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Limit Theorems for Exceedances of Sequence of Branching Processes

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Abstract. A problem of the first exceedance of a given level by the family of independent branching processes with and without immigration is considered. Using limit theorems for large deviations for processes with and without immigration limit theorems for the index of the first process exceeding some fixed or increasing levels in critical, subcritical and supercritical cases are proved. Asymptotic formulas for the expectation of the index are also obtained.

1. Introduction

We consider an ordinary Bienaymé-Galton-Watson (BGW) process that can be defined as follows. Let $X_{ki}, k \in N_0 = \{0, 1, 2, \cdots\}$, $i \in N = \{1, 2, \cdots\}$ be independent and identically distributed random variables taking values in the set N_0. We define the process $X(t)$, $t \in N$, by the following relation

$$X(0) = 1, \quad X(t) = \sum_{i=1}^{X(t-1)} X_{ti}.$$

The stochastic process $X(t)$ describes the evolution of a population of individuals who produce offspring independently.

Now we consider the sequence of BGW branching processes $\{ X_i(t), i \in N \}$ and define the "index" process

$$v(t) = min \{ k : X_k(t) > \theta(t) \},$$

for a given "level" function $\theta(t)$.

Let $f(t,s) = \sum_{i=0}^{\infty} P_k(t) s^k$ be the generating function of the number of individuals (or particles) at time *t*. We consider the sequence of BGW branching processes $\{ X_i(t), i \in N \}$ under the following assumptions:

(a) $X_i(t)$ are independent for any fixed $t \in N_0$ and $X_i(0) = 1$,

(b) $P\left\{ X_i(1) = k \,\middle|\, X_i(0) = 1 \right\} = P_k,\ k \in N_0,\ \sum_{k=0}^{\infty} P_k = 1.$

We denote

$$f(s) = f(1,s) = ES^{X(1)},$$
$$A = f'(1), \quad Q(t) = P\{X(t) > 0\}.$$

Processes $X_i(t)$, $i = 1, 2, \cdots$ can be considered as sizes of different populations existing in different regions of an area. Then the process $\nu(t)$ is the number of the first process in the family which exceeds level $\theta(t)$.

Properties of exceedances of given levels by sequences of independent and identically distributed random variables have been studied widely in the literature (see, for example, Leadbetter *et al*. (1983), Watts *et al*. (1982) and references there).

The process $\nu(t)$ was considered in [8], where some limit theorems were obtained. The proofs of these limit theorems are based on applying the so called classical limit theorems for the number of individuals, and therefore include more restricted conditions on the behaviour of the function $\theta(t)$. In the present paper, applying limit theorems for large deviations, we study the process $\nu(t)$ for a wide class of the "level" functions $\theta(t)$.

2. Processes Without Immigration

It is known that (see [5]) if $A = 1$ and $0 < \sigma^2 = f''(1) < \infty$, the following limit theorem holds for fixed $y \geq 0$:

$$\lim_{t\to\infty} e^{y} P\left\{ X(t)\, Q(t) > y \,\middle|\, X(t) > 0 \right\} = 1. \quad (1)$$

It has been proved (see [5]) that under

Condition (I): the generating function $f(s)$ be analytic on the disk $|s| < 1 + \varepsilon$ for some $\varepsilon > 0$, $A = 1$ and $0 < \sigma^2 = f''(1) < \infty$ and if for some integer $N \geq 2$,

$$0 < y = o\left(\frac{t}{\log_{(N)} t \ \log t}\right)$$

as $t \to \infty$ where $\log_{(1)} t = \log t$, $\log_{(i+1)} t = \log \log_{(i)} t$, $i = 1, 2, \cdots$,

then (1) also holds when y is not fixed.

It is also known (see [1]) that if $A = 1$, then as $t \to \infty$, $Q(t) \sim \frac{2}{\sigma^2 t}$, where $\sigma^2 = Var\, X(1)$. In this section, we prove the following assertion.

