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On Certain Representation of Topological Groups

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In this note classes of groups representations of which have either invariant vectors or invariant functionals are introduced. Connection between these classes of groups is established.

Let E be a separable topological vector space over the field of complex numbers C and $GL(E)$ be the group of all linear automorphisms of E and G be a separable topological group and $\rho : G \to GL(E)$ be a linear representation of the group G in E. We denote by E^G the subspace of $\rho(G)$-invariant elements in E, that is $E^G = \{x \in E : \rho(g)x = x \text{ for all } g \in G\}$, and denote by E' the space of continuous linear functionals on the space E.

A functional $f \in E'$ is called $\rho(G)$-invariant if $\mathrm{gf(x)} = \mathrm{f(g^{-1}x)} = \mathrm{f(x)}$ for all $g \in G$.

A linear representation ρ of topological group G in E is called continuous if the mapping $G \times E \to E$ defined by the formula $g(x) \to \rho(g)x$ is continuous.

Everywhere we consider continuous representations.

Definition 1. *A linear representation ρ is called linear reductive (or $t\alpha$-representation) if for each $x \in E^G$, $x \neq 0$ there exists continuous $\rho(G)$-invariant linear functional $f \in E'$ such that $f(x) \neq 0$. In the case of $E^G = \{0\}$ the representation ρ is also called $t\alpha$-representation.*

Definition 2. *A linear representation ρ of the group G in E is said to be $t\beta$-representation if for arbitrary nontrivial continuous $\rho(G)$-invariant functional $f \in E'$ there exists element $x \in E^G$ such that $f(x) \neq 0$.*

We recall that a closed subspace E_1 in E is said to be t-complementable if there exists a closed subspace E_2 in E such that $E_1 \cap E_2 = \{0\}$ and E is the topological direct sum of subspaces E_1 and E_2. Here one assumes that there exist continuous projections acting from E onto E_1 and E_2 [2]. In this case subspace E_2 is called t-complementable to E_1 and the notation $E = E_1 \oplus E_2$ is used.

Definition 3. *A linear representation ρ of the group G in E is said to be $t\gamma$-representation (or semi-simple), if for every $\rho(G)$-invariant subspace E_1 in E there exists on $\rho(G)$-invariant complementary subspace E_2.*

We refer to topological group G as the group of class $t\alpha$ (respectively $t\beta$ and $t\gamma$), if its every continuous linear representation is a $t\alpha$-(resp. $t\beta$- and $t\gamma$-) representation.

We refer to topological group G as the $ft\alpha$ (respectively $ft\beta$ and $ft\gamma$) group, if its every finite dimensional continuous representation is a $t\alpha$-(resp. $t\beta$- and $t\gamma$-) representation.

Proposition. *Every $t\gamma$-group is also a $t\alpha$-group.*

Proof. Let G be a $t\gamma$ -group and $0 \neq x \in E^G$. Since p is a $t\gamma$-representation, then there exists a G-invariant closed subspace E_x complementing the one dimensional invariant subspace Cx, where C denotes the field of complex number. Let us consider the linear functional f in E defined in the following way: $f(y) = 0$, at $y \in E_x$ and $f(x) = 1$. Then f is invariant with respect to G, $f \in E'$ and $f(x) \neq 0$. Consequently, G is a $t\alpha$-group.

Theorem 1. *A locally compact group G is a $t\beta$-group if and only if G is compact.*

Proof. Since G is a locally compact group, there exists a right-invariant nontrivial Haar measure dg on G. We consider the Banach space $V = L_1(G, dg)$ with the norm $\|\varphi\| = \int_G |\varphi|\, dg,\ \varphi \in V$. We give a representation G in V by setting

$$(T_g\varphi)(t) = \varphi(tg), \quad \varphi \in V,\ g,\ t \in G.$$

Since dg is right-invariant measure, then nontrivial linear functional $f(\varphi) = \int_G \varphi(t)dg$ on V is $\rho(G)$-invariant. Since the group G is a $t\beta$-group, there exists nontrivial $\varphi_0 \in V^G$ such that $f(\varphi_0) \neq 0$. But if G is not a compact group, then $V^G = \{0\}$. Consequently, G is compact.

