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The Characterization of the Spherical Timelike Curves in 3-Dimensional Lorentzian Space

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Abstract. In this paper, the characterization of the spherical curves in 3-dimensional Lorentzian space is given which corresponds to the 3-dimensional Euclidean space.

1. Introduction

As in [3], Lorentzian inner product in R^3 can be written as

$$\langle \boldsymbol{X}, \boldsymbol{Y} \rangle = -x_1 y_1 + \sum_{i=2}^{3} x_i y_i$$

where $\boldsymbol{X} = (x_1, x_2, x_3)$, $\boldsymbol{Y} = (y_1, y_2, y_3)$ and R^3 is a real standard vector space. The couple $\{R^3, \langle\ ,\ \rangle\}$ is called a 3-dimensional Lorentzian space and denoted by L^3.

A vector $\boldsymbol{X}$ is called lightlike or null if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle = 0$, timelike if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle < 0$, spacelike if $\langle \boldsymbol{X}, \boldsymbol{X} \rangle > 0$ respectively. The norm of a vector $\boldsymbol{X}$ in L^3 is defined as $\| \boldsymbol{X} \| = \sqrt{|\langle \boldsymbol{X}, \boldsymbol{X} \rangle|}$ where $\boldsymbol{X} = (x_1, x_2, x_3)$ [1].

Definition 1.1. [4] *Let α be a curve in Lorentzian space L^3. If α' is a velocity vector of α, then α is a timelike curve if $\langle \alpha', \alpha' \rangle < 0$.*

We shall assume throughout this paper that a timelike curve is parametrized by the arc length.

Definition 1.2. *Let* $\alpha \subset L^3$ *be a given timelike curve. If the Frenet vector* $\{V_1(s),\ V_2(s),\ V_3(s)\}$ *which corresponds to* $s \in I$ *is defined as*

$$\begin{aligned} &k_i : I \to R \\ &\quad s \to k_i(s) = \langle V_i'\ (s),\ V_{i+1}(s) \rangle \end{aligned}$$

then the function k_i *is called an* i^{th} *curvature function of the timelike curve* α *and the real* $k_i(s)$ *is also called an* i^{th} *curvature at the point* $\alpha(s)$.

Theorem 1.1. [2] *Let* α *be a timelike curve in* L^3. *Let* $k_i(s)$ *be an* i^{th} *curve at the point* $\alpha(s)$ *such that* $s \in I$ *is a curve parameter and* $\{V_1(s), V_2(s), V_3(s)\}$ *is a Frenet vector. Then*

$$\begin{bmatrix} V_1'(s) \\ V_2'(s) \\ V_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} V_1(s) \\ V_2(s) \\ V_3(s) \end{bmatrix} \tag{1.1}$$

where $k_1 \neq 0$ *and* $k_2 \neq 0$.

Definition 1.3. *Let* α *be a timelike curve in* L^3 *and* S^2 *be a given 2-hypersphere in* L^3. *If* $\alpha \subset S^2$ *then* α *is called a 2-timelike hyperspherical curve of* L^3.

Definition 1.4. *The sphere having sufficiently close common four points at* $m \in \alpha$ *with the timelike curve* $\alpha \subset L^3$ *is called an osculator sphere or curvature sphere of* α *at the point* $m \in \alpha$.

Now let us calculate the geometric locus of the sphere having sufficiently close common three points with α at $m \in \alpha$ of the timelike curve $\alpha \subset L^3$.

2. Results

Theorem 2.1. *Let* $\alpha \subset L^3$ *timelike curve be given with coordinate neighborhood* (I, α). *The geometric locus of the centers of the spherical curves having sufficiently close common three points with* α *providing the Frenet vectors* $\{V_1(s), V_2(s), V_3(s)\}$ *at the point* $\alpha(s),\ s \in I$ *is*

$$\boldsymbol{a}(s) = \alpha(s) - m_2(s)V_2(s) - m_3(s)V_3(s)$$

where

$$m_2 : I \to R, \quad m_2(s) = \frac{1}{k_1(s)}, \quad m_3(s) = \pm\sqrt{r^2 - \left(\frac{1}{k_1(s)}\right)^2}.$$