Theorem 1. *Suppose condition (I) is satisfied. If the function* $\phi(t)$ *is such that* $\phi(t) = o\left(\frac{t}{\log_{(N)} t \log t}\right)$ *as* $t \to \infty$ *and* $\theta(t) = \frac{\phi(t)}{Q(t)}$, *then*

$$\lim_{t\to\infty} P\left\{ \frac{\nu(t)}{E\nu(t)} \le x \right\} = \begin{cases} 1 - e^{-x} & \text{if } x \ge 0 \\ 0 & \text{otherwise}, \end{cases}$$

where $E\nu(t) \sim \frac{\sigma^2 t e^{\phi(t)}}{2}$ *as* $t \to \infty$.

Proof. It is not difficult to obtain the following relation for the probability generating function $H(t,s)$ of $\nu(t)$:

$$H(t,s) = \frac{s\left\{1 - F(t,\theta(t))\right\}}{1 - sF(t,\theta(t))} \tag{2}$$

where $F(t,x) = P(X(t) \le x)$, $0 \le s \le 1$. Putting $s = e^{\frac{-\lambda}{a(t)}}$, we find from (2) the Laplace transform of $\frac{\nu(t)}{a(t)}$:

$$Ee^{\frac{-\lambda}{a(t)}\nu(t)} = H\left(t, e^{\frac{-\lambda}{a(t)}}\right) = \frac{e^{\frac{-\lambda}{a(t)}}\left\{1 - F(t,\theta(t))\right\}}{1 - e^{\frac{-\lambda}{a(t)}} + e^{\frac{-\lambda}{a(t)}}\left\{1 - F(t,\theta(t))\right\}}.$$

Since $e^{\phi(t)} \frac{1-F(t,\theta(t))}{Q(t)} = e^{\phi(t)} P\left\{X(t)\, Q(t) > \phi(t) \,\middle|\, X(t) > 0\right\}$, using the limit theorem for large deviations for critical processes we obtain that for $\theta(t) = \frac{\phi(t)}{Q(t)}$,

$$\lim_{t\to\infty} e^{\phi(t)} \frac{1 - F(t,\theta(t))}{Q(t)} = 1. \tag{3}$$

On the other hand it follows from the simple formula $e^{-\alpha} = 1 - \alpha + o(\alpha)$, $\alpha \to 0$, that

$$e^{\frac{-\lambda}{a(t)}} = 1 - \left(\frac{\lambda}{a(t)}\right)\left(1 + o(1)\right). \tag{4}$$

Thus,

$$\lim_{t\to\infty} \frac{e^{\phi(t)}}{Q(t)} \left\{1 - e^{\frac{-\lambda}{a(t)}}\right\} = \lambda \tag{5}$$

for $a(t) = \frac{e^{\phi(t)}\sigma^2 t}{2}$. From relations (2), (3), (4) and (5),

$$\lim_{t\to\infty} Ee^{\frac{-\lambda}{a(t)}\nu(t)} = \frac{1}{\lambda+1}, \quad \lambda > 0. \tag{6}$$

Let us now consider the asymptotic behaviour of the expectation of the process $\nu(t)$. Using (2), we find the expectation of the process,

$$E\nu(t) = H'(t,1) = \frac{1}{1 - F(t,\theta(t))}. \tag{7}$$

From (3) and (7), we find that if $f(s)$ be analytic on the disk $|s| < 1 + \varepsilon$, $A = 1$, for some integer $N \geq 2$, $\phi(t) = o\left(\frac{t}{\log_{(N)} t \log t}\right)$ and $\theta(t) = \frac{\phi(t)}{Q(t)}$, then

$$E\nu(t) \sim \frac{e^{\phi(t)}}{Q(t)}.$$

The last expression in (6) is the Laplace transform of the exponential distribution with parameter 1. Hence the assertion of the theorem follows from (6) and the continuity theorem for the Laplace transform (see [2], p. 431).

In order to demonstrate the possibilities of Theorem 1 we consider some examples of the function $\theta(t)$ satisfying conditions of Theorem 1. Let $\phi(t) = \log t$. As it was noted before $Q(t) \sim \frac{2}{\sigma^2 t}$ as $t \to \infty$. Thus, in this case from Theorem 1 we obtain the following result.

