \frac{z^2 p''(z)}{p(z)} - (\frac{zp'(z)}{p(z)})^2}{c_a p(z) + c_{a+1} - c_a + \frac{zp'(z)}{p(z)}} \Bigg). \end{aligned} \tag{6.29}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$\begin{aligned} u &= r, \;\; v = \frac{1}{c_{a+1}} \left(c_a r + c_{a+1} - c_a + \frac{s}{r} \right), \\ w &= \frac{1}{c_{a+2}} \left(c_a r - c_a + \frac{s}{r} + c_{a+2} + \frac{c_a s + \frac{s}{r} + \frac{t}{r} - (\frac{s}{r})^2}{c_a r + c_{a+1} - c_a + \frac{s}{r}} \right). \end{aligned}$$

If the function $\psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ is defined by

$$\psi(r, s, t; z) := \phi(u, v, w; z),$$

then the equations (6.27), (6.28) and (6.29) show that

$$\psi\big(p(z), zp'(z), z^2 p''(z); z\big) = \phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z \right). \tag{6.30}$$

Now, it is enough to show that the admissibility condition for $\phi \in \Phi_{L,2}[\Omega, q]$ is equivalent to the admissibility condition for $\psi \in \Psi_m[\Omega, q]$ as given in Definition 6.1. Since

$$
\begin{aligned}
\frac{s}{r} &= vc_{a+1} - c_a r - c_{a+1} + c_a \\
\frac{t}{r} &= [(w-1)c_{a+2} + (1-v)c_{a+1}]vc_{a+1} - [c_a u + 1 - \frac{s}{r}]\frac{s}{r} \\
&= [(w-1)c_{a+2} + (1-v)c_{a+1}]vc_{a+1} - [vc_{a+1} - c_{a+1} + c_a + 1 - 2\frac{s}{r}]\frac{s}{r} \\
&= [(w-1)c_{a+2} + (1-v)c_{a+1}]vc_{a+1} \\
&- [vc_{a+1} - c_{a+1} + c_a + 1 - 2(vc_{a+1} - c_a r - c_{a+1} + c_a)]\frac{s}{r} \\
&= [(w-1)c_{a+2} + (1-v)c_{a+1}]vc_{a+1} + [c_{a+1}(v-1) + c_a(1-2u) - 1]\frac{s}{r},
\end{aligned}
$$

it follows that

$$
\frac{t}{s} + 1 = \frac{[(w-1)c_{a+2} + (1-v)c_{a+1}]vc_{a+1}}{c_{a+1}v - c_a u - c_{a+1} + c_a} + c_{a+1}(v-1) + c_a(1-2u).
$$

Hence $\psi \in \Psi_m[\Omega, q]$ and by Theorem 6.1, $p(z) = L_{a+1}f(z)/L_a f(z) \prec q(z)$. ∎

Remark 6.8 *If $a = c_a = \alpha_1$, then Theorem 2.7 in [19, p. 1771] follows from Theorem 6.6. If $a = n$ and $c_a = m + \lambda$, then Theorem 6.6 yields Theorem 2.7 in [18, , p. 132].*

If h is a conformal mapping from $\mathcal{U}$ onto a simply connected domain $\Omega \neq \mathbb{C}$, then $\Phi_{L,2}[\Omega, q]$ is written as $\Phi_{L,2}[h, q]$, and the following result is obtained as a consequence of Theorem 6.6.

Corollary 6.8 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,2}[\Omega, q]$. If*

$$
\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right)
$$

is analytic in $\mathcal{U}$, *and*

$$\phi\left(\frac{L_{a+1}f(z)}{L_af(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) \prec h(z),$$

then

$$\frac{L_{a+1}f(z)}{L_af(z)} \prec q(z).$$

An interesting application of Theorem 6.6 is in the case of $q(\mathcal{U})$ being the disk $|w-1| < M$. The class of admissible functions $\Phi_{L,2}[\Omega, M]$ introduced below is needed in Corollary 6.9.

Definition 6.7 *Let* $\Omega \subset \mathbb{C}$, $c_a \in \mathbb{C}$, $c_ac_{a+1}c_{a+2} \neq 0$, *and* $M > 0$. *The class of admissible functions* $\Phi_{L,2}[\Omega, M]$ *consists of functions* $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ *satisfying the admissibility condition* $\phi(u, v, w; z) \notin \Omega$ *whenever*

$$\begin{aligned} u &= 1 + Me^{i\theta}, \quad v = 1 + \frac{(1+Me^{i\theta})c_a + k}{c_{a+1}(1+Me^{i\theta})}Me^{i\theta} \\ w &= 1 + \frac{(1+Me^{i\theta})c_a + k}{c_{a+2}(1+Me^{i\theta})}Me^{i\theta} + \\ &\frac{(M+e^{-i\theta})\big(Le^{-i\theta} + kM(c_a+1) + c_akM^2e^{i\theta}\big) - k^2M^2}{c_{a+2}(M+e^{-i\theta})\big(c_aM^2e^{i\theta} + c_{a+1}e^{-i\theta} + M(c_a + c_{a+1} + k)\big)}, \end{aligned}$$

$z \in \mathcal{U}$, $\theta \in \mathbb{R}$, $\mathrm{Re}(Le^{-i\theta}) \geq (k-1)kM$ *for all real* θ, *and* $k \geq 1$.

When $a = c_a = \alpha_1$, the class $\Phi_{L,2}[\Omega, M]$ reduces to $\Phi_{H,2}[\Omega, M]$ introduced and investigated in [19]. If $a = n$ and $c_a = m + \lambda$, then the class $\Phi_{L,2}[\Omega, M]$ coincides with $\Phi_{I,2}[\Omega, M]$ studied in [18].

Corollary 6.9 *Let* $f \in \mathcal{A}_m$, *and* $\phi \in \Phi_{L,2}[\Omega, M]$. *If*

$$\phi\left(\frac{L_{a+1}f(z)}{L_af(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) \in \Omega,$$

then

$$\left|\frac{L_{a+1}f(z)}{L_af(z)} - 1\right| < M.$$

In the special case $\Omega = q(\mathcal{U}) = \{w : |w-1| < M\}$, the class $\Phi_{L,2}[\Omega, M]$ is simply denoted by $\Phi_{L,2}[M]$.

Corollary 6.10 *Let $f \in \mathcal{A}_m$, and $\phi \in \Phi_{L,2}[M]$. If*

$$\left|\phi\left(\frac{L_{a+1}f(z)}{L_af(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) - 1\right| < M,$$

then

$$\left|\frac{L_{a+1}f(z)}{L_af(z)} - 1\right| < M.$$

Remark 6.9 *When $a = c_a = \alpha_1$, Corollary 2.10 in [19, p. 1774] is easily deduced from Corollary 6.10. If $a = n$ and $c_a = m + \lambda$, then Corollary 2.10 in [18, p. 133] follows from Corollary 6.10.*

6.3 Superordination Implications of Linear Operators

The differential superordination is the dual problem of differential subordination. This section investigates three differential superordination implications for functions related to the linear operator $L_a \in \mathcal{O}_m$:

$$\Omega \subset \left\{\phi\left(L_af(z), L_{a+1}f(z), L_{a+2}f(z); z\right) : \ z \in \mathcal{U}\right\} \Rightarrow q(z) \prec L_af(z), \tag{6.31}$$

$$\Omega \subset \left\{\phi\left(\frac{L_af(z)}{z^{m-1}}, \frac{L_{a+1}f(z)}{z^{m-1}}, \frac{L_{a+2}f(z)}{z^{m-1}}; z\right) : z \in \mathcal{U}\right\} \Rightarrow q(z) \prec \frac{L_af(z)}{z^{m-1}}, \tag{6.32}$$

$$\Omega \subset \left\{\phi\left(\frac{L_{a+1}f(z)}{L_af(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) : \ z \in \mathcal{U}\right\} \Rightarrow q(z) \prec \frac{L_{a+1}f(z)}{L_af(z)}. \tag{6.33}$$

The results of the previous section combined with the corresponding results in this section yield several sandwich-type theorems.

The following results obtained by Miller and Mocanu [112] are required.

Definition 6.8 (Definition 1.2) *Let Ω be a set in $\mathbb{C}$ and $q(z) = a + a_m z^m + a_{m+1}z^{m+1} + \cdots$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_m[\Omega, q]$ consists of functions $\psi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$ satisfying the admissibility condition $\psi(r, s, t; \zeta) \in \Omega$ whenever $r = q(z), s = zq'(z)/k$ and*

$$\operatorname{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{k}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$z \in \mathcal{U}, \zeta \in \partial\mathcal{U}$ and $k \geq m \geq 1$. Denote by $\Psi'[\Omega, q] := \Psi'_1[\Omega, q]$.

