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Matrix Transformation Between Series and Sequences

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1. Preliminaries

Let X be a subset of series and Y be a subset of sequences. We shall write (X,Y) for the class of matrices $A=(a_{nk})$ of complex numbers such that for each series $x=\sum_k x_k$ in X, $(A_n x)\in Y$, where $A_n x=\sum_k a_{nk}x_k$. The sums without limits means the sum is from k = 1 to ∞. Throughout the article c, c_0, ℓ_∞ denote the spaces of all *convergent, null* and *bounded* sequences respectively. For $x\in X$, we write

$$Sx = \left(\sum_{k=1}^{n} x_k\right) = (S_n),$$

called the sequence of partial sums of $\sum_k x_k$. Now we have the two spaces of series $\gamma=\{x : Sx\in c\}$ and $\gamma_0=\{x : Sx\in c_0\}$.

A sequence $x=(x_n)$ is said to be *statistically convergent* to L, written as stat-lim $x_n = L$ if for every $\varepsilon>0$,

$$\lim_{n\to\infty} n^{-1}\left\{ number\ \ of\ k\le n\ :\ \left| x_k - L \right| \ge \varepsilon \right\} = 0.$$

The space of all statistically convergent sequences is denoted by $\bar{c}$ and $\bar{c}_0$ the space of *statistically null sequences*. For any sequence (x_n), the difference operator Δ is defined by $\Delta x_k = x_k - x_{k+1}$. In this article e_k denotes the series whose k-th term is 1 and the rest are zero.

2. Main Results

The following lemmas will be used in establishing the results of this article. First we have the well known result

Lemma 1. *(Abel's summation formula). Let* $0 \le m \le n,\ S_n = x_1 + x_2 + \cdots + x_n$, $x_0 = 0$. *Then*

$$\sum_{k=m}^{n} x_k b_k = \sum_{k=m}^{n-1} S_k \Delta b_k + S_n b_n - S_{m-1} b_m .$$

Lemma 2. [2] *The space* $\bar{c} \cap \ell_\infty$ *is a closed linear subspace of* ℓ_∞.

Lemma 3. [3] *If* $\sup_{n,k} |a_{nk}| < \infty$, $\sum_k a_{nk}$ *converges uniformly in n and* $S_n = \sum_k a_{nk}$ *for* $n \in N$, a_k = *stat-lim* a_{nk} *exists for each* $k \in N$, *then stat-lim* S_n *exists and equals to* $\sum_k a_k$.

The following two lemmas are well known, see for instance Nanda [1].

Lemma 4. $A \in (\gamma, \ell_\infty)$ *if and only if*

$$\sup_n \sum_k |\Delta a_{nk}| < \infty, \tag{2.1}$$

$$\sup_n \left| \lim_{k\to\infty} a_{nk} \right| < \infty. \tag{2.2}$$

Equivalently

$$\sup_n \sum_k |a_{nk} - a_{n,k-1}| < \infty. \tag{2.3}$$

Lemma 5. $A \in (\gamma_0, \ell_\infty)$ *if and only if* (2.1) *holds.*

The following theorems are proved in this article.

Theorem 1. $A \in (\gamma, \bar{c} \cap \ell_\infty)$ *if and only if* (2.1) *holds and*

$$\text{stat-lim } a_{nk} = \alpha_k \text{ exists for each fixed } k \in N. \tag{2.4}$$

Theorem 2. $A \in (\gamma, \bar{c} \cap \ell_\infty; P)$ *if and only if* (2.1) *and* (2.4) *with* $\alpha_k = 1$ *holds. In this transformation, the limit is preserved.*

Theorem 3. $A \in (\gamma_0, \bar{c} \cap \ell_\infty)$ *if and only if* (2.1) *holds and*

$$\text{stat-lim } \Delta a_{nk} \text{ exists for each fixed } k \in N. \tag{2.5}$$

3. Proof of Theorems

In this section we establish the theorems and deduce some new results as corollaries.

Proof of Theorem 1. The necessity of (2.1) follows from the inclusion relation $(\gamma,\ \bar{c} \cap \ell_\infty) \subset (\gamma, \ell_\infty)$. That of (2.4) follows on considering the series e_k.

Let $x \in \gamma$ and $p = \sum_k x_k$, then by Lemma 1 we have

$$A_n x = \sum_k a_{nk} x_k = p a_{n1} + \sum_k (S_k - p) \Delta a_{nk}. \tag{3.1}$$

By (2.4) we have $p a_{nk} \in \bar{c} \cap \ell_\infty$. We have $(S_k - p) \to 0$ as $k \to \infty$. Let $M = \sup_k |S_k - p|$. Then for every $\varepsilon > 0$, by (2.1) there exists m_0 such that

$$\sum_{k > m_0} |\Delta a_{nk}| < \varepsilon M^{-1}.$$

Thus $\sum_{k>m_0} |S_k - p||\Delta a_{nk}| < \varepsilon$ implies $\sum_{k>m_0} |S_k - p||\Delta a_{nk}| \to 0$ as $m_0 \to \infty$, uniformly in *n*. Also $(a_{nk})_n \in \bar{c} \cap \ell_\infty$ implies $(\Delta a_{nk})_n \in \bar{c} \cap \ell_\infty$. Thus $\sum_{k \le m_0} (S_k - p)\, \Delta a_{nk} \in \bar{c} \cap \ell_\infty$. Hence by lemma 3 $A_n x \in \bar{c} \cap \ell_\infty$.

This completes the proof of Theorem 1.

Corollary 1. $A \in (\gamma,\ \bar{c}_0 \cap \ell_\infty)$ *if and only if* (2.1) *and* (2.4) *with* $\alpha_k = 0$ *hold.*

Proof of Theorem 2. The necessity is clear from the necessity of Theorem 1.

Since $a_{nk} \to 1$ statistically, so $p a_{n1} \to p$ statistically. Also stat-lim $a_{nk} = 1$ implies stat-lim $\Delta a_{nk} = 0$. As in Theorem 1 we have $\sum_{k \ge m_0} (S_k - p)\, \Delta a_{nk}$ converges to zero uniformly in *n*. Since stat-lim $\Delta a_{nk} = 0$, so we have stat-lim $\sum_{k \le m_0} (S_k - p)\, \Delta a_{nk} = 0$. Hence $\sum_k (S_k - p)\, \Delta a_{nk}$ converges statistically to zero. Thus by (3.1) $A_n x$ converges statistically to *p*.

This completes the proof of Theorem 2.

Proof of Theorem 3. The necessity of (2.5) follows on considering the series f_k whose *k*-th term is 1 and (*k* +1)-th term is -1 and the rest are zero. That of (2.1) follows from the inclusion $(\gamma_0,\ \bar{c} \cap \ell_\infty) \subset (\gamma_0, \ell_\infty)$.

Let $x \in \gamma_0$, then (3.1) takes the form

$$A_n x = \sum_k S_k \Delta a_{nk}.$$

As in Theorem 1 we have $\sum_{k>m_0} S_k \Delta a_{nk}$ converges to zero uniformly in *n*. Since $\Delta a_{nk} \in \bar{c} \cap \ell_\infty$, so $\sum_{k \le m_0} S_k \Delta a_{nk} \in \bar{c} \cap \ell_\infty$. Hence by Lemma 3 we have $(A_n x) \in \bar{c} \cap \ell_\infty$.

This completes the proof of Theorem 3.

Corollary 2. $A \in (\gamma_0, \bar{c}_0 \cap \ell_\infty)$ *if and only if* (2.1) *holds and stat-lim* $\Delta a_{nk} = 0$ *for each fixed* $k \in N$.

References

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