Example 1. Let $f(s)$ be analytic on the disk $|s| < 1 + \varepsilon$ for some $\varepsilon > 0$. If $A = 1$, $0 < \sigma^2 = f''(1) < \infty$ and $\phi(t) = \log t$, then

$$\lim_{t\to\infty} P\left\{ \frac{2\nu(t)}{\sigma^2 t^2} \leq x \right\} = 1 - e^{-x}, \quad x \geq 0$$

and $E\nu(t) \sim \frac{\sigma^2 t^2}{2}$ as $t \to \infty$.

One can see from Example 1 that the process $\nu(t)$ may have different asymptotic behaviour depending on the level functions $\theta(t)$. We now let $\phi(t) = c \log t$.

Example 2. Let $f(s)$ be analytic on the disk $|s| < 1 + \varepsilon$ for some $\varepsilon > 0$. If $A = 1$, $0 < \sigma^2 = f''(1) < \infty$ and $\phi(t) = c \log t$ where c is a positive constant, then

$$\lim_{t\to\infty} P\left\{ \frac{2\nu(t)}{\sigma^2 t^{c+1}} \leq x \right\} = 1 - e^{-x}, \quad x \geq 0$$

and $E\nu(t) \sim \frac{\sigma^2 t^{c+1}}{2}$ as $t \to \infty$.

3. Processes With Immigration

Now we consider processes with stationary immigration. Assume that at the time of birth of tth generation, that is, at time t, there is an immigration of Y_t individuals (or particles) into the population. Then the BGW process with immigration (BGWI) is defined by a sequence of random variables $Z(t)$ which are determined by the relation

$$Z(t) = \sum_{i=1}^{Z(t-1)} X_{ti} + Y_t, \quad Z(0) = 1$$

where the $Y_1, Y_2, \cdots$ are independent and identically distributed and are independent of variables $\{X_{ti}\}$.

Now, we consider the sequence of BGWI branching processes $\{ Z_i(t), i \in N \}$ and define the "index" process $\nu(t)$ $t \in N$ by the relation

$$\nu(t) = min\{ k : Z_k(t) > \theta(t) \},$$

for a given "level" function $\theta(t)$.

We assume that processes $\{ Z_i(t), i \in N \}$ are independent and identically distributed. This means that the corresponding processes without immigration have the same offspring distribution $\{P_k \ k \geq 0\}$ and the distributions of the number of immigrants are also the same for all processes. We denote

$$Z(t) \overset{d}{=} Z_i(t), \quad i \geq 1$$

$$P\{Y_i = k\} = D_k, \quad h(s) = ES^{Y_1} = \sum_{k=0}^{\infty} D_k s^k .$$

First, we consider the subcritical case. It is known that (see [1], p. 263) if

$$A < 1 \quad \text{and} \quad 0 < h'(1) < \infty, \tag{8}$$

then $Z(t)$ have a proper limit distribution, that is

$$\lim_{t \to \infty} P\{Z(t) = k\} = \rho_k$$

exist, where $\{ \rho_k, k \geq 0 \}$ is a probability distribution and the generating function of this stationary distribution is

$$b(s) = h(s) \prod_{t=1}^{\infty} h\{f(t,s)\}.$$

Theorem 2. *If (8) is satisfied and* $\theta(t) = \theta \in N = \{1, 2, \cdots\}$, *then for any fixed* $k \in N$,

$$\lim_{t\to\infty} P\{\nu(t) = k\} = \left\{\sum_{i=0}^{\theta} \rho_i\right\}^{k-1} \left\{1 - \sum_{i=0}^{\theta} \rho_i\right\}.$$

