**BULLETIN**
*of the*
MALAYSIAN MATHEMATICAL SOCIETY

*Bull. Malaysian Math. Soc.* (Second Series) **21** (1998) 11-16

# Reiter's Condition on Foundation Semigroups

A. NASR-ISFAHANI
Department of Mathematics, University of Isfahan, Isfahan 81744, Iran
e-mail: isfahani@math.ui.ac.ir

**Abstract.** Let $S$ be a foundation semigroup with identity and with a multiplicative weight function $w$. In this paper, we show that $S$ satisfies Reiter's condition if and only if there exists a left invariant mean on the dual space of the weighted measure algebra $M_a(S, w)$. Finally, we give an example which shows that this result is not true if the hypothesis that $S$ is foundation is dropped.

## 1. Introduction

H. Reiter, in [10], introduced and studied a condition on locally compact groups $G$, known as Reiter's condition, which has been instrumental in simplifying many results in harmonic analysis: For every $\varepsilon > 0$ and every compact set $K \subseteq G$, there exists a positive function $\varphi$ in $L^1(G)$ with $\|\varphi\|_1 = 1$ such that $\|\delta_x * \varphi - \varphi\|_1 < \varepsilon$ for all $x \in K$.

This condition was demonstrated for locally compact commutative groups in [10]. Most application invoke commutativity only to insure that this condition holds, thus it is interesting to discover that Reiter's condition is equivalent to the existence of a left invariant mean on $L^\infty(G)$. For countable discrete groups this result is contained in a result of E. Følner's [5, §2]; this was pointed out by M.M. Day who also proved this equivalence (using a somewhat different terminology) for arbitrary discrete groups and semigroups [3, pp. 524-525]. For locally compact groups it was proved by H. Reiter in [11] (see also Proposition 4.1 of A. Hulanicki [6] for a similar result). Later, M. Skantharajah, in [12], proved this result for those hypergroups which possess a left Harr measure.

The aim of the present paper is to extend this equivalence to an extensive class of topological semigroups, the so-called foundation semigroups. As a consequence of this result we infer that every commutative foundation semigroup with identity, satisfies Reiter's condition. Throughout this paper, we shall be concerned with a semigroup $S$ with a weight function $w$, and with functions that are $w$-bounded. Finally, we have included an example of commutative non-foundation semigroups for which these results collapse.

## 2. Preliminaries

Throughout, $S$ denotes a locally compact Hausdorff topological semigroup. We shall assume that there is a *weight function w* on *S*. By this we mean that $w : S \to \Re_+$ with the properties that $w$ is Borel measurable and that both $w$ and $1/w$ are bounded on compact subsets of $S$ and that $w(xy) \le w(x)\, w(y)$ for all $x, y \in S$. A complex-valued function $f$ on $S$ is called *w-bounded* if there is $k > 0$ such that $|f(x)| \le k\, w(x)$ for all $x \in S$.

Recall that $M_a(S)$ denotes the space of all measures $\mu \in M(S)$, the space of all bounded complex Radon measures on $S$, for which the mappings $x \mapsto \delta_x * |\mu|$ and $x \mapsto |\mu| * \delta_x$ from $S$ into $M(S)$ are weakly continuous, where $\delta_x$ denotes the Dirac measure at $x$ (see [4], or $\tilde{L}(S)$ as in [1] ). A topological semigroup $S$ is called *foundation semigroup* if $S$ coincides with the closure of $\bigcup \{ \operatorname{supp}(\mu) : \mu \in M_a(S) \}$. The space of all complex Radon measures $\mu$ on $S$ for which $w\mu \in M(S)$ is denoted by $M(S,w)$. With the convolution

$$(\mu * \nu)(f) = \int_S \int_S f(xy)\, d\mu(x)\, d\nu(y) \qquad (1)$$

for each continuous complex-valued function $f$ on $S$ with compact support, and the norm

$$\|\mu\|_w = \|w\mu\| = \int_S w(t)\, d\,|\mu|(t)$$

$M(S,w)$ defines a convolution measure algebra. Also, the space of all measures $\mu \in M(S,w)$ such that $w\mu \in M_a(S)$ is denoted by $M_a(S,w)$ (see [7], or $\tilde{L}(S,w)$ as in [8]). By Theorem 4.6 of [8], $M_a(S,w)$ is a two-sided $L$-ideal of $M_a(S,w)$, and (1) also holds for every $w$-bounded Borel measurable function $f$ on $S$.

