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A Note on the General Hermitian Solution to $AXA^* = B$

JÜRGEN GROß
Department of Statistics, University of Dortmund, Vogelpothsweg 87, D-44221 Dortmund, Germany
e-mail: gross@amadeus.statistik.uni-dortmund.de

Abstract. Solution pairs (X, Y) to the matrix equations $AXA^* = B$ and $AX = BAY$ are considered, when X is restricted to be Hermitian and Y is restricted to be Hermitian, Hermitian nonnegative definite and Hermitian positive definite, respectively.

1. Introduction

Let $\mathcal{C}_{m\times n}$ denote the set of complex $m \times n$ matrices, and let $\mathcal{C}_m^H$ denote the set of complex Hermitian $m \times m$ matrices. Moreover let $\mathcal{C}_m^{\geq}$ denote the subset of $\mathcal{C}_m^H$ consisting of nonnegative definite matrices, and let $\mathcal{C}_m^{>}$ denote the subset of $\mathcal{C}_m^{\geq}$ consisting of positive definite matrices. The symbols A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $rk(A)$ will stand for the conjugate transpose, the range (column space), the null space and the rank, respectively, of $A \in \mathcal{C}_{m\times n}$.

The symbol A^- stands for an arbitrary generalized inverse (inner inverse) of $A \in \mathcal{C}_{m\times n}$, i.e., A^- is any matrix satisfying $AA^-A = A$.

In this note we consider the general Hermitian solution to the equation

$$AXA^* = B \tag{1.1}$$

for given matrices $A \in \mathcal{C}_{m\times n}$ and $B \in \mathcal{C}_m^H$. The usefulness of this equation has recently been emphasised by Dai and Lancaster [2], who consider solution pairs (X, Y) to the equations (1.1) and

$$AX = BAY, \tag{1.2}$$

where all combinations of restrictions $X \in \mathcal{C}_n^H$, $X \in \mathcal{C}_n^{\geq}$, $X \in \mathcal{C}_n^{>}$ and $Y \in \mathcal{C}_n^H$, $Y \in \mathcal{C}_n^{\geq}$, $Y \in \mathcal{C}_n^{>}$ are imposed on X and Y. Their approach is within the setting of real matrices, and it is based on the singular value decomposition (SVD) of the matrix A. However, from Dai and Lancaster [2] it does not become evident that equation (1.2) admits a solution $Y \in C$, where C can be $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$ or $\mathbf{C}_n^{>}$, only for a certain subset of all possible Hermitian solutions to (1.1), viz. Hermitian solutions to (1.1) satisfying $rk(AX) = rk(B)$. This fact might be of interest for instance in a situation where a Hermitian solution to (1.1) is already known, and it is desired to use this solution to (1.1) to obtain a solution to (1.2).

The following results offer the possibility for obtaining all combinations of solution pairs (X, Y) by using generalized inverses of matrices instead of using SVD of A.

2. Preliminary results

In the following we make repeated use of Corollary 6.2 in [4], which states that

$$rk(AB) = rk(B) - dim[N(A) \cap R(B)]$$

for arbitrary matrices $A \in \mathbf{C}_{m\times n}$ and $B \in \mathbf{C}_{n\times p}$. In this section we consider representations of general solutions to

$$AY = C \tag{2.1}$$

where A, $C \in \mathbf{C}_{m\times n}$ are given. Obviously (2.1) is more general than (1.2).

Lemma 1. [3, Theorem 2.1] *Let A, $C \in \mathbf{C}_{m\times n}$. Then* (2.1) *has a Hermitian solution if and only if*

$$CA^* \in \mathbf{C}_m^H \;,\; R(C) \subseteq R(A),$$

in which case a representation of the general Hermitian solution is

$$Y = A^-C + C^*(A^-)^* - A^-AC^*(A^-)^* + (I_n - A^-A)U(I_n - A^-A)^*,$$

where A^- is an arbitrary generalized inverse of A and U is an arbitrary matrix in $\mathbf{C}_n^H$.

Lemma 2. [3, Theorem 2.2] *Let A, $C \in \mathbf{C}_{m\times n}$. Then* (2.1) *has a Hermitian nonnegative definite solution if and only if*

$$CA^* \in \mathbf{C}_m^{\geq} \;,\; rk(CA^*) = rk(C)\,, \tag{2.2}$$

in which case a representation of the general Hermitian nonnegative definite solution is

$$Y = C^*(CA^*)^-C + (I_n - A^-A)U(I_n - A^-A)^*, \tag{2.3}$$

where $(\boldsymbol{CA}^*)^-$ *and* $\boldsymbol{A}^-$ *are arbitrary generalized inverses of* $\boldsymbol{CA}^*$ *and* $\boldsymbol{A}$ *respectively and* $\boldsymbol{U}$ *is an arbitrary matrix in* $\mathbf{C}_n^{\geq}$.