Let G be a compact group and dg be a measure of Haar on G, normed by the condition $\int_G dg = 1$, and $\rho :\to GL(E)$ be a linear continuous representation in complete locally convex space E and E' be the conjugated space to E.

Let $f \in (E')^G$ be an arbitrary nonzero element. There is $x \in E$ such that $f(x) \neq 0$. We consider operator

$$p = \int_G \rho(g)dg \tag{1}$$

It is known (see, for example, [1], p.150) that this operator is a projection operator on E^G (expression (1) is considered as an integral of function on G with values in $GL(E)$ (see [1], p.150)). Therefore $px = \int_G \rho(g)(x)dg = v \in E^G$. Since

$$f(v) = f\left(\int_G (g)(x)dg\right) = \int_G f((g)(x))dg = \int_G f(x)dg = f(x)\int_G dg = f(x)\cdot 1 = f(x) \neq 0,$$

therefore ρ is a $t\beta$-representation. This means that G is a β-group.

Theorem 2. *For the group G the following conditions are equivalent:*

(a) G is a group of class $t\beta$
(b) G is a group of class $t\gamma$.

Proof. $(a) \Rightarrow (b)$. Let $\rho : G \to GL(E)$ be an arbitrary continuous linear representation of the topological group G of class $t\beta$ in a space E, and E_0 be a nontrivial t-complementable $\rho(G)$-invariant subspace in E.

We prove that there exists $\rho(G)$-invariant t-complement of E_0. Let $L(E) = Hom_c(E,E) = End\ E$ be the space of all continuous linear mappings of E into E. G operates in $L(E)$ as follows : $\tilde{g}f = gfg^{-1}$, where $g \in G$, $f \in L(E)$.

In $L(E)$ we shall consider strong operator topology st: sequence f_α converges to f strongly if for all $x \in E$ the sequence $f_\alpha(x)$ converges to $f(x)$ (and it is denoted by $f_\alpha \xrightarrow{st} f$).

Since E_0 is t-complementable vector space in E, there exists a closed vector subspace E_1 of E such that E is topological direct sum of E_0 and E_1.

Let $p_0(p_1)$ is projective operator on $E_0(E_1)$ parallel to the space E_1 (respectively E_0). Then $1 = p_0 + p_1$ and these operators are continuous [2].

The equality $gp_0 = p_0gp_0$ is clear. Hence $\tilde{g}p_0 = p_0\tilde{g}p_0$; indeed $p_0\tilde{g}p_0 = p_0gp_0g^{-1} = gp_0g^{-1} = \tilde{g}p_0$.

We shall consider the following linear subspaces

$$W_0 = l.s.\{\tilde{g}p_1 : g \in G\},\ V_0 = l.s.\{(\tilde{g}-1)p_1 : g \in G\},$$

where *l.s.* means linear span.

We have

$$W_0 = V_0 + Cp_1.$$

We show that $V_0 \cap Cp_1 = \{0\}$. Indeed, $p_1E = E_1$ and at the same time

$$(\tilde{g} - 1)p_1 = \tilde{g}p_1 - p_1 = \tilde{g}(1 - p_0) - p_1 = 1 - \tilde{g}p_0 - p_1$$
$$= p_0 - \tilde{g}p_0 = p_0^2 - \tilde{g}p_0 = p_0^2 - p_0\tilde{g}p_0 = p_0(1 - \tilde{g})p_0.$$

Hence $T(E) \in E_0$ for arbitrary $T \in V_0$ and therefore $V_0 \cap Cp_1 = \{0\}$. Thus $W_0 = V_0 + Cp_1$. Now we show that $W = V \oplus Cp_1$, where $W = \overline{W_0}$, $V = \overline{V_0}$ and the closures are taken on topology *st*. Assume that the sequence $x_\alpha \in W_0$ converges to $x \in \overline{W_0} = W$. Then

$$x_\alpha = y_\alpha + c_\alpha p_1 \xrightarrow{st} x, \quad y_\alpha \in V_0,\ c_\alpha \in C.$$