We denote by $L^\infty(S; M_a(S))$ the set of all bounded complex-valued $\mu$-measurable $(\mu \in M_a(S))$ functions on $S$, and we also define

$$L^\infty(S; M_a(S,w)) = \left\{ f : \frac{f}{w} \in L^\infty(S; M_a(S)) \right\}.$$

We recall from Lemma 2 of [7] that

**Lemma 2.1.** *If $S$ is a foundation semigroup with identity and with a weight function w, then $L^\infty(S; M_a(S,w))$ with the norm $\|\cdot\|_{1/w}$ where $\|f\|_{1/w} = \|f/w\|_\infty$ $(f \in L^\infty(S; M_a(S,w)))$ and the multiplication $\odot$ where*

$$(f \odot g)(x) = \frac{f(x)\, g(x)}{w(x)} \qquad \big(f,\ g \in L^\infty(S; M_a(S,w))\big),$$

*and the involution* $f^* = \bar{f}$, *defines a commutative C*-algebra. Moreover, the mapping* $\tau : f \mapsto \tau_f \left( f \in L^\infty(S; M_a(S,w)) \right)$, *where*

$$\tau_f(\mu) = \int_S f(x)\, d\mu(x) \quad \left( \mu \in M_a(S,w) \right),$$

*defines an isometric isomorphism of* $L^\infty\left(S; M_a(S,w)\right)$ *onto the dual space of* $M_a(S,w)$.

A linear functional $m$ on $L^\infty\left(S; M_a(S,w)\right)$ is called a *mean* if $m(w) = 1$, and $m(f) \geq 0$ for each $f \in L^\infty\left(S; M_a(S,w)\right)$ with $f \geq 0$. A mean $m$ on $L^\infty\left(S; M_a(S,w)\right)$ is called a *left invariant mean* (LIM) if $m({}_x f) = w(x)\, m(f)$ for each $f \in L^\infty\left(S; M_a(S,w)\right)$ and $x \in S$, where ${}_x f \in L^\infty\left(S; M_a(S,w)\right)$ is defined by $({}_x f)(y) = f(xy)$ for all $y \in S$. Let $P(S,w)$ denotes the set of all positive measures $\eta$ in $M_a(S,w)$ with $\|\eta\|_w = 1$. For $\eta \in P(S,w)$ and $f \in L^\infty\left(S; M_a(S,w)\right)$ the function $\eta \circ f \in L^\infty\left(S; M_a(S,w)\right)$ is defined by

$$(\eta \circ f)(y) = \int_S f(xy)\, d\eta(x) \quad (y \in S).$$

A mean $m$ on $L^\infty\left(S; M_a(S,w)\right)$ is said to be a *topological left invariant mean* (TLIM) if $m(\eta \circ f) = m(f)$ for all $\eta \in P(S,w)$ and $f \in L^\infty\left(S; M_a(S,w)\right)$.

A.T.M. Lau, in [9], extended several fundamental characterizations of amenable locally compact groups to a large family of Banach algebras known as the Lau algebra. By Lemma 3 of [7], if $S$ is a foundation semigroup with identity and with a multiplicative weight function $w$, then $M_a(S,w)$ is a Lau algebra, and so the results obtained in [9] hold for $M_a(S,w)$. Here, we only state the following theorem which we will need in the next section.