Theorem 6.7 (Theorem 1.16) *Let $\psi \in \Psi'_m[\Omega, q]$ with $q(z) = a + a_1 z + a_2 z^2 + \cdots$. If $p \in \mathcal{Q}(a)$ and $\psi\big(p(z), zp'(z), z^2p''(z); z\big)$ is univalent in $\mathcal{U}$, then*

$$\Omega \subset \{\psi\big(p(z), zp'(z), z^2p''(z); z\big), z \in \mathcal{U}\}$$

implies $q(z) \prec p(z)$.

Analogous to the case of subordination, a suitable class of admissible functions is required to apply differential superordination methodology. Ali *et al.* [19] introduced the class of admissible functions $\Phi'_H[\Omega, q]$ associated with the Dziok-Srivastava linear operator given by (6.3). The class of admissible functions $\Phi'_I[\Omega, q]$ related to multiplier transformation defined by (6.5) was investigated by Ali *et al.* [18]. The following class of admissible functions is now introduced to investigate the differential superordination implication given in (6.31).

Definition 6.9 *Let $\Omega \subset \mathbb{C}$, $c_a \in \mathbb{C}, c_{a+1}c_a \neq 0$, and $q \in \mathcal{A}_m$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_L[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$*

satisfying the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ *whenever*

$$u = q(z), \quad v = \frac{zq'(z) + m(c_a - m)q(z)}{mc_a},$$
$$\operatorname{Re}\left(\frac{c_a c_{a+1} w + (m - c_a)(c_{a+1} - m)u}{c_a v + (m - c_a)u} - (c_a + c_{a+1} - 2m)\right)$$
$$\leq \frac{1}{k} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$
$$z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \text{ and } \quad k \geq m.$$

Theorem 6.8 *Let* $f \in \mathcal{A}$, $\phi \in \Phi'_L[\Omega, q]$, *and* $L_a f \in \mathcal{Q}_0$. *If*

$$\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big)$$

is univalent in $\mathcal{U}$, *then*

$$\Omega \subset \left\{\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big) : z \in \mathcal{U}\right\} \tag{6.34}$$

implies $q(z) \prec L_a f(z)$.

Proof. It follows from (6.16) and (6.34) that

$$\Omega \subset \left\{\psi\big(p(z), zp'(z), z^2 p''(z); z\big) : z \in \mathcal{U}\right\},$$

where ψ is defined by (6.15). From (6.14), it can be seen that the admissibility condition for $\phi \in \Phi'_L[\Omega, q]$ is equivalent to the admissibility condition for $\psi \in \Psi'_m[\Omega, q]$ as given in Definition 6.8. Hence $\psi \in \Psi'_m[\Omega, q]$ and by Theorem 6.7, $q(z) \prec p(z) = L_a f(z)$.

∎

For a conformal mapping h from $\mathcal{U}$ onto a simply connected domain Ω, the class $\Phi'_L[\Omega, q]$ is written as $\Phi'_L[h, q]$. Consequently, the following result is obtained by Theorem 6.8.

Corollary 6.11 *Let h be analytic in $\mathcal{U}$ and $\phi \in \Phi'_L[h,q]$. If $f \in \mathcal{A}$, $L_a f \in \mathcal{Q}_0$, and $\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big)$ is univalent in $\mathcal{U}$, then*

$$h(z) \prec \phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big),$$

implies $q(z) \prec L_a f(z)$.

The best subordinant will be determined by the next theorem.

Theorem 6.9 *Let h be analytic in $\mathcal{U}$ and $\phi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$ and ψ be given by* (6.15). *Suppose that the differential equation $\psi\big(q(z), zq'(z), z^2q''(z); z\big) = h(z)$ has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_L[h,q]$, $f \in \mathcal{A}$, $L_a f \in \mathcal{Q}_0$ and*

$$\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big)$$

is univalent in $\mathcal{U}$, then

$$h(z) \prec \phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big)$$

implies $q(z) \prec L_a f(z)$ and $q(z)$ is the best subordinant.

Proof. The proof is similar to the Theorem 6.4 and is therefore omitted. ∎

Corollary 6.11 together with Corollary 6.1 give the following sandwich-type result.

Corollary 6.12 *Let h_1 and q_1 be analytic functions in $\mathcal{U}$, h_2 be univalent function in $\mathcal{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_L[h_2, q_2] \cap \Phi'_L[h_1, q_1]$. If $f \in \mathcal{A}$, $L_a f(z) \in \mathcal{A} \cap \mathcal{Q}_0$ and $\phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big)$ is univalent in $\mathcal{U}$, then*

$$h_1(z) \prec \phi\big(L_a f(z), L_{a+1} f(z), L_{a+2} f(z); z\big) \prec h_2(z)$$

implies $q_1(z) \prec L_a f(z) \prec q_2(z)$.

The differential superordination implication (6.32) holds for the class of admissible functions $\Phi'_{L,1}[\Omega, q]$ introduced below.

Definition 6.10 *Let $\Omega \subset \mathbb{C}$, $c_a \in \mathbb{C}, c_{a+1}c_a \neq 0$ and $q \in \mathcal{A}$ with $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{L,1}[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever*

$$\begin{gathered} u = q(z), \quad v = \frac{zq'(z) + m(c_a - 1)q(z)}{mc_a}, \\ \operatorname{Re}\left(\frac{c_a c_{a+1} w + (1 - c_a)(c_{a+1} - 1)u}{c_a v + (1 - c_a)u} - c_a - c_{a+1} + 2\right) \\ \leq \frac{1}{k} \operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right), \\ z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \text{ and } \quad k \geq 1. \end{gathered}$$

Next, the dual result of Theorem 6.5 for differential superordination is obtained.

Theorem 6.10 *Let $\phi \in \Phi'_{L,1}[\Omega, q]$. If $f \in \mathcal{A}_m$, $L_a f(z)/z^{m-1} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)$$

is univalent in $\mathcal{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right) : z \in \mathcal{U}\right\} \tag{6.35}$$

implies $q(z) \prec L_a f(z)/z^{m-1}$.

Proof. It follows from (6.35) and (6.25) that

$$\Omega \subset \left\{\psi\big(p(z), zp'(z), z^2 p''(z); z\big) : z \in \mathcal{U}\right\},$$

where ψ is given by (6.24). It is seen that the admissibility condition for $\phi \in \Phi'_{L,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi \in \Psi'_m[\Omega, q]$ as given in Definition 6.8. Hence $\psi \in \Psi'_m[\Omega, q]$ and by Theorem 6.7, $q(z) \prec p(z) = L_a f(z)/z^{m-1}$. ∎

Let $\Omega = h(\mathcal{U})$ where h is a conformal mapping. In this case, the class $\Phi'_{L,1}[\Omega, q]$ is written as $\Phi'_{L,1}[h, q]$. By Theorem 6.10, the following result holds.

Corollary 6.13 *Let h be analytic in $\mathcal{U}$, $q \in \mathcal{A}$ and $\phi \in \Phi'_{L,1}[h, q]$. If $f \in \mathcal{A}_m$ and $L_a f(z)/z^{m-1} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)$$

is univalent in $\mathcal{U}$, then

$$h(z) \prec \phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)$$

implies $q(z) \prec L_a f(z)/z^{m-1}$.

From Corollaries 6.5 and 6.13, the following sandwich-type theorem is obtained.

Corollary 6.14 *Let h_1 and q_1 be analytic functions in $\mathcal{U}$, h_2 be univalent function in $\mathcal{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{L,1}[h_2, q_2] \cap \Phi'_{L,1}[h_1, q_1]$. If $f \in \mathcal{A}_m$, $L_a f(z)/z^{m-1} \in \mathcal{A} \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right)$$

is univalent in $\mathcal{U}$, then

$$h_1(z) \prec \phi\left(\frac{L_a f(z)}{z^{m-1}}, \frac{L_{a+1} f(z)}{z^{m-1}}, \frac{L_{a+2} f(z)}{z^{m-1}}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{L_a f(z)}{z^{m-1}} \prec q_2(z).$$

The class of admissible functions $\Phi'_{L,2}[\Omega, q]$ defined below will be applied to obtain the differential superordination implication given in (6.33).

