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On B^*-Quasigroups

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Abstract. The author has discussed B^*-loops in [1]. In this paper we investigate B^*-quasigroups which is a quasigroup $(G,\cdot)$ satisfying the identity $(xy\cdot z)y^\alpha = x(yz\cdot y^\alpha)$ for all $x, y, z \in G$ and some endomorphism α of $(G,\cdot)$. Connection between B^*-quasigroups and B^*-loops is examined.

1. Introduction

A binary system $(G,\cdot)$ is said to be a quasigroup if for each ordered pair $a,\ b \in G,$ there is one and only one x in G such that $ax = b$ in G and one and only one y such that $ya = b$ in G. We recall that a loop is a quasigroup which has a two sided identity element. We shall say that a quasigroup $(G,\cdot)$ is a B^*-quasigroup if and only if

$$(xy\cdot z)y^\alpha = x(yz\cdot y^\alpha)$$

for all $x,\ y,\ z$ in G and some endomorphism α of $(G,\cdot)$. For any three elements a, b, c of a binary system for which the binary operation is juxtaposition $ab\cdot c$ will mean the element $(ab)c$. A B^*-loop is a loop which is also a B^*-quasigroup. B^*-loops have been discussed in [1]. It would seem natural to investigate B^*-quasigroups. It is known [2] that an isotope of a quasigroup is a quasi-group and any quasigroup is an isotope of some loop. Our main results (Theorems 3 and 4) establish connection between B^*-quasigroups and B^*-loops.

2. Preliminary Lemmas

Lemma 1. *If* $(G,\cdot)$ is a B^**-quasigroup, then* $(G,\cdot)$ *has a unique left identity element.*

Proof. Let x be a fixed element in G. Since $(G,\cdot)$ is quasigroup there exists an element $e \in G$ such that $ex = x$. Now $x^\alpha \in G$ and for each $a \in G$ there exist elements $y,\ z \in G$ such that $yx^\alpha = a$ and $xz = y$. Then $ea = e(yx^\alpha) = e(xz\cdot x^\alpha)$ $= (ex\cdot z)x^\alpha = (xz)x^\alpha = yx^\alpha = a$. So, e is a left identity of $(G,\cdot)$. The fact that e is unique is the consequence of $(G,\cdot)$ being a quasigroup.

From now onward, e will denote the unique left identity for a B^*-quasigroup $(G,\cdot)$ and for each x in G x^ν and x^μ shall designate those unique elements in G such that $x^\nu x = xx^\mu = e$. Moreover, the mapping $R(x)$ defined by $yR(x) = yx$ for every y in G, is a permutation on G if G is a quasigroup (Bruck [2], p. 54).

Lemma 2. *If* $(G,\cdot)$ is a B^**-quasigroup, then*

(i) $x^\mu = x^\nu$ *for all* $x \in G$,

(ii) $(G,\cdot)$ *satisfies the right inverse property.*

Proof

(i) For x in G, we have $x(x^\mu x \cdot (x^\mu)^\alpha) = (xx^\mu \cdot x)(x^\mu)^\alpha = ex \cdot (x^\mu)^\alpha = x(x^\mu)^\alpha$. This gives $x^\mu x \cdot (x^\mu)^\alpha = (x^\mu)^\alpha$ and we have $x^\mu x = e = x^\nu x$. Hence, $x^\mu = x^\nu$ for all x in G.

(ii) For $x, y \in G$, we have $(xy \cdot y^\mu) y^\alpha = x(yy^\mu \cdot y^\alpha) = x \cdot ey^\alpha = xy^\alpha$. This gives $(xy)y^\mu = x$, for all $x, y \in G$ which proves (ii).

Remarks

(i) In view of Lemma 2, let $x^{-1} = x^\mu = x^\nu$ for any element x of a B^*-quasigroup $(G,\cdot)$.

(ii) $e \cdot e = e$ gives $e^{-1} = e$ and Lemma 2 (ii) gives $(xe)e^{-1} = x$ and hence $(xe)e = x$. Thus we have $R(e)^2 = I_G$ the identity mapping on G.

3. Main Results

Theorem 3. *If* $(G,\cdot)$ is a B^**-quasigroup with respect to an endomorphism* α*, then* $(G,\circ)$ *is a* B^**-loop where* $x \circ y = xe \cdot y$ *for all* x, y in G.