Proof. Let (I, α) be a coordinate neighborhood, and $s \in I$ be curve parameter for α. Let also $\boldsymbol{a}$ be the center and r be the radius of the sphere having sufficiently close common three points with α. In accordance to this, let us consider

$$f : I \to R$$
$$s \to f(s) = \langle \alpha(s) - \boldsymbol{a},\ \alpha(s) - \boldsymbol{a} \rangle - r^2. \tag{2.1}$$

Since

$$f(s) = f'(s) = f''(s) = 0 \tag{2.2}$$

at the point $\alpha(s)$, then the sphere

$$S^2 = \{ x \mid x \in L^3, \langle x - \boldsymbol{a},\ x - \boldsymbol{a} \rangle = r^2 \}$$

with the timelike curve α at this point passes sufficiently close three points. According to this, considering (2.1) and (2.2) together,

$$f'(s) = \langle V_1(s),\ \alpha(s) - \boldsymbol{a} \rangle = 0$$

is obtained. From this, since $f''(s) = 0$ then,

$$\langle V_1'(s),\ \alpha(s) - \boldsymbol{a} \rangle + \langle V_1(s),\ \alpha'(s) \rangle = 0.$$

Considering (1.1) with this, then

$$k_1(s) \langle V_2(s),\ \alpha(s) - \boldsymbol{a} \rangle - 1 = 0.$$

On the other hand, for the base $\{V_1(s),\ V_2(s),\ V_3(s)\}$,

$$\alpha(s) - \boldsymbol{a} = m_1(s) V_1(s) + m_2(s) V_2(s) + m_3(s) V_3(s) \tag{2.3}$$

is obtained. However, by using (2.2), then

$$\langle \alpha(s) - \boldsymbol{a}, V_1(s) \rangle = -m_1(s) \Rightarrow m_1(s) = 0 \tag{2.4}$$

and

$$\langle \alpha(s) - \boldsymbol{a},\ V_2(s) \rangle = m_2(s) \Rightarrow m_2(s) = \frac{1}{k_1(s)}. \tag{2.5}$$

By using $f(s) = 0$, then

$$\langle \alpha(s) - \boldsymbol{a},\ \alpha(s) - \boldsymbol{a} \rangle = r^2 \Rightarrow -m_1^2(s) + m_2^2(s) + m_3^2(s) = r^2. \tag{2.6}$$

Considering (2.4) in (2.6), then

$$m_2^2(s) + m_3^2(s) = r^2. \tag{2.7}$$

Considering also (2.5) in (2.7), then

$$m_3(s) = \pm\sqrt{r^2 - \left(\frac{1}{k_1(s)}\right)^2} = \lambda. \tag{2.8}$$

Therefore, substituting (2.4), (2.5), (2.8) in (2.3), then

$$\boldsymbol{a} = \alpha(s) - m_2(s)V_2(s) - \lambda V_3(s).$$

Here $\boldsymbol{a}$ and r change when the spheres change. Hence, $m_3(s) = \lambda \in R$ is a parameter. This completes the proof of the theorem.

Corollary 2.1. *Let the timelike curve $\alpha \subset L^3$ be given with neighborhood coordinate (I, α). Then the centers of the spheres which pass sufficiently close common three points with α at the points $\alpha(s) \in \alpha$ are located on a straight line.*

Proof. By Theorem 2.1, the equation

$$\boldsymbol{a} = \alpha(s) - m_2(s)V_2(s) - \lambda V_3(s)$$

where the parameter $\lambda \in R$ denotes a line.

Definition 2.1. *The geometric locus of the centers of the spheres which have sufficiently close common three points with $\alpha \in L^3$ timelike curve at the point $m \in \alpha$ is defined as the curvature axis.*