By applying p_1 to

$$y_\alpha + c_\alpha p_1 \xrightarrow{st} x$$

we get

$$p_1(y_\alpha + c_\alpha p_1) = p_1 y_\alpha + c_\alpha p_1 = 0 + c_\alpha p_1 \xrightarrow{st} p_1 x.$$

Since $c_\alpha p_1 \in Cp_1$, then $p_1 x \in Cp_1$. On the other hand from the convergence $x_\alpha \xrightarrow{st} x$ we get that

$$p_0 x_\alpha = p_0(y_\alpha + c_\alpha p_1) = p_0 y_\alpha + pc_\alpha p_1 = y_\alpha + 0 \xrightarrow{st} p_0 x,$$

i.e., $p_0 x \in \overline{V_0} = V$. Therefore $x = (p_0 + p_1)x = p_0 x + p_1 x \in V + Cp_1$.

Assume that $z \in \overline{V_0} = V$, then there exists $y_\alpha \in V_0$, such that $y_\alpha \xrightarrow{st} z$. From here we have

$$p_0 y_\alpha = y_\alpha \xrightarrow{st} p_0 z \quad \text{and} \quad z = p_0 z,$$

i.e., $z(E) \subset E_0$. But $Cp_1(E) \subset E_1$. Hence $V \cap Cp_1 = \{0\}$. Thus $W = V \oplus Cp_1$.

Since V is *st*-closed and $\dim Cp_1 = 1$, there exists $f \in W'$ such that $\ker f = V$. Let ω be an arbitrary element of W, i.e., $\omega = \lambda p_1 + \omega_1$, $\lambda \in C$, $\omega_1 \in V$. Then $f(\omega) = \lambda f(p_1) + f(\omega_1) = \lambda f(p_1)$.

It can be assumed that $f(p_1) = 1$. Taking it into account, we get that $\lambda = f(\omega)$ and $\omega_1 = \omega - f(\omega)p_1$.

Thus $\omega = f(\omega)p_1 + (\omega - f(\omega)p_1)$. From the construction of W it follows that this space is $\rho(G)$-invariant. We consider restriction of the representation of the group G to W and show that f is a $\rho(G)$-invariant functional. Indeed, $f(\lambda p_1 + v) = \lambda$, where $\upsilon \in V$. Therefore

$$f(\tilde{g}(\lambda p_1 + \upsilon)) = f(\lambda \tilde{g} p_1 + \tilde{g}\upsilon) = f(\lambda p_1 - \lambda p_1 + \lambda \tilde{g} p_1 + \tilde{g}\upsilon)$$
$$= f(\lambda p_1 - \lambda(1 - \tilde{g})p_1 + \tilde{g}\upsilon) = f(\lambda p_1 + \lambda(\tilde{g} - 1)p_1 + \tilde{g}\upsilon) = \lambda,$$

because $\lambda(\tilde{g} - 1)p_1 + \tilde{g}\upsilon \in V$. This means that f is $\rho(G)$-invariant.

Since G is $t\beta$-group, for this functional there exists a $\rho(G)$-invariant element $p_2 = p_1 + Q_0 \in W(Q_0 \in V)$ such that $f(p_2) = 1 \neq 0$. We define $L = \overline{p_2(E_1)}$ (closure in topology *st*).

Since p_2 is $\rho(G)$-invariant, L is also $\rho(G)$-invariant: Indeed $\tilde{g}p_2 = gp_2g^{-1} = p_2$, i.e., $gp_2 = p_2 g$. Taking it into account we get that

$$g(p_2(E) = gp_2g^{-1}(g(E)) = p_2(g(E)) \subset p_2E.$$

By using the continuity of g and taking the limit we get that $\rho(L) \subset L$, i.e., L is $\rho(G)$-invariant.