Corollary. *Let* $\boldsymbol{A}$, $\boldsymbol{C} \in \mathbf{C}_{m\times n}$ *such that* (2.2) *is satisfied. Then for any matrix* $\boldsymbol{Y}$ *from* (2.3), $rk(\boldsymbol{Y}) = rk(\boldsymbol{C}) + rk[(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{U}]$.

Proof. Suppose that (2.2) is satisfied. Then $\boldsymbol{C}^*(\boldsymbol{CA}^*)^-\boldsymbol{C}$ is invariant with respect to the choice of generalized inverse $(\boldsymbol{CA}^*)^-$ and Hermitian nonnegative definite with $rk[\boldsymbol{C}^*(\boldsymbol{CA}^*)^-\boldsymbol{C}] = rk(\boldsymbol{C})$, cf. [3, p. 580].

Let now $\boldsymbol{y} = \boldsymbol{C}^*(\boldsymbol{CA}^*)^-\boldsymbol{Ca} = (\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{Ub}$ for some $n \times 1$ vectors $\boldsymbol{a}, \boldsymbol{b}$. Then $\boldsymbol{Ay} = \boldsymbol{AC}^*(\boldsymbol{CA}^*)^-\boldsymbol{Ca} = \boldsymbol{Ca} = \boldsymbol{0}$, implying that $\boldsymbol{y} = \boldsymbol{0}$. Hence

$$\mathcal{R}\left[\boldsymbol{C}^*(\boldsymbol{CA}^*)^-\boldsymbol{C}\right] \cap \mathcal{R}\left[\left(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A}\right)\boldsymbol{U}\left(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A}\right)^*\right] = \{\boldsymbol{0}\},$$

which implies $rk(\boldsymbol{Y}) = rk[\boldsymbol{C}^*(\boldsymbol{CA}^*)^-\boldsymbol{C}] + rk(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{U}]$, cf. [4, Theorem 11].

Lemma 3. *Let* $\boldsymbol{A}, \boldsymbol{C} \in \mathbf{C}_{m\times n}$. *Then* (2.1) *has a Hermitian positive definite solution if and only if*

$$\boldsymbol{CA}^* \in \mathbf{C}_m^{\geq}, \; rk(\boldsymbol{CA}^*) = rk(\boldsymbol{C}), \; rk(\boldsymbol{C}) = rk(\boldsymbol{A}), \tag{2.4}$$

in which case a representation of the general Hermitian positive definite solution is given by (2.3), *where* $\boldsymbol{U} \in \mathbf{C}_n^{\geq}$ *satisfies* $\mathcal{N}(\boldsymbol{U}) \cap \mathcal{N}[(\boldsymbol{A}^-\boldsymbol{A})^*] = \{\boldsymbol{0}\}$.

Proof. When $\boldsymbol{Y}$ is a Hermitian positive definite solution to (2.1), obviously $rk(\boldsymbol{A}) = rk(\boldsymbol{C})$ and (2.4) follows from Lemma 2. On the other hand, when (2.4) is satisfied, every matrix $\boldsymbol{Y}$ from (2.3) is a Hermitian nonnegative definite solution to (2.1). When $rk(\boldsymbol{C}) = rk(\boldsymbol{A})$ and $\boldsymbol{U}$ is chosen to be in $\mathbf{C}_n^{>}$, then from the Corollary $\boldsymbol{Y}$ has full rank, i.e., $\boldsymbol{Y}$ is a Hermitian positive definite solution to (2.1).

When (2.4) is satisfied, Lemma 2 and the Corollary show that $\boldsymbol{Y}$ is a Hermitian positive definite solution to (2.1) if and only if $\boldsymbol{Y}$ is of the form (2.3) with $rk(\boldsymbol{C}) + rk(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{U}] = n$, the latter being equivalent to $rk(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})\boldsymbol{U}] = n - rk(\boldsymbol{A}) = rk(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})$. Applying Corollary 6.2 in [4] and observing $\mathcal{R}[(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})^*] = \mathcal{N}[(\boldsymbol{A}^-\boldsymbol{A})^*]$ completes the proof.

3. Solutions to special matrix equations

In this section we give *all* solutions $X \in \mathbf{C}_n^H$ to (1.1) which guarantee existence of solutions $Y \in C$ to (1.2), where C is $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$ or $\mathbf{C}_n^{>}$. Moreover, we demonstrate that the results of the previous section can be used to obtain *all* solutions $Y \in C$ to (1.2) for *each* solution $X \in \mathbf{C}_n^H$ to (1.1) which guarantees existence of solutions to (1.2) in C.