Theorem 2.2. *Let $\alpha \subset L^3$ timelike curve is given by (I, α) coordinate neighborhood. If*

$$\boldsymbol{a}(s) = \alpha(s) - m_2(s)V_2(s) - m_3(s)V_3(s)$$

is the center of the osculator sphere at the point $\alpha(s) \in \alpha$, then

$$m_2(s) = \frac{1}{k_1(s)} \quad , \quad m_3(s) = \frac{m_2'(s)}{k_2(s)} .$$

Proof. The proof of the theorem is similar to the proof of Theorem 2.1. Osculator sphere with α timelike curve have sufficiently close common four points. Therefore, since $f''(s) = 0$ in (2.2), thus $f'''(s) = 0$. If $f'''(s) = 0$ then

$$k_1'(s)\left\langle V_2(s), \alpha(s) - \boldsymbol{a} \right\rangle + k_1(s)\left[\left\langle V_2'(s),\ \alpha(s) - \boldsymbol{a} \right\rangle + \left\langle V_2(s), \alpha'(s) \right\rangle\right] = 0\,.$$

Considering (1.1), (2.4) and (2.5) in the last expression,

$$k_1'(s)m_2(s) + k_1(s)k_2(s)m_3(s) = 0$$

or

$$m_3(s) = -\frac{k_1'(s)m_2(s)}{k_1(s)k_2(s)}\,.$$

Using (2.5) in the last equation yields

$$m_3(s) = \frac{m_2'(s)}{k_2(s)}.$$

Corollary 2.2. *Let α timelike curve in L^3 is given by (I, α) neighbouring coordinate. If r is the radius of the osculator sphere at $\alpha(s) \in \alpha$, then*

$$r = \left[m_2^2(s) + m_3^2(s)\right]^{1/2}\,.$$

Proof. If the center of the osculator sphere at $\alpha(s)$ is $\boldsymbol{a}$, then by Theorem 2.1,

$$\boldsymbol{a}(s) = \alpha(s) - m_2(s)V_2(s) - m_3(s)V_3(s).$$

Thus

$$r = \left\| \boldsymbol{a} - \alpha(s) \right\| = \left[m_2^2(s) + m_3^2(s)\right]^{1/2}.$$

Theorem 2.3. *Let S_0^2 be a sphere centered at 0 and also $\alpha \subset S_0^2$ be a spherical curve. In this case, since (I, α) is a neighbouring coordinate for α and $s \in I$ is a parameter of arc, then*

$$\left\langle \alpha(s),\ V_i(s) \right\rangle = m_i(s)\,, \quad 2 \le i \le 3.$$

Proof. Since $\alpha(s) \in S_0^2$ for all $s \in I$, and r is a radius, then

$$\left\langle \alpha(s), \alpha(s) \right\rangle = r^2.$$

Taking derivatives both sides of the equation with respect to s, then

$$\langle V_2(s), \alpha(s) \rangle = m_2(s).$$

Again, taking derivatives both sides of the equation with respect to s and using (1.1) and (2.4) in here, then

$$\frac{m_2'(s)}{k_2(s)} = \langle V_3(s), \alpha(s) \rangle.$$

Thus by Theorem 2.2,

$$m_3(s) = \langle V_3(s), \alpha(s) \rangle.$$

Theorem 2.4. *Let* $S_0^2 \subset L^3$ *be a sphere centered at* 0*. If* α *is a timelike curve on* S_0^2*, then the osculator sphere of* α *timelike curve at every point is* S_0^2*.*

Proof. Let α timelike curve with (I, α) neighbouring coordinate such that $s \in I$ is a parameter of arc. By Theorem 2.1,

$$\boldsymbol{a} = \alpha(s) - m_2(s)V_2(s) - m_3(s)V_3(s).$$

By Theorem 2.3, this expression can be written as

$$\boldsymbol{a} = \alpha(s) - \langle \alpha(s), V_2(s) \rangle V_2(s) - \langle \alpha(s), V_3(s) \rangle V_3(s).$$

On the other hand, we know that

$$\alpha(s) = \sum_{i=1}^{3} \varepsilon_i \langle \alpha(s), V_i(s) \rangle V_i(s),$$

$$\varepsilon_i = \begin{cases} -1, & i = 1 \\ 1, & i = 2, 3. \end{cases}$$

Substituting (2.4) in the last equation, then

$$\alpha(s) = \langle \alpha(s), V_2(s) \rangle V_2(s) + \langle \alpha(s), V_3(s) \rangle V_3(s).$$

Hence,

$$\boldsymbol{a} = \alpha(s) - \alpha(s) = 0.$$

On the other hand,

$$d(\alpha(s), 0) = 0.$$

This completes the proof of the theorem.