We note that $T(E_0) = 0$ for any $T \in V_0$ and therefore $T(E_0) = 0$ for all $T \in V$. From here $Q_0(E_0) = 0$ and $p_1(E_0) = 0$. Then

$$p_2(E_0) = (p_1 + Q_0)E_0 = p_1E_0 + Q_0E_0 = 0$$

Thus

$$p_2E = p_2(E_0 + E_1) = p_2E_0 + p_2E_1 = p_2E_1.$$

From here we get that $L = \overline{p_2(E_1)}$. For every $x \in E$ we have

$$x = p_0x + p_1x = p_0x + (p_2 - Q_0)x = (p_0 - Q_0)x + p_2x.$$

Taking into account $(p_0 - Q_0)x \in E_0$ and $p_2x \in L$, we get that $E = E_0 + L$. If $z \in E_0 \cap p_2E_1$, then $z = p_2y\,(y \in E_1)$ and $z \in E_0$, in particular $p_1z = 0$. Then

$$0 = p_1(p_2y) = p_1(p_1 + Q_0)y = (p^2 + p_1Q_0)y = (p_1 + p_1Q_0)$$
$$= p_1y + p_1Q_0y = p_1y = y.$$

From here we get $y = 0$ and $z = 0$. Hence $E_0 \cap p_2E_1 = \{0\}$.

Now we assume that $z \in E_0 \cap L$, then there exists $z_n \in p_2E = p_2E_1$ such that $z_n \xrightarrow{st} z$. Since $z \in E_0$, then $p_0z = z$, $p_1z = 0$. Hence $p_1z_n \to p_1z = 0$. Next $z_n = p_2y_n$, where $y_n \in E_1$. Hence $0 \xleftarrow{st} p_1z_n = p_1(p_2y_n) = p_1y_n = y_n$, i.e., $y_n \xrightarrow{st} 0$.

From here we get $z \xleftarrow{st} z_n = p_2 y_n = p_1 y_n = y_n \xrightarrow{st} 0$, and therefore $z = 0$. It means that $E_0 \cap L = \{0\}$.

Thus $E = E_0 \oplus L$ i.e., it is topological direct sum of $\rho(G)$-invariant subspaces E_0 and L.

$(b) \Rightarrow (a)$. We assume that $0 \neq f \in E'$, f is $\rho(G)$-invariant continuous linear functional, $V = ker\, f$.

Subspace $V = ker\, f$ is obviously $\rho(G)$-invariant. Then from the condition $(t\gamma)$ it follows that there exists closed $\rho(G)$ invariant subspace $H \subset E$ such that $E = V \oplus H$. Since $codim\ V = 1$, then $dim\, H = 1$, i.e., $H = \{\lambda \upsilon\}_{\lambda \in C}$, $\upsilon \neq 0$ and $f(\upsilon) \neq 0$.

Let $g\upsilon = \lambda_1 \upsilon$. We have that $0 \neq f(\upsilon) = f(g\upsilon) = f(\lambda_1 \upsilon) = \lambda_1 f(\upsilon)$. Hence $\lambda_1 = 1$, and therefore $g\upsilon = \upsilon$ for every $g \in G$. Thus G is $t\beta$-group. Theorem 2 is proved.

Corollary. *Every $t\beta$-group is a $t\alpha$-group.*

The proof follows from the Proposition and Theorem 2.

Theorem 3. *For the group G the following conditions are equivalent:*

(a) G is a $ft\alpha$-group,
(b) G is a $ft\beta$-group,
(c) G is a $ft\gamma$-group.

Proof. $(a) \Leftrightarrow (c)$ is analogous to the proof of the proposal 2.2.4 (see [3] p.27). That $(b) \Leftrightarrow (c)$ can be proved in the same way as in Theorem 2.

References

1. A.A. Kirillov, *Elements of Representation Theory*, Nauka, Moscow, 1978.
2. W. Rudin, *Functional Analysis*, Mir, Moscow, 1975.
3. T. Springer, *Invariant Theory*, Mir, Moscow, 1981.