**Theorem 2.2.** *Let S be a foundation semigroup with identity and with a multiplicative weight function w. Then the following are equivalent:*

*(i)* *There exists a* LIM *on* $L^\infty\left(S; M_a(S,w)\right)$;

*(ii)* *There exists a net* $(\eta_\alpha)$ *in* $P(S,w)$ *such that* $\| \eta * \eta_\alpha - \eta_\alpha \|_w \to 0$ *for each* $\eta \in P(S,w)$;

*(iii)* *There exists a* TLIM *on* $L^\infty\left(S; M_a(S,w)\right)$.

*Proof.* The equivalence of (i) and (iii) follows easily from Theorem 4.2.4 of [4]. By Theorem 4.1 and Theorem 4.6 of [9], (ii) and (iii) are also equivalent.

## 3. Reiter's Condition on *S*

We say that $S$ satisfies *Reiter's condition* if for every $\varepsilon > 0$ and every compact set $K \subseteq S$, there exists a measure $\eta \in P(S, w)$ such that

$$\left\| \frac{1}{w(x)} \delta_x * \eta - \eta \right\|_w < \varepsilon \quad \text{for all} \quad x \in K.$$

We now give the main result of this paper.

**Theorem 3.1.** *Let $S$ be a foundation semigroup with identity and with a multiplicative weight function $w$. Then $S$ satisfies Reiter's condition if and only if there exists a* LIM *on* $L^\infty(S; M_a(S, w))$.

*Proof.* Suppose that $S$ satisfies Reiter's condition. Then it is easy to see that there exists a net $(\eta_\alpha)$ in $P(S, w)$ such that $\| (1/w(x)) \delta_x * \eta_\alpha - \eta_\alpha \|_w \to 0$ for all $x \in S$. Therefore, every weak* cluster point of $(\eta_\alpha)$ in the dual space of $L^\infty(S; M_a(S, w))$ is a LIM on $L^\infty(S; M_a(S, w))$.

Conversely, suppose that there exists a LIM on $L^\infty(S; M_a(S, w))$. Let $\varepsilon > 0$ and compact set $K \subseteq S$ be given. Choose a compact neighbourhood $U$ of $e$, the identity of $S$. Since $S$ is a foundation semigroup, there exists a measure $\nu \in M_a(S)$ such that $U \cap \operatorname{supp}(\nu) \neq \varnothing$. Put

$$\eta_1 = \frac{\chi_U}{|\nu|(U) w} |\nu|,$$

where $\chi_U$ denotes the characteristic function of $U$. Then $\eta_1 \in P(S, w)$. Therefore, by Theorem 3.13 of [13], the mapping $x \mapsto \delta_x * w\eta_1$ from $S$ into $M_a(S)$ is $\|\cdot\|$-continuous. Hence the mapping $x \mapsto (1/w(x)) \delta_x * \eta_1$ from $S$ into $M_a(S, w)$ is $\|\cdot\|_w$-continuous. Thus the set $\{ (1/w(x)) \delta_x * \eta_1 : x \in K \}$ is $\|\cdot\|_w$-compact in $M_a(S, w)$. This implies that there exists a finite set $\{x_1, \cdots, x_n\}$ in $S$ such that

$$\left\{ \frac{1}{w(x)} \delta_x * \eta_1 : x \in K \right\} \subseteq \bigcup_{i=1}^{n} \left\{ \mu \in M_a(S, w) : \left\| \mu - \frac{1}{w(x_i)} \delta_{x_i} * \eta_1 \right\|_w < \frac{\varepsilon}{3} \right\}.$$

Now, by Theorem 2.2, there is a measure $\eta_2 \in P(S,w)$ such that

$$\| \eta_1 * \eta_2 - \eta_2 \|_w < \varepsilon/3$$

and also

$$\left\| \frac{1}{w(x_i)} \delta_{x_i} * \eta_1 * \eta_2 - \eta_2 \right\|_w < \varepsilon/3 \quad \text{for all} \quad 1 \leq i \leq n.$$