Definition 6.11 *Let $\Omega \subset \mathbb{C}$, $c_a \in \mathbb{C}$, $c_{a+2}c_{a+1}c_a \neq 0$, and $q \in \mathcal{A}_0$ with $q(z) \neq 0$, $zq'(z) \neq 0$. The class of admissible functions $\Phi'_{L,2}[\Omega, q]$ consists of functions $\phi : \mathbb{C}^3 \times \overline{\mathcal{U}} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u, v, w; \zeta) \in \Omega$ whenever*

$$u = q(z), \quad v = \frac{1}{c_{a+1}}\left(c_a q(z) + c_{a+1} - c_a + \frac{zq'(z)}{mq(z)}\right),$$

$$\operatorname{Re}\left(\frac{vc_{a+1}\big((w-1)c_{a+2} + (1-v)c_{a+1}\big)}{c_{a+1}v - c_a u - c_{a+1} + c_a} + c_{a+1}(v-1) + c_a(1-2u)\right)$$

$$\leq \frac{1}{k}\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right),$$

$$z \in \mathcal{U}, \quad \zeta \in \partial\mathcal{U} \text{ and } \quad k \geq 1.$$

Theorem 6.11 *Let $\phi \in \Phi'_{L,2}[\Omega, q]$. If $f \in \mathcal{A}_m$, $L_{a+1}f(z)/L_a f(z) \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right)$$

is univalent in $\mathcal{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) : z \in \mathcal{U}\right\}$$

implies

$$q(z) \prec \frac{L_{a+1}f(z)}{L_a f(z)}.$$

Corollary 6.15 *Let $q \in \mathcal{A}_0$, h be analytic in $\mathcal{U}$ and $\phi \in \Phi'_{L,2}[h, q]$. If $f \in \mathcal{A}_m$,*

$L_{a+1}f(z)/L_a f(z) \in \mathcal{Q}_1$ *and*

$$\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right)$$

is univalent in $\mathcal{U}$*, then*

$$h(z) \prec \phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right)$$

implies

$$q(z) \prec \frac{L_{a+1}f(z)}{L_a f(z)}.$$

Corollaries 6.15 and 6.8 can be combined to show the following sandwich-type result.

Corollary 6.16 *Let* h_1 *and* q_1 *analytic functions in* $\mathcal{U}$*,* h_2 *be univalent function in* $\mathcal{U}$*,* $q_2 \in \mathcal{Q}_1$ *with* $q_1(0) = q_2(0) = 1$ *and* $\phi \in \Phi_{L,2}[h_2, q_2] \cap \Phi'_{L,2}[h_1, q_1]$*. If* $f \in \mathcal{A}_m$*,* $L_{a+1}f(z)/L_a f(z) \in \mathcal{A} \cap \mathcal{Q}_1$ *and*

$$\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right)$$

is univalent in $\mathcal{U}$*, then*

$$h_1(z) \prec \phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \frac{L_{a+1}f(z)}{L_a f(z)} \prec q_2(z).$$

6.4 Applications

In this section, several examples are given by considering appropriate admissible functions.

Example 6.1 *Let $M > 0$ and $0 \neq c_a \in \mathbb{C}$. If $f \in \mathcal{A}_m$ satisfies*

$$\left| L_{a+1}f(z) + \left(\frac{m}{c_a} - 1\right) L_a f(z) \right| < \frac{Mm}{|c_a|},$$

then $|L_a f(z)| < M$.

Let $\phi(u,v) = v + (m/c_a - 1)u$, and $\Omega = h(\mathcal{U})$ where $h(z) = Mz/|c_a|$. Since

$$\left| \phi\left(Me^{i\theta}, \frac{k + c_a - m}{c_a} Me^{i\theta} \right) \right| = \frac{kM}{|c_a|} \geq \frac{mM}{|c_a|}$$

where $\theta \in \mathbb{R}$, and $k \geq m$, the admissibility condition (6.19) is satisfied. Hence the required result follows from Corollary 6.3.

Remark 6.10 *If $a = c_a = \alpha_1$, then Example 6.1 yields Corollary 2.5 in [19, p. 1768].*

Example 6.2 *Let $M > 0$, and $\operatorname{Re} c_a > 0$. If $f \in \mathcal{A}_m$ satisfies one of the following*

(1) $|L_{a+1}f(z)| < M$,

(2) $|L_{a+1}f(z) + L_a f(z)| < 2M$,

then $|L_a f(z)| < M$.

(1) *Let $\phi(u,v) = v$, and $\Omega = h(\mathcal{U})$ where $h(z) = Mz$. To use Corollary 6.3, it is needed to show that the admissibility condition (6.19) is satisfied. This follows since*

$$\left| \phi\left(Me^{i\theta}, \frac{k + c_a - m}{c_a} Me^{i\theta} \right) \right| = \left| \frac{k + c_a - m}{c_a} Me^{i\theta} \right| \geq M$$

where $\theta \in \mathbb{R}$ and $k \geq m$.

(2) *Let* $\phi(u,v) = v + u$, *and* $\Omega = h(\mathcal{U})$ *where* $h(z) = 2Mz$. *Since the admissibility condition* (6.19) *is satisfied by*

$$\left|\phi\left(Me^{i\theta}, \frac{k+c_a-m}{c_a}Me^{i\theta}\right)\right| = \left|Me^{i\theta} + \frac{k+c_a-m}{c_a}Me^{i\theta}\right| \geq 2M$$

where $\theta \in \mathbb{R}$ *and* $k \geq m$, *the result follows from Corollary 6.3*

Example 6.3 *Let* $M > 0$, $c_a, c_{a+1} \in \mathbb{R}^+$, *and* $c_a c_{a+1} \neq 0$. *If* $f \in \mathcal{A}_m$ *satisfies one of the following*

(1) $|L_a f(z) + L_{a+1} f(z) + L_{a+2} f(z)| < 3M$,

(2) $|L_{a+2} f(z) + L_{a+1} f(z)| < 2M$,

then $|L_a f(z)| < M$.

(1) *Let* $\phi(u,v,w) = u + v + w$. *It is shown that the admissibility condition given in Corollary 6.3 is satisfied.*

$$\begin{aligned}
&\phi\Bigg(Me^{i\theta}, \frac{k+c_a-m}{c_a}Me^{i\theta}, \\
&\qquad \frac{L + (c_a + c_{a+1} - 2m + 1)kMe^{i\theta} + (c_a - m)(c_{a+1} - m)Me^{i\theta}}{c_a c_{a+1}}\Bigg) \\
&\qquad \geq M + \frac{k+c_a-m}{c_a}M \\
&\qquad\qquad + \frac{\mathrm{Re}(Le^{-i\theta}) + (c_a + c_{a+1} - 2m + 1)kM + (c_a - m)(c_{a+1} - m)M}{c_a c_{a+1}} \\
&\qquad \geq M + \frac{k+c_a-m}{c_a}M \\
&\qquad\qquad + \frac{kM(k-1) + (c_a + c_{a+1} - 2m + 1)kM + (c_a - m)(c_{a+1} - m)M}{c_a c_{a+1}} \\
&\qquad \geq 3M
\end{aligned}$$

where $\theta \in \mathbb{R}$ *and* $k \geq m$. *Hence by Corollary 6.3, the required result follows.*

(2) *Take* $\phi(u,v,w) = w + v$ *and* $\Omega = h(\mathcal{U})$ *where* $h(z) = 2Mz$. *To apply Corollary 6.3, it is needed to show that the admissibility condition* (6.19) *is satisfied. This follows since*

$$\begin{aligned}
\phi\Bigg(&Me^{i\theta}, \frac{k + c_a - m}{c_a} Me^{i\theta}, \\
&\frac{L + (c_a + c_{a+1} - 2m + 1)kMe^{i\theta} + (c_a - m)(c_{a+1} - m)Me^{i\theta}}{c_a c_{a+1}}\Bigg) \\
&\geq \frac{\operatorname{Re}(Le^{-i\theta}) + (c_a + 2c_{a+1} - 2m + 1)kM + (c_a - m)(2c_{a+1} - m)M}{c_a c_{a+1}} \\
&\geq \frac{(k-1)kM + (c_a + 2c_{a+1} - 2m + 1)kM + (c_a - m)(2c_{a+1} - m)M}{c_a c_{a+1}} \\
&\geq 2M
\end{aligned}$$

where $\theta \in \mathbb{R}$ *and* $k \geq m$.

Example 6.4 *Let* $M > 0$ *and* $\operatorname{Re} c_a > 0$. *If* $f \in \mathcal{A}_m$ *satisfies* $\left|L_{a+1} f(z)/z^{m-1}\right| < M$, *then* $\left|L_a f(z)/z^{m-1}\right| < M$.

To show the result, it suffices to take $\phi(u,v,w;z) = v$ *in Corollary 6.7.*

Example 6.5 *If* $M > 0$, $c_a, c_{a+1} \in \mathbb{R}$ *and* $f \in \mathcal{A}_m$ *satisfies*

$$\begin{aligned}
\left| c_a c_{a+1} \frac{L_{a+2} f(z)}{z^{m-1}} + c_a \frac{L_{a+1} f(z)}{z^{m-1}} - (c_a - 1)(c_{a+1} - 1) \frac{L_a f(z)}{z^{m-1}} \right| \\
< M(2c_a + c_{a+1} - 1),
\end{aligned}$$

then $|L_a f(z)/z^{m-1}| < M$.

Take $\phi(u,v,w;z) = c_a c_{a+1} w + c_a v - (c_a - 1)(c_{a+1} - 1)u$ *and* $\Omega = h(\mathcal{U})$ *where*

$$h(z) = Mz(2c_a + c_{a+1} - 1).$$

The result follows by Corollary 6.6.