Proof. As mentioned above $(G,\circ)$ being an isotope of a quasigroup $(G,\cdot)$ is a quasigroup. Now for every x in G, $x \circ e = (xe)e = x$ and $e \circ x = ee \cdot x = x$. Hence, $(G,\circ)$ is a loop. For all x, y in G $(x \circ y)^\alpha = (xe \cdot y)^\alpha = (xe)^\alpha \cdot y^\alpha = x^\alpha e \cdot y^\alpha = x^\alpha \circ y^\alpha$ showing that α is an endomorphism of the loop $(G,\circ)$. Moreover, for all x, y, z in G we have

$$\begin{aligned}(xe)\circ\big((y\circ z)\circ y^\alpha\big) &= (xe)e\cdot\big((ye\cdot z)e\cdot y^\alpha\big) = x\big((ye\cdot z)e^\alpha\cdot y^\alpha\big)\\ &= x\big(y(ez\cdot e)\cdot y^\alpha\big) = \big((xy)(ze)\big)y^\alpha,\end{aligned}$$

and

$$\begin{aligned}\big(((xe)\circ y)\circ z\big)\circ y^\alpha &= \big(((xe)e\cdot y)e\cdot z\big)e\cdot y^\alpha = \big(((xy\cdot e)z\big)e^\alpha\big)y^\alpha\\ &= \big((xy)(ez\cdot e^\alpha)\big)y^\alpha = \big((xy)(ze)\big)y^\alpha.\end{aligned}$$

Hence, we have

$$\big(((xe)\circ y)\circ z\big)\circ y^{\alpha} = (xe)\circ\big((y\circ z)\circ y^{\alpha}\big).$$

Showing that $(G,\circ)$ is a B^*-loop with respect to the endomorphism α of it.

Theorem 4. *Suppose $(G,\circ)$ is a B^*-quasigroup with respect to an endomorphism α. Suppose δ is an automorphism of $(G,\circ)$ such that $\delta^2 = I_G$ and $\alpha\delta = \delta\alpha$. If we define $x\cdot y = x\delta\circ y$ for all x, y in G then $(G,\cdot)$ is a quasigroup such that the same map α from G to G is an endomorphism of this quasigroup $(G,\cdot)$ also making it a B^*-quasigroup.*

Proof. Clearly $(G,\cdot)$ is a quasigroup. For all x, y in G, $(x\cdot y)^{\alpha} = (x\delta\circ y)^{\alpha}$ $= (x\delta)^{\alpha}\circ y^{\alpha} = x^{\alpha}\delta\circ y^{\alpha} = (x^{\alpha}\cdot y^{\alpha})$ showing that α is an endomorphism of the quasigroup $(G,\cdot)$. Moreover, for all x, y, z in G we have

$$\begin{aligned}(xy\cdot z)y^{\alpha} &= \big((x\delta\circ y)\delta\circ z\big)\delta\circ y^{\alpha} = \big((x\delta\circ y)\delta^2\circ z\delta\big)\circ y^{\alpha}\\ &= \big((x\delta\circ y)\circ z\delta\big)\circ y^{\alpha} = x\delta\circ\big((y\circ z\delta)\circ y^{\alpha}\big)\\ &= x\delta\circ\big((y\delta^2\circ z\delta)\circ y^{\alpha}\big) = x\delta\circ\big((y\delta\circ z)\,\delta\circ y^{\alpha}\big)\\ &= x(yz\cdot y^{\alpha})\end{aligned}$$

showing that $(G,\cdot)$ is a B^*-quasigroup with respect to the endomorphism α of it.

In the next theorem we show that all B^*-quasigroups can be obtained from B^*-loops in the manner described in Theorem 4.

Theorem 5. *If $(G,\cdot)$ is a B^*-quasigroup with respect to an endomorphism α, then there is a loop $(G,\circ)$ for which α is an endomorphism making it a B^*-loop and an automorphism δ of $(G,\circ)$ with property that $\delta^2 = I_G$, $\alpha\delta = \delta\alpha$ and $x\cdot y = x\delta\circ y$ for all x, y in G.*

Proof. If e is the left identity of the B^*-quasigroup $(G,\cdot)$ then, from Theorem 3 and its proof, $(G,\circ)$, where $x\circ y = xe\cdot y$ for all x, y in G, is a loop having e as two sided identity and α an endomorphism making it a B^*-loop . Take δ to be $R(e)$. Then by above remark (ii), $\delta^2 = R(e)^2 = I_G$. Now $R(e)$ is a permutation of G. Moreover, for all x, y in G, $(x\circ y)\delta = (xe\cdot y)e = (xe\cdot y)e^{\alpha} = x(ey\cdot e^{\alpha}) = x(ye) = (xe\cdot e)(ye)$ $= (x\delta\cdot e)y\delta = x\delta\circ y\delta$ showing that δ is an automorphism of $(G,\circ)$ and $\delta^2 = I_G$.

Also we observe that $x\delta \circ y = (xe \cdot e)y = x \cdot y$ for all x, y in G. Moreover, for every x in G, $(x\delta)^{\alpha} = (xe)^{\alpha} = x^{\alpha} \cdot e^{\alpha} = x^{\alpha} \cdot e = (x^{\alpha})\delta$ showing that $\delta\alpha = \alpha\delta$.

References

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2. R.H. Bruck, A survey of binary systems, Springer-Verlag, 1972.

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