We start with the following proposition, which can be seen as a complement to Corollary 1 in [5, p. 25].

Proposition 1. *Let* $A \in \mathbf{C}_{m \times n}$ *and* $B \in \mathbf{C}_m^H$. *Then* (1.1) *has a Hermitian solution if and only if* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, *in which case a representation of the general Hermitian solution is*

$$X = A^- B(A^-)^* + Z - A^- AZ(A^- A)^* , \qquad (3.1)$$

where A^- *is an arbitrary generalized inverse of* A, *and* Z *is an arbitrary matrix in* $\mathbf{C}_n^H$.

Proof. When (1.1) has a solution, then $\mathcal{R}(B) = \mathcal{R}(AXA^*) \subseteq \mathcal{R}(A)$. On the other hand, when $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, then there exists a matrix G such that $B = AG = G^* A^*$. This gives $AG = AG(A^-)^* A^*$ and $G^* A^* = AA^- G^* A^*$. Hence $B = \frac{1}{2}(AG(A^-)^* A^* + AA^- G^* A^*) = A\frac{1}{2}(G(A^-)^* + A^- G^*)A^*$, showing that $\frac{1}{2}(G(A^-)^* + A^- G^*)$ is a Hermitian solution to (1.1).

When $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, i.e., $AA^- B = B = B(AA^-)^*$, it is easily seen that (3.1) is a Hermitian solution to (1.1). To observe that (3.1) is the general solution, note that any Hermitian solution to (1.1) may be written as (3.1) with $Z = X - A^- B(A^-)^*$.

When we assume that X is a Hermitian solution to (1.1), it is not difficult to derive Hermitian, Hermitian nonnegative definite and Hermitian positive definite solutions to (1.2) from the results in the previous section. Clearly, from Proposition 1, a necessary and sufficient condition for existence of a Hermitian solution to (1.1) with $A \in \mathbf{C}_{m \times n}$ and $B \in \mathbf{C}_m^H$ is $\mathcal{R}(B) \subseteq \mathcal{R}(A)$.

Proposition 2. *Let* $A \in \mathbf{C}_{m \times n}$ *and* $B \in \mathbf{C}_m^H$ *such that* $\mathcal{R}(B) \subseteq \mathcal{R}(A)$, *and let* X *be a Hermitian solution to* (1.1). *Then* (1.2) *has a solution* in C, *where* C *is* $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$, *or* $\mathbf{C}_n^{>}$, *if and only if* $rk(AX) = rk(B)$. *Representations of the general solutions in is* $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$, *and* $\mathbf{C}_n^{>}$, *are obtainable from Lemmas* 1, 2 *and* 3, *respectively*.

Proof. From Lemma 1, (1.2) has a Hermitian solution if and only if $\boldsymbol{B}^2 \in \mathbf{C}_m^H$, which is obviously satisfied, and $\mathcal{R}(\boldsymbol{AX}) \subseteq \mathcal{R}(\boldsymbol{BA})$. The latter is easily seen to be equivalent to $rk(\boldsymbol{AX}) = rk(\boldsymbol{B})$.

From Lemma 2, (1.2) has a Hermitian nonnegative definite solution if and only if $\boldsymbol{B}^2 \in \mathbf{C}_m^{\geq}$, which is obviously satisfied, and $rk(\boldsymbol{B}^2) = rk(\boldsymbol{AX})$. The latter is clearly equivalent to $rk(\boldsymbol{AX}) = rk(\boldsymbol{B})$.

From Lemma 3, (1.2) has a Hermitian positive definite solution if and only if $\boldsymbol{B}^2 \in \mathbf{C}_m^{\geq}$, $rk(\boldsymbol{B}^2) = rk(\boldsymbol{AX})$ and $rk(\boldsymbol{AX}) = rk(\boldsymbol{BA})$. When $rk(\boldsymbol{B}^2) = rk(\boldsymbol{AX})$, i.e., $rk(\boldsymbol{B}) = rk(\boldsymbol{AX})$, the condition $rk(\boldsymbol{AX}) = rk(\boldsymbol{BA})$ is equivalent to $rk(\boldsymbol{B}) = rk(\boldsymbol{BA})$. But this is equivalent to $\mathcal{N}(\boldsymbol{A}^*) \cap \mathcal{R}(\boldsymbol{B}) = \{\boldsymbol{0}\}$ from Corollary 6.2 in [4]. In view of $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$, the latter is clearly satisfied.

