

Spacelike Maximal Surfaces with Constant Scalar Normal Curvature in a Normal Contact Lorentzian Manifold

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Abstract. If the scalar normal curvature of a spacelike maximal surface in a 5-dimensional normal contact Lorentzian manifold with constant ϕ -sectional curvature is constant, then the surface is totally geodesic or nonpositively curved.

1. Introduction

On some odd dimensional manifolds, the normal contact Riemannian metric structure (or Sasakian structure) can be defined. The study of manifolds with this structure has a long history.

If we change the Riemannian metric of the Sasakian structure to a Lorentzian one, we can define the normal contact Lorentzian structure. This definition was given at the starting time of the study of the Sasakian structure. But practical study of it has not been given sufficiently yet (*cf.* [5], [7]). In [3], [4], we study the fundamental properties of manifolds with the normal contact Lorentzian structure.

In this paper, we shall study the scalar normal curvature for spacelike maximal surfaces in a 5-dimensional normal contact Lorentzian manifold of constant ϕ -sectional curvature and prove.

Theorem. *Let \bar{M}^5 be a 5-dimensional normal contact Lorentzian manifold with constant ϕ -sectional curvature k and M^2 a spacelike maximal surface with vector field ξ normal to M^2 . Assume that the scalar normal curvature K_N of M^2 in \bar{M}^5 is constant. Then M^2 is totally geodesic with Gauss curvature $K = \frac{k-3}{4}$ or a nonpositive curved surface.*

2. Spacelike Submanifold

Let \bar{M} be a $(2n+1)$ -dimensional ($n \geq 2$) manifold. The normal contact Lorentzian structure (ϕ, ξ, η, g) of \bar{M} is given by a $(1,1)$ -type skew-symmetric tensor field ϕ , a vector field ξ , a 1-form η and a Lorentzian metric g as

$$\begin{aligned}
\phi^2 X &= -X + \eta(X)\xi, & \phi\xi &= 0, \\
g(\xi, \xi) &= -1, & \eta(X) &= -g(X, \xi), \\
(\bar{\nabla}_X \eta)Y &= g(\phi X, Y), & \bar{\nabla}_X \xi &= \phi X, \\
(\bar{\nabla}_X \phi)Y &= -\eta(Y)X - g(X, Y)\xi,
\end{aligned} \tag{2.1}$$

where X is a vector field of \bar{M} and $\bar{\nabla}$ is the covariant derivative with respect to g ([3], [4]). When the curvature tensor field of $K(X, Y)Z$ of \bar{M} has the following form

$$\begin{aligned}
4K(X, Y)Z &= (k-3)(g(Y, Z)X - g(X, Z)Y) \\
&+ (k+1)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(X, Z)\eta(Y)\xi) \\
&- g(Y, Z)\eta(X)\xi + g(\phi Y, Z)\phi X + g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z,
\end{aligned} \tag{2.2}$$

\bar{M} is called a *space of constant ϕ -sectional curvature k* .

Let M be an n -dimensional submanifold of \bar{M} . By ∇ we denote the covariant derivative of M determined by the induced metric on M . Let $X(\bar{M})$ (resp. $X(M)$) be the Lie algebra of vector fields on \bar{M} (resp. M) and $X^\perp(M)$ the set of all vector fields normal to M .

The Gauss-Weingarten formulas are given by

$$\begin{aligned}
\bar{\nabla}_X Y &= \nabla_X Y + B(X, Y), & \bar{\nabla}_X N &= -A^N(X) + D_X N, \\
X, Y &\in X(M), & N &\in X^\perp(M),
\end{aligned} \tag{2.3}$$

where D is the normal connection [6]. B is called the second fundamental form tensor and A the shape operator, and they satisfy

$$g(A^N(X), Y) = g(B(X, Y), N). \tag{2.4}$$

If the induced metric on M is positive-definite, then M is called a *spacelike submanifold*.

Let M be a spacelike submanifold of \bar{M} with the vector field ξ normal to M , then from (2.1), (2.3) and (2.4), M satisfies following properties.

Proposition. *Let M be an m -dimensional spacelike submanifold in a normal contact Lorentzian manifold \bar{M}^{2n+1} with structure (ϕ, ξ, η, g) . Then*

- (i) *The dimension m of M satisfies $m \leq n$.*
- (ii) *The shape operator of ξ direction is identically zero.*
- (iii) *If $X \in X(M)$ then $\phi X \in X^\perp(M)$.*
- (iv) *If $m=n$, then $A^{\phi X}(Y) = A^{\phi Y}(X)$, for $X, Y \in X(M)$.*

3. Local Formulas

We consider a spacelike surface M^2 in a 5-dimensional normal contact Lorentzian manifold \bar{M} . Let $\{e_1, e_2, e_3, e_4, \xi\}$ be an orthonormal frame field on \bar{M}^5 so that

$$e_1, e_2 \in X(M), \quad e_{1^*} := \phi e_1 = e_3, \quad e_{2^*} := \phi e_2 = e_4.$$