Example 6.6 *Let $M > 0$, $0 \le c_{a+1} - c_a \le 1$, and $c_{a+1} \neq 0$. If $f \in \mathcal{A}_m$ satisfies one of the following*

(1)

$$\left|\frac{L_{a+2}f(z)}{L_{a+1}f(z)} - \frac{L_{a+1}f(z)}{L_a f(z)}\right| < \frac{M}{|c_{a+1}|}\left|\frac{1}{1+M} + c_a - c_{a+1}\right|,$$

(2)

$$\left|\frac{L_{a+1}f(z)}{L_a f(z)}\left(\frac{L_{a+2}f(z)}{L_{a+1}f(z)} - \frac{L_{a+1}f(z)}{L_a f(z)}\right)\right| < \frac{M}{|c_{a+1}|}\big|1-(1+M)(c_{a+1}-c_a)\big|,$$

(3)

$$\left|\frac{\dfrac{L_{a+2}f(z)}{L_{a+1}f(z)}}{\dfrac{L_{a+1}f(z)}{L_a f(z)}} - 1\right| < \frac{M}{(1+M)|c_{a+1}|}\left|\frac{1}{1+M} + c_a - c_{a+1}\right|,$$

(4) Let $\delta \ge 0$.

$$\left|\delta\frac{L_{a+2}f(z)}{L_{a+1}f(z)} + (1-\delta)\frac{L_{a+1}f(z)}{L_a f(z)} - 1\right| < \frac{M}{|c_{a+1}|}\left|\frac{\delta}{1+M} + \delta c_a + c_{a+1}(1-\delta)\right|,$$

In particular,

$$\left|\frac{L_{a+2}f(z)}{L_{a+1}f(z)} - 1\right| < \frac{M}{|c_{a+1}|}\left|\frac{1}{1+M} + c_a\right|,$$

then

$$\left|\frac{L_{a+1}f(z)}{L_a f(z)} - 1\right| < M.$$

It suffices to show that the admissibility condition in Definition 6.7 for appropriate admissible function ϕ is satisfied so that the required result follows from Corollary 6.9.

(1) *Take $\phi(u,v,w;z) = v - u$ and $\Omega = h(\mathcal{U})$ where*

$$h(z) = \frac{Mz}{|c_{a+1}|}\left|\frac{1}{1+M} + c_a - c_{a+1}\right|.$$

Hence

$$\begin{aligned}
|\phi\,(u, v, w; z)| &= \left|1 + \frac{(1+Me^{i\theta})c_a + k}{c_{a+1}(1+Me^{i\theta})}Me^{i\theta} - 1 - Me^{i\theta}\right| \\
&= \frac{M}{|c_{a+1}|}\left|\frac{k + (1+Me^{i\theta})(c_a - c_{a+1})}{1+Me^{i\theta}}\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|\frac{k - (1+M)(c_{a+1} - c_a)}{1+M}\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|\frac{1}{1+M} + c_a - c_{a+1}\right|
\end{aligned}$$

where $z \in \mathcal{U}$, $\theta \in \mathbb{R}$ *and* $k \geq 1$.

(2) *Let* $\phi(u, v, w; z) = u(v-u)$ *and* $\Omega = h(\mathcal{U})$ *where*

$$h(z) = \frac{Mz}{|c_{a+1}|}\big|1 - (1+M)(c_{a+1} - c_a)\big|.$$

Thus

$$\begin{aligned}
|\phi(u, v, w; z)| &= \frac{M}{|c_{a+1}|}\left|k + (c_a - c_{a+1})(1+Me^{i\theta})\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|k - (c_{a+1} - c_a)(1+M)\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|1 - (c_{a+1} - c_a)(1+M)\right|.
\end{aligned}$$

(3) *Take* $\phi(u, v, w; z) = (v-u)/u$ *and* $\Omega = h(\mathcal{U})$ *where*

$$h(z) = \frac{Mz}{(1+M)|c_{a+1}|}\left|\frac{1}{1+M} + c_a - c_{a+1}\right|.$$

(4) *Let* $\phi(u, v, w; z) = \delta v + (1-\delta)u$ *and* $\Omega = h(\mathcal{U})$ *where*

$$h(z) = \frac{Mz}{|c_{a+1}|}\left|\frac{\delta}{1+M} + \delta c_a + c_{a+1}(1-\delta)\right|.$$

Thus

$$\begin{aligned}
|\phi(u,v,w;z)-1| &= \left|\delta + \frac{(1+Me^{i\theta})c_a+k}{c_{a+1}(1+Me^{i\theta})}M\delta e^{i\theta} + (1-\delta)(1+Me^{i\theta})-1\right| \\
&= M\left|\frac{(1+Me^{i\theta})c_a+k}{c_{a+1}(1+Me^{i\theta})}\delta + (1-\delta)\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|\frac{k\delta}{1+Me^{i\theta}} + c_a\delta + c_{a+1}(1-\delta)\right| \\
&\geq \frac{M}{|c_{a+1}|}\left|\frac{\delta}{1+M} + \delta c_a + c_{a+1}(1-\delta)|\right|
\end{aligned}$$

where $z \in \mathcal{U}$, $\theta \in \mathbb{R}$ and $k \geq 1$.

Remark 6.11 *If $c_a = 1$ and $m = 1$, then Example 6.6* (2) *reduces to [19, Example 2.2, p. 1775] while for $c_a = 1$ and $m = 1$, Example 6.6* (3) *reduces to [19, Example 2.3, p. 1775]. If $a = c_a = \alpha_1$, then [19, Corollary 2.11, p. 1774] follows from Example 6.6* (1)*.*

6.5 Dominant for Functions with Positive Real Part

In this section, the class $\Psi_L[\Omega, q]$ is considered where $q(z) = (1+z)/(1-z)$. If $|\zeta| = 1$, then $q(\zeta) = i\rho$ ($\rho \in \mathbb{R}$), $\zeta q'(\zeta) = -(1+\rho^2)/2$ and $\operatorname{Re}\big(1+\zeta q''(\zeta)/q'(\zeta)\big) = 0$ [111, p. 26]. First, the class of admissible functions $\Psi_L[\Omega, q]$ for this particular q is introduced.

Definition 6.12 *Let $\Omega \subset \mathbb{C}$. The class of admissible functions $\Phi_L[\Omega, 1]$ consists of functions $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ satisfying the admissibility condition $\phi(u,v,w;z) \notin \Omega$ whenever*

$$\begin{aligned}
u = i\rho,\ \ v &= 1 - \frac{c_a}{c_{a+1}} + \frac{c_a\rho^2-\sigma}{c_{a+1}\rho}i, \\
w = 1 - \frac{c_a}{c_{a+2}} &+ \frac{c_a\rho^2-\sigma}{c_{a+2}\rho}i + \frac{\rho(\mu+\sigma)+(\nu\rho+c_a\sigma\rho^2+\sigma^2)i}{c_{a+2}\rho(\sigma-c_a\rho^2+(c_{a+1}-c_a)i\rho)},
\end{aligned}$$

$$\sigma \leq -m(1+\rho^2)/2, \quad \sigma + \mu \leq 0, \quad z \in \mathcal{U} \quad \text{and} \quad m \geq 1.$$

If $\Omega = q(\mathcal{U}) = \{z : \operatorname{Re} z > 0\}$, *then the class of* $\Phi_L[q(\mathcal{U}), 1]$ *consists of functions* $\phi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$ *satisfying* $\operatorname{Re}\phi(u, v, w, ; z) \leq 0$ *whenever* $\sigma \leq -m(1+\rho^2)/2 \leq 0$, $\sigma + \mu \leq 0$, $c_a c_{a+1} c_{a+2} \neq 0$, *and* $z \in \mathcal{U}$.

In view of Definition 6.12, next theorem is a immediate consequence of Theorem 6.6 where $q(z) = (1+z)/(1-z)$.

Theorem 6.12 *Let* $\phi \in \Phi_L[\Omega, 1]$. *If* $f \in \mathcal{A}_m$ *satisfies*

$$\left\{\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) : z \in \mathcal{U}\right\} \subset \Omega,$$

then

$$\operatorname{Re}\frac{L_{a+1}f(z)}{L_a f(z)} > 0.$$

In particular, if $\phi \in \Phi_L[q(\mathcal{U}), 1]$ *where* $q(z) = (1+z)/(1-z)$, *then*

$$\operatorname{Re}\phi\left(\frac{L_{a+1}f(z)}{L_a f(z)}, \frac{L_{a+2}f(z)}{L_{a+1}f(z)}, \frac{L_{a+3}f(z)}{L_{a+2}f(z)}; z\right) > 0 \Rightarrow \operatorname{Re}\frac{L_{a+1}f(z)}{L_a f(z)} > 0.$$

Example 6.7 *If* $f \in \mathcal{A}_m$ *and* $c_{a+1} > 0$, *then*

$$\operatorname{Re}\left(\frac{\dfrac{L_{a+2}f(z)}{L_{a+1}f(z)}}{\dfrac{L_{a+1}f(z)}{L_a f(z)}}\right) > \frac{2c_a + m}{2c_{a+1}} \Rightarrow \operatorname{Re}\frac{L_{a+1}f(z)}{L_a f(z)} > 0.$$

Let $\phi(u, v) = v/u$. *Since*

$$\begin{aligned}\operatorname{Re}\phi\left(i\rho, 1 - \frac{c_a}{c_{a+1}} + \frac{c_a\rho^2 - \sigma}{c_{a+1}\rho}i\right) &= \frac{c_a}{c_{a+1}} - \frac{\sigma}{\rho^2 c_{a+1}} \\ &\geq \frac{c_a}{c_{a+1}} + \frac{m}{2c_{a+1}},\end{aligned}$$

whenever $\sigma \leq -m(1+\rho^2)/2$, *and* $m \geq 1$, *the admissibility condition in Definition*

6.12 is satisfied. Hence the result follows from Theorem 6.12.