Let $x \in K$. Then $\| (1/w(x)) \delta_{x_i} * \eta_1 - (1/w(x_i)) \delta_{x_i} * \eta_1 \|_w < \varepsilon/3$ for some $1 \leq i \leq n$. Thus

$$\left\| \frac{1}{w(x)} \delta_x * \eta_1 * \eta_2 - \eta_1 * \eta_2 \right\|_w \leq \left\| \frac{1}{w(x)} \delta_x * \eta_1 * \eta_2 - \frac{1}{w(x_i)} \delta_{x_i} * \eta_1 * \eta_2 \right\|_w$$

$$+ \left\| \frac{1}{w(x_i)} \delta_{x_i} * \eta_1 * \eta_2 - \eta_2 \right\|_w + \| \eta_1 * \eta_2 - \eta_2 \|_w < \varepsilon .$$

Now, if we put $\eta = \eta_1 * \eta_2$, then since $w$ is multiplicative, we have $\|\eta\|_w = \|\eta_1\|_w \|\eta_2\|_w = 1$. Therefore $\eta \in P(S,w)$ and $\| (1/w(x)) \delta_x * \eta - \eta \|_w < \varepsilon$ for all $x \in K$.

**Corollary 3.2.** *If S is a commutative foundation semigroup with identity and with a multiplicative weight function w, then S satisfies Reiter's condition.*

*Proof.* Since $M_a(S,w)$ is a commutative Lau algebra, from the Example 1 and Theorem 4.1 of [9] and Theorem 2.2, it follows that there exists a LIM on $L^\infty(S; M_a(S,w))$. Now, an application of Theorem 3.1 completes the proof.

The following example shows that Theorem 3.1 and Corollary 3.2 are not valid in general for non-foundation semigroups.

**Example 3.3.** Let $S = ([0,1], \min)$. Then with the usual topology of the real line and the multiplicative weight function $w(x) = 1$ $(x \in S)$, we have $M_a(S,w) = \{0\}$. Thus $L^\infty(S; M_a(S,w))$ is the set of all bounded complex-valued functions on *S*, and so there exists a LIM on $L^\infty(S; M_a(S,w))$ (see for example Corollary 2.3.8 of [2] ). On the other hand, since $P(S,w) = \varnothing$, the commutative non-foundation semigroup $S$ does not satisfy Reiter's condition.

**Acknowledgement.** The author wishes to thank Dr. M. Lashkarizadeh-Bami for a preprint of [7].

## References

1. A.C. Baker and J.W. Baker, Algebra of measures on a locally compact semigroup III, *J. London Math. Soc.* **4** (1972), 685-695.
2. J.F. Bergland, H.D. Junghenn, and P. Milnes, *Analysis on semigroups,* Wiley, New York, 1989.
3. M.M. Day, Amenable semigroups, *Illinois J. Math.* **1** (1957), 509-544.
4. H.A.M. Dzinotyweyi, *The analogue of the group algebra for topological semigroups,* Pitman, 1984.
5. E. Følner, On groups with full Banach mean value, *Math. Scand.* **3** (1955), 243-254.
6. A. Hulanicki, Means and Følner conditions on locally compact groups, *Studia Math.* **27** (1966), 87-104.
7. M. Lashkarizadeh-Bami, Positive functionals on Lau Banach *-algebras with application to negative definite functions on foundation semigroups, *Semigroup Forum* **55** (1997),177-184.
8. M. Lashkarizadeh-Bami, Representations of foundation semigroups and their algebras, *Canad. J. Math.* **37** (1985), 785-809.
9. A.T.M. Lau, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, *Fund. Math.* **118** (1983), 161-175.
10. H. Reiter, Investigations in harmonic analysis, *Trans. Amer. Math. Soc.* **73** (1952), 401-427.
11. H. Reiter, On some properties of locally compact groups, *Indag. Math.* **27** (1965), 697-701.
12. M. Skantharajah, Amenable hypergroups, *Illinois J. Math.* **36** (1992), 15-46.
13. G.L.G. Sleijpen, Locally compact semigroups and continuous translations of measures, *Proc. London Math. Soc.* **37** (1978), 75-97.

1991 AMS subject classification: Primary 43A07, Secondary 43A10, 43A15