Proof. We consider again the relation (2) for the probability generating function $H(t, s)$ of $\nu(t)$ but here $F(t, z) = P\{Z(t) \le z\}$. Using the above limit theorem for the subcritical processes with immigration, we obtain that for any fixed θ, $H(t, s)$ tends to

$$\frac{s\left(1 - \sum_{i=0}^{\theta} \rho_i\right)}{1 - s\sum_{i=0}^{\theta} \rho_i}$$

as $t \to \infty$ and for any $0 < s < 1$, which is the generating function of

$$\sum_{k \le x} p^{k-1}(1-p)$$

with

$$p = \sum_{i=0}^{\theta} \rho_i .$$

Example 3. Let the limiting distribution be the geometric distribution that is $\rho_i = p(1-p)^i$. Then

$$\lim_{t\to\infty} P\{\nu(t) = k\} = \{(1-p) - (1-p)^{\theta+1}\}^{k-1} \{p + (1-p)^{\theta+1}\}.$$

Now, we consider the critical case. It is known (see [1], p. 265) and proved (see [9]) that if $A = 1$,

$$f''(1) = \sigma^2 < \infty , \quad 0 < h'(1) < \infty \tag{9}$$

then $\frac{2Z(t)}{\sigma^2 t}$ converges in distribution to a random variable with gamma density function

$$w(x) = \frac{1}{\Gamma\left(\frac{2h'(1)}{\sigma^2}\right)} x^{\frac{2h'(1)}{\sigma^2}-1} e^{-x}, \quad x \in (0, \infty).$$

Notice that when $h'(1) = \frac{\sigma^2}{2}$, then $w(x) = e^{-x}$.

Theorem 3. *If* $A = 1$, *(9) is satisfied and* $\theta(t) = \frac{\sigma^2 t}{2} c$, $c \in (0, \infty)$, *then*

$$\lim_{t \to \infty} P\{\nu(t) = k\} = \{W(c)\}^{k-1}\{1 - W(c)\},$$

where $W(c)$ *is the distribution function of* $w(c)$.

Proof. The proof of this theorem is similar to theorem 2 except that $p = W(c)$ in this case.

In the supercritical case, it is known (see [1], p. 264) that if $A > 1$, then there exists a sequence of constants C_t such that $\left\{\frac{Z(t)}{C_t}\right\}$ converges with probability one to a random variable V. If $E \log Y_1 < \infty$, then $P\{V < \infty\} = 1$ and V has an absolutely continuous distribution on $0, \infty)$. If $E \log Y_1 < \infty$, then $P\{V < \infty\} = 0$.

Theorem 4. *If* $A > 1$ *and* $\theta(t) = C_t v$, $v \in (0, \infty)$, *then*

$$\lim_{t \to \infty} P\{\nu(t) = k\} = \{R(v)\}^{k-1}\{1 - R(v)\},$$

where $R(v) = P(V \le v)$.

Proof. The proof of this theorem is also similar to theorems 2 and 3 except that $p = R(v)$.

Let us now consider the asymptotic behaviour of the expectation of the process $\nu(t)$. In the subcritical case, again using the limit theorem for the BGWI process $Z(t)$, we find that if $A < 1$ and for $\theta(t) = \theta$,

$$E\nu(t) \sim \left(1 - \sum_{i=0}^{\theta} \rho_i\right)^{-1}.$$

If the process is critical for which (9) holds and for $\theta(t) = \frac{\sigma^2 t}{2} c$, then

$$E\nu(t) \sim (1 - W(c))^{-1}.$$

At last in the supercritical case if $A > 1$ and for $\theta(t) = C_t v$ one can find that

$$E\nu(t) \sim (1 - R(v))^{-1}.$$

It has been proved in [7] that if $A=1$, $0<f''(1)=\sigma^2<\infty$, $h'(1)=h>0$ and $\alpha=\frac{2h}{\sigma^2}$, then for fixed $x>0$,

$$P\left\{\frac{2Z(t)}{\sigma^2 t} > x\right\} \to \frac{1}{\Gamma(\alpha)}\int_x^\infty y^{\alpha-1}e^{-y}\,dy$$

as $t\to\infty$ where $\Gamma(\alpha)$ is the gamma function. Moreover, by [6], the asymptotic behaviour of the probability $P\left\{\frac{2Z(t)}{\sigma^2} > x\right\}$ has been studied when $t\to\infty$ and $x\to\infty$. More exactly it is shown that under