Example 6.8 *Let* $0 < c_a < c_{a+1}$ *and* $\delta > 0$. *If* $f \in \mathcal{A}_m$ *satisfies*

$$\operatorname{Re}\left(\delta \frac{L_{a+2}f(z)}{L_{a+1}f(z)} + (1-\delta)\frac{L_{a+1}f(z)}{L_a f(z)}\right) > \delta\left(1 - \frac{c_a}{c_{a+1}}\right),$$

then

$$\operatorname{Re}\left(\frac{L_{a+1}f(z)}{L_a f(z)}\right) > 0,$$

and, in particular,

$$\operatorname{Re}\left(\frac{L_{a+2}f(z)}{L_{a+1}f(z)}\right) > 0 \Rightarrow \operatorname{Re}\frac{L_{a+1}f(z)}{L_a f(z)} > 0.$$

To verify the admissibility condition in Definition 6.12, take $\phi(u,v) = \delta v + (1-\delta)u$. *Since*

$$\begin{aligned}\operatorname{Re}\phi\left(i\rho, 1 - \frac{c_a}{c_{a+1}} + \frac{c_a\rho^2 - \sigma}{c_{a+1}\rho}i\right) &= \operatorname{Re}\delta\left(1 - \frac{c_a}{c_{a+1}} + \frac{c_a\rho^2 - \sigma}{c_{a+1}\rho}i + (1-\delta)i\rho\right)\\ &= \delta\left(1 - \frac{c_a}{c_{a+1}}\right),\end{aligned}$$

the result follows from Theorem 6.12.

Example 6.9 *Let* $0 < c_a < c_{a+1}$. *If* $f \in \mathcal{A}_m$ *satisfies*

$$\operatorname{Re}\left(\frac{L_{a+1}f(z)}{L_a f(z)} + \frac{L_{a+2}f(z)}{L_{a+1}f(z)} - \left(\frac{L_{a+1}f(z)}{L_a f(z)}\right)^2\right) > 1 - \frac{c_a}{c_{a+1}},$$

then

$$\operatorname{Re}\frac{L_{a+1}f(z)}{L_a f(z)} > 0.$$

Let $\phi(u,v) = u + v - u^2$. *The admissibility condition given in Definition 6.12* *follows from*

$$\begin{aligned}\operatorname{Re}\phi\left(i\rho, 1-\frac{c_a}{c_{a+1}}+\frac{c_a\rho^2-\sigma}{c_{a+1}\rho}i\right) &= \operatorname{Re}\left(i\rho+1-\frac{c_a}{c_{a+1}}+\frac{c_a\rho^2-\sigma}{c_{a+1}\rho}i-(i\rho)^2\right)\\ &\geq 1-\frac{c_a}{c_{a+1}}.\end{aligned}$$

Then Theorem 6.12 proves the result.

CHAPTER 7

RADIUS CONSTANTS FOR ANALYTIC FUNCTIONS WITH FIXED SECOND COEFFICIENT

7.1 Introduction

This chapter deals with radius properties of analytic functions with fixed second coefficient. Let $\mathcal{M}$ be a set of functions and $\mathcal{P}$ be a property which functions in $\mathcal{M}$ may or may not possess in a disk $|z| < r$. The least upper bound of all numbers r such that every function $f \in \mathcal{M}$ has the property $\mathcal{P}$ in the disk $\mathcal{U}_r = \{z : |z| < r\}$ is the radius for the property $\mathcal{P}$ in the set $\mathcal{M}$.

de Branges [37] proved the Bieberbach conjecture that $|a_n| \leq n$ $(n \geq 2)$ for $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. However, the inequality $|a_n| \leq n$ $(n \geq 2)$ need not imply f is univalent; for example, $f(z) = z + 2z^2$ is not a member of $\mathcal{S}$.

Gavrilov [58] showed that the radius of univalence for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ is the real root $r_0 \approx 0.1648$ of the equation $2(1-r)^3-(1+r) = 0$, and the result is sharp for $f(z) = 2z - z/(1-z)^2$. Gavrilov also proved that the radius of univalence for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq M$ $(n \geq 2)$ is $1 - \sqrt{M/(1+M)}$. The inequality $|a_n| \leq M$ holds for functions $f \in \mathcal{A}$ satisfying $|f(z)| \leq M$, and for these functions, Landau [90] proved that the radius of univalence is $M - \sqrt{M^2-1}$. Yamashita [209] showed that the radius of univalence obtained by Gavrilov [58] is also the radius of starlikeness for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ or $|a_n| \leq M$ $(n \geq 2)$. Additionally Yamashita [209] determined that the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ is the real root $r_0 \approx 0.0903$ of the equation $2(1-r)^4-(1+4r+r^2) = 0$, while the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq M$ $(n \geq 2)$ is the real root of

$$(M+1)(1-r)^3 - M(1+r) = 0.$$

Recently Kalaj *et al.* [74] obtained the radii of univalence, starlikeness, and convexity for harmonic mappings satisfying similar coefficient inequalities.

This chapter studies the class $\mathcal{A}_b$ consisting of functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (|a_2| = 2b,\ 0 \le b \le 1, z \in \mathcal{U}).$$

Univalent functions in $\mathcal{A}_b$ have been studied in [9,13,122,161]. In [161], Ravichandran obtained the sharp radii of starlikeness and convexity of order α for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \le n$ or $|a_n| \le M$ $(M > 0)$, $n \ge 3$. The radius constants for uniform convexity and parabolic starlikeness for functions $f \in \mathcal{A}_b$ satisfying $|a_n| \le n$, $n \ge 3$ were also obtained.

In [92], Lewandowski *et al.* proved that an analytic function f satisfying

$$\operatorname{Re}\left(\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > 0 \quad (z \in \mathcal{U}) \tag{7.1}$$

is starlike. The class of such functions is easily extended to

$$\operatorname{Re}\left(\alpha\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right) > \beta \quad (\alpha \ge 0,\ \beta < 1,\ z \in \mathcal{U}), \tag{7.2}$$

and has subsequently been investigated in [97, 133, 134, 142, 157, 160, 162]. For $-\alpha/2 \le \beta < 1$, Li and Owa [93] proved that functions satisfying (7.2) are starlike. In chapter 5, sufficient conditions of convexity and starlikeness of positive order were obtained for functions satisfying (7.2). Several coefficient inequalities related to this class were also investigated.

Another related class is the class of analytic functions satisfying

$$\operatorname{Re}\frac{zf'(z)}{f(z)} < \beta \quad (\beta > 1,\ z \in \mathcal{U}).$$

This class was studied in [128, 138, 204, 206]. In [96], Liu *et al.* extended the class to functions satisfying

$$\operatorname{Re}\left(\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right)<\beta \quad (\alpha\geq 0,\ \beta>1,\ z\in\mathcal{U}). \tag{7.3}$$

Now functions satisfying (7.2) or (7.3) evidently belongs to the class

$$\mathcal{L}(\alpha,\beta):=\left\{f\in\mathcal{A}:\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\prec\frac{1+(1-2\beta)z}{1-z},\ \beta\in\mathbb{R}\setminus\{1\},\ \alpha\geq 0\right\}. \tag{7.4}$$

Denote by $\mathcal{L}_0(\alpha,\beta)$ its subclass consisting of functions $f\in\mathcal{A}$ satisfying

$$\left|\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}-1\right|\leq|1-\beta| \quad (\beta\in\mathbb{R}\setminus\{1\},\ \alpha\geq 0,\ z\in\mathcal{U}).$$

A sufficient condition for functions $f\in\mathcal{A}$ to belong to the class $\mathcal{L}(\alpha,\beta)$ is given in the following lemma.

Lemma 7.1 [97,197] *Let $\beta\in\mathbb{R}\setminus\{1\}$, and $\alpha\geq 0$. If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n\in\mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty}\left(\alpha n^2+(1-\alpha)n-\beta\right)|a_n|\leq|1-\beta|, \tag{7.5}$$

then $f\in\mathcal{L}(\alpha,\beta)$.

Next let $\mathcal{ST}[A,B]$ denote the class of Janowski starlike functions $f\in\mathcal{A}$ satisfying the subordination

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz} \quad (-1\leq B<A\leq 1).$$

This class was introduced by Janowski [73]. Certain well-known subclasses of starlike functions are special cases of $\mathcal{ST}[A,B]$ for suitable choices of the parameters A and B. For example, for $0\leq\beta<1$, $\mathcal{ST}(\beta):=\mathcal{ST}[1-2\beta,-1]$ is the familiar class of starlike functions of order β, and $\mathcal{ST}_\beta := \mathcal{L}_0(0,\beta) = \mathcal{ST}[1-\beta,0]$. Janowski [73] obtained the exact value of the radius of convexity for $\mathcal{ST}[A,B]$.

