

Semi-Invariant Submanifolds of a Lorentzian Para-Sasakian Manifold

BHAGWAT PRASAD

Department of Mathematics, M. M. T. D. College, Ballia, India

Abstract. Recently Matsumoto [1] introduced the idea of Lorentzian para contact structure and studied its several properties. In the present paper we studied the integrability condition of the distribution on semi-invariant submanifolds of LP-Sasakian manifold.

1. Introduction

Let \bar{M} be an n -dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞ -vector valued linear function ϕ , a C^∞ vector field ξ , 1-form η and Lorentzian metric g of type $(0, 2)$ such that for each $p \in \bar{M}$, the tensor $g_p : T_p\bar{M} \times T_p\bar{M} \rightarrow R$ is a non-degenerate inner product of signature $(-, +, +, \dots, +)$, where $T_p\bar{M}$ denotes the tangent vector space of \bar{M} at p and R is the real number space, which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \quad (1.1)$$

$$\begin{aligned} \eta(\xi) &= -1, \\ g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \\ g(X, \xi) &= \eta(X), \end{aligned} \quad (1.2)$$

for all vector fields X, Y tangent to \bar{M} . Such structure (ϕ, ξ, η, g) is termed as *Lorentzian para contact* [1].

In a Lorentzian para-contact structure the following holds

$$\begin{aligned} \phi\xi &= 0, \quad \eta(\phi X) = 0, \\ \text{rank}(\phi) &= n-1. \end{aligned}$$

A Lorentzian para contact manifold \bar{M} is called *Lorentzian para-Sasakian* (LP-Sasakian) manifold if [1]

$$(\bar{\nabla}_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (1.3)$$

and from (1.3), we find

$$\bar{\nabla}_X \xi = \phi X \quad (1.4)$$

$\forall X, Y$ tangent to \bar{M} , where $\bar{\nabla}$ is the Riemannian connection with respect to g .

Let us put

$$\Phi(X, Y) = g(\phi X, Y)$$

then the tensor field Φ is symmetric (0,2)-tensor field. Thus we have,

$$\Phi(X, Y) = \Phi(Y, X),$$

and

$$\Phi(X, Y) = (\bar{\nabla}_X \eta)(Y).$$

Definition 1.1. *The submanifold M of the LP-Sasakian manifold \bar{M} is said to be semi-invariant if it is endowed with the pair of orthogonal distribution (D, D^\perp) satisfying the conditions*

- (i) $TM = D \oplus D^\perp \oplus \{\xi\}$,
- (ii) the distribution D is invariant under ϕ , that is

$$\phi D_x = D_x, \text{ for each } x \in M,$$

- (iii) the distribution D^\perp is anti-invariant under ϕ , that is

$$\phi D_x^\perp \subset T_x M^\perp, \text{ for each } x \in M.$$

The distribution D (respectively D^\perp) is called the *horizontal* (respectively *vertical*) *distribution*. A semi-invariant submanifold M is said to be *invariant* (respectively *anti-invariant*) *submanifold* if we have $D_x^\perp = \{0\}$ respectively $(D_x = 0)$ for each $x \in M$. We say that M is a *proper semi-invariant submanifold* if it is a semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold.

We denote by same symbol g both metrics on \bar{M} and M . The projection morphisms of TM to D and D^\perp are denoted by P and Q respectively. For any $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, we have

$$X = PX + QX + \eta(X)\xi \quad (1.5)$$

$$\phi N = BN + CN \quad (1.6)$$

where BN (respectively CN) denotes the tangential (respectively normal) component of ϕN .

The equations of Gauss and Weingarten for the immersion of M in \bar{M} are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (1.7)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (1.8)$$

for any $X, Y \in \Gamma(TM)$ and $N \in TM^\perp$, where ∇ is the Levi-Civita connection on M , ∇^\perp is the linear connection induced by $\bar{\nabla}$ on the normal bundle TM^\perp , h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N . Also we have

$$g(h(X, Y), N) = g(A_N X, Y) \quad (1.9)$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(TM^\perp)$.

2. Basic Lemmas

For $X, Y \in \Gamma(TM)$, we put

$$u(X, Y) = \nabla_X \phi P Y - A_{\phi Q Y} X. \quad (2.1)$$

We begin with the following lemma.

Lemma 2.1. *Let M be a semi-invariant submanifold of LP-Sasakian manifold \bar{M} . Then we have*

$$P(u(X, Y)) = \phi P \nabla_X Y + \eta(Y) P X + 2\eta(X) \eta(Y) P \xi + g(X, Y) P \xi \quad (2.2)$$

$$Q(u(X, Y)) = B h(X, Y) + \eta(Y) Q X + 2\eta(X) \eta(Y) (Q \xi) + g(X, Y) Q \xi \quad (2.3)$$

$$h(X, \phi P Y) + \nabla_X^\perp \phi Q Y = \phi Q \nabla_X Y + C h(X, Y) \quad (2.4)$$

$$\eta(u(X, Y)) = -g(\phi X, \phi Y), \quad (2.5)$$

for all $X, Y \in TM$.

Proof. By using the decompositions (1.5), (1.6), (1.7), (1.8) in (1.3), we obtain (2.2), (2.3), (2.4) and (2.5) respectively.