Theorem 2.5. *Let the timelike curve α in L^3 be given with neighbouring coordinate (I, α). The radius of the osculator sphere at the point $\alpha(s)$ for all $s \in I$ such that $m_3(s) \neq 0$, $k_2(s) \neq 0$ is constant if and only if the centers of the osculator sphere are the same.*

Proof. By Corollary 2.2, then

$$r^2(s) = m_2^2(s) + m_3^2(s).$$

By taking the derivative of this equation with respect to s, and considering Theorem 2.2 for this equation, then

$$k_2(s)\, m_2(s) + m_3'(s) = 0\,, \qquad (r \text{ constant}). \tag{2.9}$$

On the other hand, since

$$\boldsymbol{a}(s) = \alpha(s) - m_2(s)V_2(s) - m_3(s)\,V_3(s)$$

then

$$D_{\dot{\alpha}}\boldsymbol{a}(s) = \alpha'(s) - m_2'(s)\,V_2(s) - m_3'(s)\,V_3(s) - m_2(s)\,V_2'(s) - m_3(s)\,V_3'(s)\,.$$

Considering (1.1), (2.5) and Theorem 2.2 together in the equation yields

$$D_{\dot{\alpha}}\,\boldsymbol{a}(s) = -\big(m_3'(s) + k_2(s)m_2(s)\big)V_3(s). \tag{2.10}$$

Substituting (2.9) into (2.10), then $D_{\dot{\alpha}}\boldsymbol{a}(s) = 0$ is obtained. Hence, if $D_{\dot{\alpha}}\,\boldsymbol{a}(s) = 0$ for all $s \in I$, then $\boldsymbol{a}(s) = \text{constant}$.

Conversely, let $a(s)$ be constant for all $s \in I$. Considering the equation $\big\langle \alpha(s) - \boldsymbol{a}(s), \alpha(s) - \boldsymbol{a}(s)\big\rangle = r^2$, taking derivative of this equation with respect to s and if necessary calculations are made, then

$$-r(s)\frac{dr}{ds} = 0.$$

Here, either $r(s) = 0$ or $dr/ds = 0$. If $r(s) = 0$, then by Corollary 2.2, $m_2^2(s) = m_3^2(s) = 0$. But this contradicts the theorem. Therefore $dr/ds = 0$ thus $r(s) = \text{constant}$ for all $s \in I$.

Theorem 2.6. *Let α be a timelike curve in L^3 with (I, α) neighbouring coordinate and $k_2(s) \neq 0$, $m_3(s) \neq 0$ for all $s \in I$. Then, α is a spherical curve if and only if the centers of the osculator spheres at the point $\alpha(s)$ for all $s \in I$ are located at the same point.*

Proof. Let α be a timelike curve on S_b^2 which have the radius r and centered at any point b. By Theorem 2.3, the proof is clear.

Conversely, let the centers of the osculator curve be the point b in $\alpha(s) \in \alpha$ for all $s \in I$. Then by Theorem 2.5 all osculator spheres have the same radius r. Therefore

$$d(\alpha(s),\, b) = r$$

for all $s \in I$. This completes the proof of the theorem.

Theorem 2.7. *Let α timelike curve in L^3 be given with (I, α) neighbouring coordinate. If $m_3(s) \neq 0$, $k_2(s) \neq 0$ such that s is a parameter of arc, then α is a spherical curve if and only if*

$$m_3'(s) + m_2(s)k_2(s) = 0.$$

Proof. Let α timelike curve be a spherical curve. By Theorem 2.6, for all $s \in I$, the center $\boldsymbol{a}(s)$ of the osculator spheres are constant. Additionally, equation (2.9) yields

$$m_3'(s) + k_2(s)m_2(s) = 0\,.$$

Conversely, let $k_2(s)m_2(s) + m_2'(s) = 0$. By Theorem 2.4,

$$D_{\dot{\alpha}}\, \boldsymbol{a}(s) = 0$$

for all $s \in I$. Therefore $\boldsymbol{a}(s) = \text{constant}$. Thus, by Theorem 2.6, curve α is a spherical curve.

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