Another result that will be required in our investigation is the following result of Goel and Sohi [59].

Lemma 7.2 [59] *Let $-1 \leq B < A \leq 1$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies the inequality*

$$\sum_{n=2}^{\infty} \big((1-B)n - (1-A)\big)|a_n| \leq A - B, \tag{7.6}$$

then $f \in \mathcal{ST}[A,B]$.

The Taylor coefficients of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ are known to satisfy certain coefficient inequality. For instance, starlike, and close-to-convex functions are bounded by $|a_n| \leq n$ $(n \geq 2)$ ([126], [163]).

A domain D is convex in the direction of the imaginary axis if its intersection with each parallel line to the imaginary axis is connected (or empty). A function f is convex in the direction of the imaginary axis if $f(\mathcal{U})$ is a domain convex in the direction of the imaginary axis. Goodman [61, p. 210] showed that the Taylor coefficients of convex functions in the direction of imaginary axis satisfy $|a_n| \leq n$ $(n \geq 2)$.

For starlike functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ of order $1/2$, it is known that $|a_n| \leq 1$ $(n \geq 2)$ [177]. Loewner [99] showed that a convex function f satisfies the sharp inequality $|a_n| \leq 1$ $(n \geq 2)$. Sakaguchi [174] introduced the class of starlike functions with respect to symmetric points and proved that if f is univalent and starlike with respect to symmetric points, then f satisfies $|a_n| \leq 1$ $(n \geq 2)$ and the result is sharp .

A function f is said to be close-to-convex with argument β if there is a $\phi \in \mathcal{CV}$ such that

$$\operatorname{Re} \frac{f'(z)}{e^{i\beta}\phi'(z)} > 0.$$

Goodman [64] showed that if f is close-to-convex with argument β, then $|a_n| \leq 1+(n-1)\cos\beta$. The class of uniformly starlike and uniformly convex functions are respectively defined by

$$\mathcal{UCV} := \left\{ f \in \mathcal{S} : \ \operatorname{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0, \ \zeta, z \in \mathcal{U} \right\}, \tag{7.7}$$

$$\mathcal{UST} := \left\{ f \in \mathcal{S} : \ \operatorname{Re}\left(\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)}\right) > 0, \ \zeta, z \in \mathcal{U} \right\}. \tag{7.8}$$

The Taylor coefficients of uniformly starlike functions are bounded by $|a_n| \leq 2/n$ $(n \geq 2)$ [63], and the uniformly convex functions by $|a_n| \leq 1/n$ $(n \geq 2)$ [62]. Also, functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ with $\operatorname{Re} f'(z) > 0$ satisfy $|a_n| \leq 2/n$ $(n \geq 2)$.

This chapter studies functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying either $|a_n| \leq cn + d$ $(c, \ d \geq 0)$ or $|a_n| \leq c/n$ $(c > 0)$ for $n \geq 3$. Sharp $\mathcal{L}(\alpha,\beta)$-radius and $\mathcal{ST}[A,B]$-radius are obtained for these classes. Several known radius constants are shown to be specific cases of the results obtained in this chapter.

7.2 Radius Constants

In the present section, the sharp $\mathcal{L}_0(\alpha,\beta)$-radius and $\mathcal{L}(\alpha,\beta)$-radius of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$ for $n \geq 3$ are obtained.

Theorem 7.1 *Let $\beta \in \mathbb{R} \setminus \{1\}$, and $\alpha \geq 0$. The $\mathcal{L}(\alpha,\beta)$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$, $c, d \geq 0$ for $n \geq 3$ is the real root in $(0,1)$ of the equation*

$$\begin{aligned} \big[(c+d)(1-\beta) + |1-\beta| + (2\alpha+2-\beta)(2(c-b)+d)r\big](1-r)^4 \\ = c\alpha(1+4r+r^2) + \big((1-\alpha)c + \alpha d\big)(1-r^2) \\ + \big((1-\alpha)d - \beta c\big)(1-r)^2 - \beta d(1-r)^3. \end{aligned} \tag{7.9}$$

For $\beta < 1$, this number is also the $\mathcal{L}_0(\alpha,\beta)$-radius of $f \in \mathcal{A}_b$. The results are sharp.

Proof. For $0 \le r_0 < 1$, the following identities hold:

$$\sum_{n=3}^{\infty} r_0^n = \frac{1}{1-r_0} - 1 - r_0, \tag{7.10}$$

$$\sum_{n=3}^{\infty} n r_0^n = \frac{1}{(1-r_0)^2} - 1 - 2r_0, \tag{7.11}$$

$$\sum_{n=3}^{\infty} n^2 r_0^n = \frac{1+r_0}{(1-r_0)^3} - 1 - 4r_0, \tag{7.12}$$

$$\sum_{n=3}^{\infty} n^3 r_0^n = \frac{1+4r_0+r_0^2}{(1-r_0)^4} - 1 - 8r_0. \tag{7.13}$$

The number r_0 is the $\mathcal{L}(\alpha,\beta)$-radius of function $f \in \mathcal{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathcal{L}(\alpha,\beta)$. Therefore, by Lemma 7.1, it is sufficient to verify the inequality

$$\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| r_0^{n-1} \le |1-\beta|, \tag{7.14}$$

where r_0 is the real root in $(0,1)$ of (7.9). Using (7.10), (7.11), (7.12), and (7.13) for $f \in \mathcal{A}_b$ lead to

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| r_0^{n-1} \\
&\leq 2(2\alpha + 2 - \beta) b r_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)(cn+d) r_0^{n-1} \\
&= 2(2\alpha + 2 - \beta) b r_0 + c\alpha \left(\frac{1 + 4r_0 + r_0^2}{(1-r_0)^4} - 1 - 8r_0 \right) \\
&\qquad + \big((1-\alpha)c + \alpha d\big) \left(\frac{1+r_0}{(1-r_0)^3} - 1 - 4r_0 \right) \\
&\qquad + \big((1-\alpha)d - \beta c\big) \left(\frac{1}{(1-r_0)^2} - 1 - 2r_0 \right) \\
&\qquad - \beta d \left(\frac{1}{1-r_0} - 1 - r_0 \right) \\
&= (c+d)(\beta - 1) - (2\alpha + 2 - \beta)\big(2(c-b) + d\big) r_0 \\
&\qquad + \Big(c\alpha(1 + 4r_0 + r_0^2) + \big((1-\alpha)c + \alpha d\big)(1 - r_0^2) \\
&\qquad + \big((1-\alpha)d - \beta c\big)(1-r_0)^2 - \beta d (1-r_0)^3 \Big) / (1-r_0)^4 \\
&= |1 - \beta|.
\end{aligned}
$$

For $\beta < 1$, consider the function

$$
f_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} (cn+d) z^n = (c+1)z + 2(c-b)z^2 - \frac{cz}{(1-z)^2} - \frac{dz^3}{1-z}. \quad (7.15)
$$

At the point $z = r_0$ where r_0 is the root in $(0,1)$ of (7.9), f_0 satisfies

$$
\operatorname{Re} \left(\alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} \right) = 1 - \frac{N(r_0)}{D(r_0)} = \beta, \quad (7.16)
$$

where

$$N(r_0) = -2(c-b)(2\alpha+1)r_0 + \frac{2cr_0(2\alpha+1)}{(1-r_0)^3} + \frac{6c\alpha r_0^2}{(1-r_0)^4} + \frac{2dr_0^2(3\alpha+1)}{1-r_0} + \frac{dr_0^3(6\alpha+1)}{(1-r_0)^2} + \frac{2dr_0^4\alpha}{(1-r_0)^3},$$

$$D(r_0) = c+1+2(c-b)r_0 - \frac{c}{(1-r_0)^2} - \frac{dr_0^2}{1-r_0}.$$

This shows that r_0 is the sharp $\mathcal{L}(\alpha,\beta)$-radius for $f \in \mathcal{A}_b$. For $\beta < 1$, equation (7.16) shows that the rational expression $N(r_0)/D(r_0)$ is positive, and therefore

$$\left|\alpha \frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} - 1\right| = 1 - \beta.$$

Thus, r_0 is the sharp $\mathcal{L}_0(\alpha,\beta)$-radius for $f \in \mathcal{A}_b$ when $\beta < 1$.