Lemma 2.2. *Let M be a semi-invariant submanifold of LP-Sasakian manifold \bar{M} , then we have*

$$\nabla_X \xi = \phi X, \quad h(X, \xi) = 0, \quad \text{for any } X \in \Gamma(D); \quad (2.6)$$

$$\nabla_Y \xi = 0, \quad h(Y, \xi) = \phi Y, \quad \text{for any } Y \in \Gamma(D^\perp); \quad (2.7)$$

$$\nabla_\xi \xi = 0, \quad h(\xi, \xi) = 0. \quad (2.8)$$

Proof. In consequence of (1.4) and (1.5), we obtain

$$\bar{\nabla}_X \xi = \nabla_X \xi + h(X, \xi)$$

or

$$\nabla_X \xi + h(X, \xi) = \phi P X + \phi Q X. \quad (2.9)$$

Thus, (2.6)–(2.8) follows from (2.9).

Lemma 2.3. *Let M be a semi-invariant submanifold of LP-Sasakian manifold \bar{M} . Then we have*

$$A_{\phi X} Y = -A_{\phi Y} X, \quad (2.10)$$

for all $X, Y \in \Gamma(D^\perp)$.

Proof. By using (1.2), (1.7), (1.9), we get

$$\begin{aligned} g(A_{\phi X} Y, Z) &= g(h(Y, Z), \phi X) = g(\bar{\nabla}_Z Y, \phi X) \\ &= g(\phi \bar{\nabla}_Z Y, X) = g(\bar{\nabla}_Z \phi Y, X) \\ &= -g(\phi Y, \bar{\nabla}_Z X) = -g(\phi Y, h(X, Z)) \\ &= -g(A_{\phi Y} X, Z), \end{aligned}$$

for all $X, Y \in \Gamma(D^\perp)$ and $Z \in \Gamma(TM)$, which proves (2.10).

Lemma 2.4. *Let M be a semi-invariant submanifold of LP-Sasakian manifold \bar{M} . Then we find*

$$\begin{aligned} \nabla_\xi U &\in \Gamma(D), \quad \text{for any } U \in \Gamma(D), \\ \nabla_\xi V &\in \Gamma(D^\perp), \quad \text{for any } V \in \Gamma(D^\perp). \end{aligned}$$

The proof is obvious.

Lemma 2.5. *Let M be a semi-invariant submanifold of LP-Sasakian manifold \bar{M} . Then we obtain*

$$[X, \xi] \in \Gamma(D), \quad \text{for any } X \in \Gamma(D), \quad (2.11)$$

$$[Y, \xi] \in \Gamma(D^\perp), \quad \text{for any } Y \in \Gamma(D^\perp). \quad (2.12)$$

The proof follows from Lemma 2.4.

3. Integrability of distribution on a semi-invariant submanifold in a LP-Sasakian manifold

Theorem 3.1. *Let M be a semi-invariant submanifold in LP-Sasakian manifold \bar{M} . Then the distribution D is integrable if and only if*

$$h(X, \phi Y) = h(Y, \phi X).$$

Proof. We have from (2.6)

$$\begin{aligned} g([X, Y], \xi) &= g(\nabla_X Y - \nabla_Y X, \xi) \\ &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= -g(Y, \phi X) + g(X, \phi Y) \\ &= 0, \end{aligned}$$

for all $X, Y \in \Gamma(D)$.

In consequence of (2.4), we find

$$h(X, \phi Y) - h(Y, \phi X) = \phi Q[X, Y],$$

which proves the theorem.

Corollary 3.1. *The distribution $D \oplus \{\xi\}$ is integrable if and only if $h(X, \phi Y) = h(Y, \phi X)$ is satisfied.*

Proof follows from Theorem 3.1 and (2.11).

Theorem 3.2. *Let M be semi-invariant submanifold in LP-Sasakian manifold \bar{M} . Then the distribution D^\perp is never integrable.*

Proof. From (2.1), we have for $X, Y \in \Gamma(D^\perp)$

$$u(X, Y) = -A_{\phi Y} X.$$

Operating ϕ in (2.2) and using (1.1), we get

$$P\nabla_X Y = \phi P(A_{\phi Y} X), \quad (3.1)$$

for any $X, Y \in \Gamma(D^\perp)$. By virtue of Lemma 2.3, (3.1) reduces to

$$P([X, Y]) = 2\phi P(A_{\phi Y} X),$$

which proves the statement.

Corollary 3.2. *The distribution on $D^\perp \oplus \{\xi\}$ is never integrable.*

Proof follows from Theorem 3.2 and (2.12).

References

1. K. Matsumoto, On Lorentzian paracontact manifolds, *Bull. Yamogata Univ. Nat. Sci.* **12** (1989).
2. A. Bejancu and N. Papaghiue, Almost semi-invariant submanifolds of a Sasakian manifold, *Bull. Math. dela Soc. Sci., Math. dela R.S. Roumanie* **28** (1984), 1.
3. N. Papaghiue, Some results on almost semi-invariant submanifolds in Sasakian manifolds, *Bull. Math. dela Soc. Sci., Math. dela R.S. Roumanie* **28** (1984), 3.
4. M. Kobayashi, Semi-invariant submanifold of a certain class of almost contact manifold, *Tensor, N.S.* **43** (1986).
5. S. Prasad and R.H. Ojha, Lorentzian para contact submanifolds, *Publ. Math. Debrecen* **44** (1994), 215-223.