We shall make use of the following convention on the ranges of indices:

$$\begin{aligned} 1 \leq A, B, \dots \leq 5, & \quad 1 \leq i, j, \dots \leq 2, \\ 3 \leq i^*, j^*, \dots \leq 4, & \quad 3 \leq \alpha, \beta, \dots \leq 5. \end{aligned}$$

Let $\{w^1, w^2, w^{1^*}, w^{2^*}, w^5\}$ be the field of dual frames. Then the structure equations of \bar{M} are given by

$$\begin{aligned} dw^A &= -\sum \varepsilon_B w_B^A \wedge w^B, \quad w_B^A + w_A^B = 0, \\ dw_B^A &= -\sum \varepsilon_C w_C^A \wedge w_B^C + \Phi_B^A, \\ \Phi_B^A &= \frac{1}{2} \varepsilon_C \varepsilon_D \sum K_{BCD}^A w^C \wedge w^D, \\ K_{BCD}^A + K_{BDC}^A &= 0. \end{aligned}$$

Restricting these forms to M^2 , we have $w^\alpha = 0$. Since $0 = dw^\alpha = -\sum w_i^\alpha \wedge w^i$, by Cartan's lemma we may write

$$w_i^\alpha = \sum h_{ij}^\alpha w^j, \quad h_{ij}^\alpha = h_{ji}^\alpha.$$

From these formulas we obtain

$$\begin{aligned} dw^i &= -\sum w_j^i \wedge w^j, \quad w_j^i + w_i^j = 0, \\ dw_j^i &= \sum w_k^i \wedge w_j^k + \Omega_j^i, \\ \Omega_j^i &= \frac{1}{2} \sum R_{jkl}^i w^k \wedge w^\ell, \\ R_{jkl}^i &= K_{jkl}^i + \sum \varepsilon_\alpha (h_{ik}^\alpha h_{j\ell}^\alpha - h_{i\ell}^\alpha h_{jk}^\alpha). \\ dw_\beta^\alpha &= -\sum \varepsilon_\gamma w_\gamma^\alpha \wedge w_\beta^\gamma + \Omega_\beta^\alpha, \\ \Omega_\beta^\alpha &= \frac{1}{2} \sum R_{\beta k\ell}^\alpha w^k \wedge w^\ell, \\ R_{\beta k\ell}^\alpha &= K_{\beta k\ell}^\alpha + \sum (h_{ik}^\alpha h_{i\ell}^\beta - h_{i\ell}^\alpha h_{ik}^\beta). \end{aligned}$$

An immersion is said to be *maximal* if $\sum_i h_{ii}^\alpha = 0$ for all α .

We define h_{ijk}^α and $h_{ijk\ell}^\alpha$ by

$$\begin{aligned}\sum h_{ijk}^\alpha w^k &= dh_{ij}^\alpha - \sum h_{i\ell}^\alpha w_j^\ell - \sum h_{ij}^\alpha w_i^\ell + \sum \varepsilon_\beta h_{ij}^\beta w_\beta^\alpha, \\ \sum h_{ijk\ell}^\alpha w^\ell &= dh_{ijk}^\alpha - \sum h_{ljk}^\alpha w_i^\ell - \sum h_{i\ell k}^\alpha w_j^\ell - \sum h_{ij\ell}^\alpha w_k^\ell + \sum \varepsilon_\beta h_{ijk}^\beta w_\beta^\alpha.\end{aligned}\quad (3.1)$$

The Laplacian Δh_{ij}^α is given by

$$\Delta h_{ij}^\alpha = \sum h_{ijkk}^\alpha.$$

When M^2 is maximal in \bar{M}^5 , that is $\sum h_{kk}^\alpha = 0$ for all α , Δh_{ij}^α can be written as

$$\begin{aligned}\Delta h_{ij}^\alpha &= \sum (2K_{\beta ki}^\alpha h_{jk}^\beta - K_{k\beta k}^\alpha h_{ij}^\beta + 2K_{\beta kj}^\alpha h_{ki}^\beta) + \sum (K_{mik}^m h_{mj}^\alpha + K_{kjk}^m h_{mi}^\alpha + 2K_{ijk}^m h_{mk}^\alpha) \\ &\quad + \sum (2h_{km}^\alpha h_{ki}^\beta h_{mj}^\beta - h_{mk}^\alpha h_{km}^\beta h_{ij}^\beta - h_{mi}^\alpha h_{mk}^\beta h_{kj}^\beta - h_{mj}^\alpha h_{ki}^\beta h_{mk}^\beta).\end{aligned}\quad (3.2)$$

The scalar normal curvature K_N of M^2 is defined by

$$K_N = \sum \varepsilon_\alpha \varepsilon_\beta S_{\beta ij}^\alpha S_{\beta ij}^\alpha, \quad S_{\beta ij}^\alpha = \sum (h_{ik}^\alpha h_{jk}^\beta - h_{jk}^\alpha h_{ik}^\beta).$$