For $\beta > 1$,

$$f_0(z) = z + 2bz^2 + \sum_{n=3}^{\infty}(cn+d)z^n = (1-c)z + 2(b-c)z^2 + \frac{cz}{(1-z)^2} + \frac{dz^3}{1-z} \quad (7.17)$$

shows sharpness of the result. The proof is similar to the case $\beta < 1$, and is thus omitted. ∎

For $c = 1$, $d = 0$ and $\alpha = 0$, Theorem 7.1 reduces to the radius of starlikeness of order β for functions in $\mathcal{A}_b$ as follows;

Corollary 7.1 [161, Theorem 2.1, p. 3] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ *and* $|a_n| \le n$ *for* $n \ge 3$. *Then* f *satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \le 1 - \beta \quad (|z| \le r_0) \quad (7.18)$$

where $r_0 = r_0(\beta)$ is the real root in $(0,1)$ of the equation

$$1-\beta+(1+\beta)r = 2\big(1-\beta+(2-\beta)(1-b)r\big)(1-r)^3.$$

The number $r_0(\beta)$ is also the radius of starlikeness of order β. The number $r_0(1/2)$ is the radius of parabolic starlikeness. The results are sharp.

For $\beta = 0$, and $0 \leq b \leq 1$, Corollary 7.1 yields the radius of starlikeness obtained by Yamashita [209].

Corollary 7.2 [209] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfy the coefficient inequality $|a_n| \leq n$ for $n \geq 2$. The radius of starlikeness of f is the real root $r_0 \approx 0.1648$ of the equation $2(1-r)^3 - (1+r) = 0$.*

Proof. Choose $\beta = 0$ in Corollary 7.1, and let r_b be the real root of

$$1+r = \big(2+4(1-b)r\big)(1-r)^3 \quad (0 \leq b \leq 1). \tag{7.19}$$

Thus every function $f \in \mathcal{A}_b$ is starlike in each disk $C_r = \{z : |z| < r\}$ for every $r < r_b$. Differentiating implicitly from (7.19) with respect to b gives

$$\frac{\partial r}{\partial b} = \frac{4r(1-r)^3}{4(1-b)(1-r)^3 - 3(1-r)^2(2+4(1-b)r) - 1} \leq 0.$$

It shows that if b varies from 0 to 1, then the radius starlikeness of functions $f \in \mathcal{A}_b$ is decreasing. Therefore,

$$r_1 < \cdots < r_{3/4} < r_{1/2} < r_{1/4} < \cdots < r_0,$$

In view of the facts that

$$\mathcal{A} = \bigcup_{0\leq b\leq 1} \mathcal{A}_b, \quad C_{r_1} = \bigcap_{0\leq b\leq 1} C_{r_b},$$

it follows that every function $f \in \mathcal{A}$ is starlike in each disk C_r for every $r < r_1$.

∎

In the case $\alpha = 0$, $c = 0$ and $d = M$, Theorem 7.1 leads to Theorem 2.5 in [161, p. 5].

Corollary 7.3 [161, Theorem 2.5, p. 5] *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ and $|a_n| \leq M$ for $n \geq 3$. Then f satisfies* (7.18) *where $r_0 = r_0(\beta)$ is the real root in $(0,1)$ of the equation*

$$M(1 - \beta + \beta r) = \big((1 + M)(1 - \beta) - (2 - \beta)(2b - M)r\big)(1 - r)^2.$$

The number $r_0(\beta)$ is also the radius of starlikeness of order β. The number $r_0(1/2)$ is the radius of parabolic starlikeness. The results are sharp.

Next, the sharp $\mathcal{L}(\alpha, \beta)$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$, $c > 0$ for $n \geq 3$ is obtained.

Theorem 7.2 *Let $\beta \in \mathbb{R} \setminus \{1\}$, and $\alpha \geq 0$. The $\mathcal{L}(\alpha, \beta)$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$ for $n \geq 3$ and $c > 0$ is the real root in $(0, 1)$ of*

$$\begin{aligned} &\left[c(1 - \beta) + |1 - \beta| + (2\alpha + 2 - \beta)r\left(\frac{c}{2} - 2b\right)\right](1 - r)^2 \\ &= c\alpha + (1 - \alpha)c(1 - r) + \beta c(1 - r)^2 \frac{\log(1 - r)}{r}. \end{aligned} \tag{7.20}$$

For $\beta < 1$, this number is also the $\mathcal{L}_0(\alpha, \beta)$-radius of $f \in \mathcal{A}_b$. The results are sharp.

Proof. By Lemma 7.1, r_0 is the $\mathcal{L}(\alpha, \beta)$-radius of functions $f \in \mathcal{A}_b$ when (7.14) holds for the real root r_0 of equation (7.20) in $(0, 1)$. Using (7.10) and (7.11) together with

$$\sum_{n=3}^{\infty} \frac{r_0^n}{n} = -\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2}, \tag{7.21}$$

for $f \in \mathcal{A}_b$ imply that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big)|a_n| r_0^{n-1} \\
&\leq 2(2\alpha + 2 - \beta) b r_0 + \sum_{n=3}^{\infty} \big(\alpha n^2 + (1-\alpha)n - \beta\big) \left(\frac{c}{n}\right) r_0^{n-1} \\
&= 2(2\alpha + 2 - \beta) b r_0 + c\alpha \left(\frac{1}{(1-r_0)^2} - 1 - 2r_0\right) \\
&\qquad + (1-\alpha)c \left(\frac{1}{1-r_0} - 1 - r_0\right) \\
&\qquad - \beta c \left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\
&= c(\beta - 1) + (2\alpha + 2 - \beta) r_0 \left(2b - \frac{c}{2}\right) \\
&\qquad + \frac{c\alpha r_0 + (1-\alpha)c(1-r_0)r_0 + \beta c(1-r_0)^2 \log(1-r_0)}{(1-r_0)^2 r_0} \\
&= |1-\beta|.
\end{aligned}$$

To verify sharpness for $\beta < 1$, consider the function

$$f_0(z) = z - 2bz^2 - \sum_{n=3}^{\infty} \frac{c}{n} z^n = (1+c)z + \left(\frac{c}{2} - 2b\right) z^2 + c \log(1-z). \tag{7.22}$$

At the point $z = r_0$ where r_0 is the root in $(0,1)$ of equation (7.20), f_0 satisfies

$$\begin{aligned}
&\operatorname{Re}\left(\alpha\frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)}\right) \\
&= 1 - \frac{-\left(\frac{c}{2} - 2b\right) r_0(2\alpha+1) + \frac{c r_0 \alpha}{(1-r_0)^2} + \frac{c}{1-r_0} + \frac{c\log(1-r_0)}{r_0}}{(1+c) + \left(\frac{c}{2} - 2b\right) r_0 + \frac{c\log(1-r_0)}{r_0}} \qquad (7.23) \\
&= \beta.
\end{aligned}$$

Thus r_0 is the sharp $\mathcal{L}(\alpha,\beta)$-radius of $f \in \mathcal{A}_b$. Since $\beta < 1$, the rational expression in (7.23) is positive, and therefore

$$\left|\alpha\frac{z^2 f_0''(z)}{f_0(z)} + \frac{z f_0'(z)}{f_0(z)} - 1\right| = 1 - \beta,$$

which shows r_0 is the sharp $\mathcal{L}_0(\alpha,\beta)$-radius of $f \in \mathcal{A}_b$. For $\beta > 1$, sharpness of the result is demonstrated by the function f_0 given by

$$f_0(z) = z + 2bz^2 + \sum_{n=3}^{\infty} \frac{c}{n} z^n = (1-c)z + \left(2b - \frac{c}{2}\right) z^2 - c\log(1-z).$$

∎

For $\alpha = 0$, Theorem 7.2 coincides with the radius of starlikeness of order β for $f \in \mathcal{A}_b$ obtained by Ravichandran [161].

Corollary 7.4 [161, Theorem 2.8, p. 6] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ *and* $|a_n| \leq M/n$ *for* $n \geq 3$. *Then* f *satisfies the inequality* (7.18) *where* $r_0 = r_0(\beta)$ *is the real root in* $(0,1)$ *of the equation*

$$2M(1 + \beta(1-r)\log(1-r) = r\big(2(1+M)(1-\beta) + (2-\beta)(M-4b)r\big)(1-r).$$

The number $r_0(\beta)$ is also the radius of starlikeness of order β. The number $r_0(1/2)$ is the radius of parabolic starlikeness. The results are sharp.

7.3 Radius of Janowski Starlikeness

The next result finds the sharp $\mathcal{ST}[A,B]$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$, $c, d \geq 0$ for $n \geq 3$.

Theorem 7.3 *Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A,B]$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq cn + d$ for $n \geq 3$ and $c, d \geq 0$ is the real root in $(0,1)$ of*

$$\begin{aligned} &[(A-B)(c+d+1) - (2b-2c-d)\big(2(1-B)-(1-A)\big)r](1-r)^3 \\ &\quad = c(1-B)(1+r) + \big(d(1-B) - c(1-A)\big)(1-r) - (1-A)d(1-r)^2. \end{aligned} \tag{7.24}$$

The result is sharp.