The above result focuses our interest on Hermitian solutions to

$$\boldsymbol{AXA}^* = \boldsymbol{B}, \quad rk(\boldsymbol{AX}) = rk(\boldsymbol{B}). \tag{3.2}$$

Clearly, for each solution $\boldsymbol{X} \in \mathbf{C}_n^H$ to (3.2), all solutions in $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$ or $\mathbf{C}_n^{>}$ to (1.2) are obtainable from Proposition 2.

Proposition 3. *Let* $\boldsymbol{A} \in \mathbf{C}_{m\times n}$ *and* $\boldsymbol{B} \in \mathbf{C}_m^H$. *Then* (3.2) *has a Hermitian solution if and only if* $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$, *in which case a representation of the general Hermitian solution is* (3.1)*, where* $\boldsymbol{Z} \in \mathbf{C}_n^H$ *is an arbitrary Hermitian solution to*

$$\boldsymbol{G}^*\boldsymbol{G}\ \boldsymbol{Z}\ \boldsymbol{H}^*\boldsymbol{H} = \boldsymbol{0} \tag{3.3}$$

with $\boldsymbol{G} = (\boldsymbol{I}_m - \boldsymbol{BB}^-)\boldsymbol{A}$ *for an arbitrary generalized inverse* $\boldsymbol{B}^-$ *of* $\boldsymbol{B}$, *and* $\boldsymbol{H} = (\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})$ *for* $\boldsymbol{A}^-$ *from* (3.1). *The general Hermitian solution to* (3.3) *is obtainable from Theorem* 2.4 *in* [3].

Proof. Necessity of $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$ for (3.2) to have a solution is clear. When $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$, then $\boldsymbol{X} = \boldsymbol{A}^-\boldsymbol{B}(\boldsymbol{A}^-)^*$ is a Hermitian solution to (1.1) with $rk(\boldsymbol{AX}) = rk[\boldsymbol{B}(\boldsymbol{A}^-)^*] = rk(\boldsymbol{A}^-\boldsymbol{B}) = rk(\boldsymbol{B}) - dim[\mathcal{N}(\boldsymbol{A}^-) \cap \mathcal{R}(\boldsymbol{B})]$, see Corollary 6.2 in [4] for the last equality. Clearly $\mathcal{N}(\boldsymbol{A}^-) \cap \mathcal{R}(\boldsymbol{B}) = \{\boldsymbol{0}\}$ when $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$.

When $\mathcal{R}(\boldsymbol{B}) \subseteq \mathcal{R}(\boldsymbol{A})$, any matrix (3.1) satisfies $rk(\boldsymbol{AX}) = rk(\boldsymbol{B})$ if and only if $\mathcal{R}(\boldsymbol{AX}) \subseteq \mathcal{R}(\boldsymbol{B})$, since $\mathcal{R}(\boldsymbol{B}) = \mathcal{R}(\boldsymbol{AXA}^*) \subseteq \mathcal{R}(\boldsymbol{AX})$ is always satisfied. It is clear that for any matrix (3.1) we have $\mathcal{R}(\boldsymbol{AX}) \subseteq \mathcal{R}(\boldsymbol{B})$ if and only if $\mathcal{R}[\boldsymbol{AZ} - \boldsymbol{AZ}(\boldsymbol{A}^-\boldsymbol{A})^*] \subseteq \mathcal{R}(\boldsymbol{B})$. This is equivalent to $\boldsymbol{BB}^-\boldsymbol{AZ}(\boldsymbol{I}_n - \boldsymbol{A}^-\boldsymbol{A})^*$

$= AZ(I_n - A^- A)^*$ for an arbitrary generalized inverse B^- of B. The latter is easily seen to be equivalent to (3.3).

It is easy to see that when X is a Hermitian nonnegative definite solution to (1.1), the condition $rk(AX) = rk(B)$ is met. Hence (1.2) admits solutions in $\mathbf{C}_n^H$, $\mathbf{C}_n^{\geq}$ or $\mathbf{C}_n^{>}$ for *all* Hermitian nonnegative definite solutions to (1.1), provided Hermitian nonnegative definite solutions to (1.1) exist.

Representations of the general Hermitian nonnegative definite and Hermitian positive definite solutions to (1.1) are given in Baksalary [1]. Hence by applying Baksalary [1] and Proposition 3, all combinations of solution pairs (X, Y) with $X \in \mathbf{C}_n^H$, $X \in \mathbf{C}_n^{\geq}$, $X \in \mathbf{C}_n^{>}$ and $Y \in \mathbf{C}_n^H$, $Y \in \mathbf{C}_n^{\geq}$, $Y \in \mathbf{C}_n^{>}$ are obtainable from Proposition 2.

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