The covariant derivative of $S_{\beta ij}^\alpha$ is defined by

$$S_{\beta ij\ell}^\alpha = \sum (h_{ik\ell}^\alpha h_{jk}^\beta + h_{ik}^\alpha h_{jk\ell}^\beta - h_{jk\ell}^\alpha h_{ik}^\beta - h_{jk}^\alpha h_{ik\ell}^\beta).$$

Then for the Laplacian of K_N , we have the following formula

$$\begin{aligned}\frac{1}{2} \Delta K_N &= \sum \varepsilon_\alpha \varepsilon_\beta (S_{\beta ijk}^\alpha)^2 + \sum S_{\beta ij}^\alpha (h_{ik\ell}^\alpha h_{j\ell}^\beta - h_{ik\ell}^\beta h_{j\ell}^\alpha) \\ &\quad + 4 \sum \varepsilon_\alpha \varepsilon_\beta S_{\beta ij}^\alpha (\Delta h_{ik}^\alpha) h_{jk}^\beta.\end{aligned}$$

4. Proof of Theorem

Let M^2 be a spacelike maximal surface in a normal contact Lorentzian manifold \bar{M}^5 of constant ϕ -sectional curvature k . Then, from Proposition, we obtain

$$(h_{ij}^*) = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad (h_{ij}^{2*}) = \begin{bmatrix} 0 & -a \\ -a & 0 \end{bmatrix}, \quad (h_{ij}^{\tilde{5}}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (4.1)$$

From (3.2), it follows that

$$\begin{aligned}\Delta h_{11}^{1*} &= \frac{3k-5}{4}a - 6a^3, & \Delta h_{12}^{1*} &= \Delta h_{21}^{1*} = 0, \\ \Delta h_{22}^{1*} &= -\frac{3k-5}{4}a + 6a^3, & \Delta h_{11}^{2*} &= \Delta h_{22}^{2*} = 0, \\ \Delta h_{12}^{2*} &= \Delta h_{21}^{2*} = -\frac{3k-5}{4}a + 6a^3, \\ \Delta h_{11}^5 &= \Delta h_{12}^5 = \Delta h_{21}^5 = \Delta h_{22}^5 = 0,\end{aligned}$$

by virtue of (2.2).

From (3.1), we have

$$\begin{aligned}h_{111}^{1*} &= -h_{221}^{1*} = -h_{121}^{2*} = -h_{221}^{2*} \\ &= -h_{122}^{1*} = -h_{212}^{1*} = -h_{112}^{2*} = h_{222}^{2*} = a_{,1}, \\ h_{112}^{1*} &= -h_{222}^{1*} = -h_{122}^{2*} = -h_{212}^{2*} \\ &= h_{121}^{1*} = h_{211}^{1*} = h_{111}^{2*} = -h_{221}^{2*} = a_{,2}, \\ h_{112}^5 &= h_{121}^0 = h_{221}^0 = h_{222}^0 = 0, \\ h_{122}^0 &= h_{221}^0 = -a.\end{aligned}$$

Since

$$S_{1^*11}^{2*} = S_{1^*22}^{2*} = 0, \quad S_{1^*12}^{2*} = -S_{1^*21}^{2*} = 2a^2,$$

by virtue of (4.1), we obtain

$$\begin{aligned}\sum (S_{\beta ij}^\alpha)^2 &= \sum (S_{m^*ijk}^{\ell^*})^2 + 2\varepsilon_0 \sum (S_{\beta ij}^0)^2 \\ &= 4\left(((2a^2)_{,1})^2 + ((2a^2)_{,2})^2 \right) - 32a^4,\end{aligned}$$

$$\sum S_{\beta ij}^\alpha \left(h_{ik\ell}^\alpha h_{jk\ell}^\beta - h_{ik\ell}^\beta h_{jk\ell}^\alpha \right) = 2\left(((2a^2)_{,1})^2 + ((2a^2)_{,2})^2 \right),$$

$$\sum S_{\beta ij}^\alpha (\Delta h_{jk}^\alpha) h_{jk}^\beta = 8a^4 \left(\frac{3k-5}{4} - 6a^2 \right),$$

so that

$$\frac{1}{2} \Delta K_N = 8\left(((2a^2)_{,1})^2 + ((2a^2)_{,2})^2 \right) + 32a^4 \left(\frac{3k-9}{4} - 6a^2 \right).$$

If we assume $K_N \equiv \text{constant}$, then since a is continuous, this equation reduces to $a \equiv 0$ or $a^2 \geq \frac{k-3}{8}$ everywhere. On the other hand, the Gauss curvature K of M^2 is given by $K = \frac{k-3}{4} - 2a^2$. Hence if $a^2 \geq \frac{k-3}{8}$ then $K \leq 0$. This completes the proof.

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