Condition (II): the generating function $f(s)$ and $h(s)$ are analytic in the disk $|s|<1+\varepsilon$ for some $\varepsilon>0$, the maximal step of the lattice of the distribution of $f(s)$ is equal to 1 and $f'(1)=1$, $f''(1)=\sigma^2>0$, $0<h'(1)=h<\infty$ for the case where $0<x=o\left(\frac{t}{\log t}\right)$ as $t\to\infty$,

it has been proved that

$$\lim_{t\to\infty} P\left\{\frac{2Z(t)}{\sigma^2 t} \geq x\right\} = \frac{1}{\Gamma(\alpha)}\int_x^\infty y^{\alpha-1}e^{-y}\,dy. \tag{10}$$

This result allows us to consider the case when $\theta(t)$ tends to infinity quicker than a linear function. Namely we shall prove one general limit theorem for $\nu(t)$.

Theorem 5. *Suppose condition (II) is satisfied. If the function* $\phi(t)$ *is such that* $0<\phi(t)=o\left(\frac{t}{\log t}\right)$ *as* $t\to\infty$ *and* $\theta(t)=\frac{\sigma^2 t}{2}\phi(t)$ *then*

$$\lim_{t\to\infty} P\{\nu(t)\,U(\phi(t))\leq x\} = 1-e^{-x},\quad x>0$$

as $t\to\infty$ *where* $U(\phi(\mathrm{t}))=\frac{1}{\Gamma(\alpha)}\int_{\phi(t)}^\infty y^{\alpha-1}e^{-y}\,dy$.

Proof. The proof is similar to what have been done in the previous theorems. Putting $s=e^{-\lambda U(\phi(t))}$ in the probability generating function $H(t,s)$ of $\nu(t)$, we obtain the Laplace transform of $U(\phi(t))\,\nu(t)$:

$$Ee^{-\lambda U(\phi(t))\nu(t)} = H\left(t, e^{-\lambda U(\phi(t))}\right) = \frac{e^{-\lambda U(\phi(t))}\{1-F(t,\theta(t))\}}{1-e^{-\lambda U(\phi(t))}+e^{-\lambda U(\phi(t))}\{1-F(t,\theta(t))\}}.$$

Since $1-F(t,\theta(t) = P\left\{ Z(t) > \frac{\sigma^2 t}{2}\phi(t)\right\} = P\left\{ \frac{2Z(t)}{\sigma^2 t} > \phi(t)\right\}$, using relation (10), we obtain that

$$\lim_{t\to\infty} \left\{1-F(t,\theta(t))\right\} = (1+o(1))\,U(\theta(t)).$$

On the other hand, again it follows from the simple formula $e^{-\alpha} = 1-\alpha+o(\alpha),\ \alpha\to 0$, that

$$e^{-\lambda U(\phi(t))} = 1 - \lambda U(\phi(t)) + o\big(\lambda U(\phi(t))\big), \tag{12}$$

$$= (1 + o(1)). \tag{13}$$

Thus we have from relations (11), (12) and (13),

$$\lim_{t\to\infty} Ee^{-\lambda U(\phi(t))\,\nu(t)} = \frac{1}{\lambda+1}, \quad \lambda > 0.$$

The last expression is the Laplace transform of the exponential distribution.

Theorem 5 is a source of different limit theorems for $\nu(t)$. For example, if the quantity $\frac{2h'(1)}{\sigma^2}$ is an integer, we can use the following asymptotic formula for the tail of the gamma distribution:

$$U(x) = e^{-x}\sum_{k=0}^{m}\frac{1}{k!}x^k$$

$$\sim e^{-x}\frac{x^m}{m!} \text{ as } x\to\infty.$$

Thus, we have the following result.

Example 4. Suppose condition (II) is satisfied. If $\phi(t) = \frac{\sqrt{t}}{\log t}$ and $\theta(t) = \frac{\sigma^2 t}{2}\phi(t)$, then as $t\to\infty$

$$\lim_{t\to\infty} P\left\{ \nu(t)\frac{e^{-\phi(t)}(\phi(t))^m}{m!} \le x \right\} = 1 - e^{-x}, \quad x > 0$$

for $m = \frac{2h'(1)}{\sigma^2}$ integer.

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