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The A.M.-G.M. Inequality as a Rearrangement Inequality

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Abstract. This paper shows that the arithmetic mean-geometric mean inequality is a direct consequence of a rearrangement inequality of Hardy *et al*.

1. Introduction

In this paper, it is shown that the classical arithmetic mean-geometric mean inequality can be obtained as a corollary to a seemingly unrelated rearrangement inequality of Hardy *et al.* [1, Theorem 368, p. 261].

2. Preliminaries

If $\boldsymbol{x} = \left(x_1, x_2, \cdots, x_n\right) \in \boldsymbol{R}^n$ is any n-tuple of real numbers, we denote by $\boldsymbol{x}^* = (x_1^*, x_2^*, \cdots, x_n^*)$ (respectively $\boldsymbol{x}' = (x_1', x_2', \cdots, x_n')$) the n-tuple in $\boldsymbol{R}^n$ whose components are those of $\boldsymbol{x}$ arranged in nonincreasing (respectively nondecreasing) order of magnitude, i.e., $x_1^* \geq x_2^* \geq \cdots \geq x_n^*$ (respectively $x_1' \leq x_2' \leq \cdots \leq x_n'$) and $x_i^* = x_{\pi(i)}$ (respectively $x_i' = x_{\rho(i)}$), $1 \leq i \leq n$, for some permutation π (respectively ρ) of the integers $1, 2, \cdots, n$.

If $\boldsymbol{a} = (a_1, a_2, \cdots, a_n) \in \boldsymbol{R}^n$ and $\boldsymbol{b} = (b_1, b_2, \cdots, b_n) \in \boldsymbol{R}^n$, then the following rearrangement inequalities of Hardy *et al.* [1, Theorem 368, p. 261] hold:

$$\sum_{i=1}^{n} a_i^* b_i' \leq \sum_{i=1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^* b_i^* \tag{1}$$

where equality on the right (respectively left) holds if and only if $\boldsymbol{a}$ and $\boldsymbol{b}$ are *similarly* (respectively *oppositely*) *ordered*, i.e., if and only if $(a_i - a_j)(b_i - b_j) \geq 0$ (respectively $(a_i - a_j)(b_i - b_j) \leq 0$) for all integers i and j such that $1 \leq i \leq n$ and $1 \leq j \leq n$.

3. The arithmetic mean-geometric mean inequality

We shall now show that the following arithmetic mean-geometric mean inequality can be obtained as a rearrangement inequality by demonstrating that it is a direct consequence of the rearrangement inequality of Hardy *et al.* given in (1).

Theorem. *For any integer* $n \ge 1$, let $x_1, x_2, \cdots, x_n$ *be* n *nonnegative numbers. Then*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \ge \sqrt[n]{x_1 x_2 \cdots x_n}$$

where equality holds if and only if $x_1 = x_2 = \cdots = x_n$.

Proof. Clearly the theorem is equivalent to entailing that

$$x_1 + x_2 + \cdots + x_n \ge n \tag{2}$$

whenever $x_1 x_2 \cdots x_n = 1$ where $x_i > 0,\ i = 1, 2, \cdots, n$ and equality holds in (2) if and only if $x_i = 1,\ i = 1, 2, \cdots, n$.

Assume that $x_1 x_2 \cdots x_n = 1$ where $x_i > 0,\ i = 1, 2, \cdots, n$. Let y_0 be any given positive number. Define

$$y_k = \frac{y_0}{x_1 x_2 \cdots x_k} \quad \text{for } k = 1, 2, \cdots, n.$$

Then it is easy to see that $y_n = y_0$ and $x_k = \frac{y_{k-1}}{y_k},\ k = 1, 2, \cdots, n$, and so

$$x_1 + x_2 + \cdots + x_n = \frac{y_n}{y_1} + \frac{y_1}{y_2} + \cdots + \frac{y_{n-1}}{y_n}. \tag{3}$$

Let $\boldsymbol{a} = (a_1, a_2, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \cdots, b_n)$ where $a_i = y_{i-1},\ b_i = \frac{1}{y_i}$, $1 \le i \le n$. Then it is obvious that

$$\boldsymbol{b}' = \left(\frac{1}{y_1^*}, \frac{1}{y_2^*}, \cdots, \frac{1}{y_n^*}\right) \text{ if } \boldsymbol{a}^* = (y_1^* y_2^*, \cdots, y_n^*).$$

Thus, from (1), we have

$$\frac{y_n}{y_1} + \frac{y_1}{y_2} + \cdots + \frac{y_{n-1}}{y_n} = \sum_{i=1}^{n} a_i b_i \ge \sum_{i=1}^{n} a_i^* b_i' = \sum_{i=1}^{n} y_i^* \left(\frac{1}{y_i^*}\right) = n \tag{4}$$

whence (2) follows in view of (3).

To establish the condition for equality, we first note that there is no loss in generality in assuming that

$$x_1 \geq x_2 \geq \cdots \geq x_n . \tag{5}$$

If $x_1 = x_2 = \cdots = x_n (= 1)$, then obviously equality holds in (2). On the other hand, if not all of $x_1, x_2, \cdots, x_n$ are equal, then there is at least one strict inequality in (5), say $x_k > x_{k+1}$ for some k, $1 \leq k < n-1$.

There is no loss in generality in assuming that $x_i \neq 1$, $i = 1, 2, \cdots, n$, since any x_i with value 1 can just be deleted from the left of the inequality (2) which can then be adjusted accordingly with $n-1$ replacing n on the right. Moreover, we can also choose k in such a way that $x_k > 1 > x_{k+1}$. Thus

$$x_k = \frac{y_{k-1}}{y_k} > 1 > \frac{y_k}{y_{k+1}} = x_{k+1} \tag{6}$$

and

$$(a_i - a_j)(b_i - b_j) = (y_{i-1} - y_{j-1})\left(\frac{1}{y_i} - \frac{1}{y_j}\right), \quad 1 \leq i \leq n, \quad 1 \leq j \leq n.$$

Now choose $i = k$, $j = k+1$. Then

$$\begin{aligned}(a_k - a_{k+1})(b_k - b_{k+1}) &= (y_{k-1} - y_k)\left(\frac{1}{y_k} - \frac{1}{y_{k+1}}\right) \\ &= \left(\frac{y_{k-1}}{y_k} - 1\right)\left(1 - \frac{y_k}{y_{k+1}}\right) \\ &= (x_k - 1)(1 - x_{k+1}) > 0\end{aligned}$$

in view of (6) so that $\boldsymbol{a}$ and $\boldsymbol{b}$ cannot be oppositely ordered and so the inequality on the left of (1) is strict.

It follows that the inequality in (4) and consequently the inequality in (2) are strict.

References

1. G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge, 1959.