Proof. It is evident that r_0 is the $\mathcal{ST}[A,B]$-radius of $f \in \mathcal{A}_b$ if and only if $f(r_0 z)/r_0 \in \mathcal{ST}[A,B]$. Hence, by Lemma 7.2, it suffices to show that

$$\sum_{n=2}^{\infty} \big((1-B)n - (1-A)\big)|a_n| r_0^{n-1} \leq A - B \quad (-1 \leq B < A \leq 1), \tag{7.25}$$

where r_0 is the root in $(0,1)$ of equation (7.24). From (7.10), (7.11), and (7.12) for function $f \in \mathcal{A}_b$, it follows that

$$\begin{aligned}
&\sum_{n=2}^{\infty} \big((1-B)n-(1-A)\big)|a_n|r_0^{n-1} \\
&\leq 2\big(2(1-B)-(1-A)\big)br_0 + \sum_{n=3}^{\infty} \big((1-B)n-(1-A)\big)(cn+d)r_0^{n-1} \\
&= 2\big(2(1-B)-(1-A)\big)br_0 + c(1-B)\left(\frac{1+r_0}{(1-r_0)^3}-1-4r_0\right) \\
&\qquad + \big(d(1-B)-c(1-A)\big)\left(\frac{1}{(1-r_0)^2}-1-2r_0\right) \\
&\qquad - (1-A)d\left(\frac{1}{1-r_0}-1-r_0\right) \\
&= (B-A)(c+d) + (2b-2c-d)\big(2(1-B)-(1-A)\big)r_0 \\
&\qquad + \Big(c(1-B)(1+r_0) + \big(d(1-B)-c(1-A)\big)(1-r_0) \\
&\qquad - (1-A)d(1-r_0)^2\Big)/(1-r_0)^3 \\
&= A-B.
\end{aligned}$$

The function f_0 given by (7.15) shows that the result is sharp. Indeed, at the point $z = r_0$ where r_0 is the root in $(0,1)$ of equation (7.24), the function f_0 satisfies

$$\left|\frac{zf_0'(z)}{f_0(z)}-1\right| = \frac{-2(c-b)r_0 + \dfrac{2dr_0^2}{1-r_0} + \dfrac{dr_0^3}{(1-r_0)^2} + \dfrac{2cr_0}{(1-r_0)^3}}{c+1+2(c-b)r_0 - \dfrac{c}{(1-r_0)^2} - \dfrac{dr_0^2}{1-r_0}},$$

and

$$\left|A - B\frac{zf_0'(z)}{f_0(z)}\right| = \frac{(c+1)(A-B) + 2(c-b)r_0(A-2B)}{c+1+2(c-b)r_0 - \dfrac{c}{(1-r_0)^2} - \dfrac{dr_0^2}{1-r_0}}$$
$$- \frac{\dfrac{c(A-B)}{(1-r_0)^2} + \dfrac{2cr_0B}{(1-r_0)^3} - \dfrac{dr_0^2(A-3B)}{1-r_0} + \dfrac{dr_0^3B}{(1-r_0)^2}}{c+1+2(c-b)r_0 - \dfrac{c}{(1-r_0)^2} - \dfrac{dr_0^2}{1-r_0}}.$$

Then (7.24) yields

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \left|A - B\frac{zf_0'(z)}{f_0(z)}\right| \quad (-1 \leq B < A \leq 1,\ z = r_0), \tag{7.26}$$

or equivalently $f_0 \in \mathcal{ST}[A, B]$. ∎

Applying Theorem 7.3 to the particular case $c = 1$, and $d = 0$ yields the following corollary.

Corollary 7.5 *Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A,B]$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq n$ for $n \geq 3$ is the real root in $(0,1)$ of*

$$2\big((A-B) - (b-1)\big(2(1-B) - (1-A)\big)r\big)(1-r)^3 = (1-B)(1+r) - (1-A)(1-r).$$

The result is sharp.

For $c = 0$, and $d = M$, Theorem 7.3 leads to the next corollary.

Corollary 7.6 *Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A,B]$-radius of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq M$ for $n \geq 3$, and*

$M \geq 0$ is the real root in $(0,1)$ of

$$\big((A-B)(M+1)-(2b-M)\big(2(1-B)-(1-A)\big)r\big)(1-r)^3$$
$$= M(1-B)(1-r)-(1-A)M(1-r)^2.$$

The result is sharp.

Remark 7.1 *The $\mathcal{ST}[1-2\beta,-1]$-radius for function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \mathcal{A}_b$ satisfying $|a_n| \leq cn+d$, $c,d \geq 0$ coincides with the radius of starlikeness of order β for function $f \in \mathcal{A}_b$ obtained in Corollary 7.1. The radius of starlikeness of order $1/2$ for function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ is the real root $r_0 \approx 0.1203$ of the equation $2(1-r)^3 = 1+3r$, while this radius constant for function $f \in \mathcal{A}_b$ satisfying $|a_n| \leq n$ $(n \geq 3)$ is the real root $r_0 \approx 0.2062$ of the equation $(1-r)^3(2+6r) = 1+3r$.*

The last theorem will obtain the $\mathcal{ST}[A,B]$-radius of $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$ for $n \geq 3$ and $c>0$.

Theorem 7.4 *Let $-1 \leq B < A \leq 1$. The $\mathcal{ST}[A,B]$-radius of $f(z)=z+\sum_{n=2}^{\infty}a_nz^n \in \mathcal{A}_b$ satisfying the coefficient inequality $|a_n| \leq c/n$ for $n \geq 3$ and $c>0$ is the real root in $(0,1)$ of the equation*

$$\begin{aligned}\left((c+1)(A-B)-\big(2(1-B)-(1-A)\big)r\left(2b-\frac{c}{2}\right)\right)(1-r)& \\ = c(1-B)+c(1-A)(1-r)\frac{\log(1-r)}{r}&\end{aligned} \tag{7.27}$$

The result is sharp.

Proof. By Lemma 7.2, condition (7.25) assures that r_0 is the $\mathcal{ST}[A,B]$-radius of $f \in \mathcal{A}_b$ where r_0 is the real root of (7.27). Therefore, using (7.10) and (7.21) for

$f \in \mathcal{A}_b$ yield

$$\begin{aligned}
&\sum_{n=2}^{\infty} \big((1-B)n-(1-A)\big)|a_n| r_0^{n-1} \\
&\leq 2\big(2(1-B)-(1-A)\big) b r_0 + \sum_{n=3}^{\infty} \big((1-B)n-(1-A)\big) \left(\frac{c}{n}\right) r_0^{n-1} \\
&= 2\big(2(1-B)-(1-A)\big) b r_0 + c(1-B)\left(\frac{1}{1-r_0} - 1 - r_0\right) \\
&\qquad - c(1-A)\left(-\frac{\log(1-r_0)}{r_0} - 1 - \frac{r_0}{2}\right) \\
&= c(B-A) + \big(2(1-B)-(1-A)\big) r_0 \left(2b - \frac{c}{2}\right) \\
&\qquad + \frac{c(1-B)r_0 + c(1-A)(1-r_0)\log(1-r_0)}{(1-r_0)r_0} \\
&= A - B.
\end{aligned}$$

The result is sharp for the function f_0 given by (7.22). Indeed, f_0 satisfies

$$\left|\frac{zf_0'(z)}{f_0(z)} - 1\right| = \frac{-\left(\frac{c}{2} - 2b\right) r_0 + \frac{c}{1-r_0} + \frac{c\log(1-r_0)}{r_0}}{(1+c) + \left(\frac{c}{2} - 2b\right) r_0 + \frac{c\log(1-r_0)}{r_0}},$$

and

$$\left|A - B\frac{zf_0'(z)}{f_0(z)}\right| = \frac{(1+c)(A-B) + (A-2B)\left(\frac{c}{2} - 2b\right) r_0 + \frac{cB}{1-r_0} + \frac{cA\log(1-r_0)}{r_0}}{(1+c) + \left(\frac{c}{2} - 2b\right) r_0 + \frac{c\log(1-r_0)}{r_0}},$$

at the point $z = r_0$ where r_0 is the root in $(0,1)$ of equation (7.27). From (7.27), the function f_0 is seen to satisfy (7.26), and hence the result is sharp. ∎

Remark 7.2 *(1) Let* $f(z) = z + \sum_{n=3}^{\infty} a_n z^n \in \mathcal{A}_b$ *satisfy* $|a_n| \leq 1/n$ $(n \geq 3)$. *The radius of starlikeness is the real root* $r_0 \approx 0.6180$ *of the equation* $(2+r)(1-r) = 1$, *and the radius of starlikeness of order 1/2 is the real root*

$r_0 \approx 0.5241$ of

$$(1-r)\left(2+\frac{3r}{2}\right)r = 2r-(1-r)\log(1-r).$$

(2) Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{A}$ satisfies $|a_n| \leq 1/n$ $(n \geq 2)$. The radius of starlikeness is the real root $r_0 \approx 0.2324$ of the equation $(2-3r)(1-r) = 1$, and the radius of starlikeness of order 1/2 is the real root $r_0 \approx 0.1583$ of the equation

$$\left(2-\frac{9r}{2}\right)(1-r)r = 2r-(1-r)\log(1-r).$$

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