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Confidence Intervals Based on Corrected Likelihood Ratio Tests for Parallel Systems with Covariates

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Abstract. The problem of interval estimation for the regression parameter of the model of parallel systems with covariates is investigated. Intervals based on inverted likelihood ratio tests were considered. Two corrections to the likelihood ratio statistic were examined, the Bartlett correction and the mean and variance correction (Dicicco [4]). A simulation study is conducted to investigate the coverage probability and the symmetry of upper and lower error probabilities for various degrees of time censoring. It is found that the uncorrected intervals perform equally well as that of the corrected ones, except when the sample size is small and the censoring is heavy. In this case, intervals based on the mean and variance correction perform very well.

1. Introduction

Recently, much effort has been directed towards the study of small sample corrections of the likelihood ratio statistic. Dicicco [4] and Barndoff-Nielsen [2] proposed a Bartlett correction to the likelihood ratio statistic and a mean and variance correction to the signed squared root of the likelihood ratio statistic. These corrections improve the quality of the asymptotic approximations to the sampling distributions of the likelihood ratio statistic and its signed squared root.

Confidence intervals based on corrected likelihood ratio tests have been used very little in applications due to difficulties in obtaining the corrections factors needed to implement them (Doganaksoy [5]). Dicicco [4] have obtained an approximation to these factors that depend only on observed log-likelihood derivatives.

Doganaksoy and Schmee [6] conducted a simulation study to evaluate the performance of the corrected confidence intervals for complete and right censored samples for the smallest extreme value simple linear regression models. Their results show that the intervals obtained from the uncorrected likelihood ratio statistic usually perform as those obtained from the corrected likelihood ratio statistics. However, the mean and variance correction considerably improves the convergence to the nominal error probability, especially for very small samples. The Bartlett correction tends to give intervals with conservative error probability with small samples.

In this paper we shall investigate the finite sample performance of these corrections applied to the problem of interval estimation for the parameters of the regression model based on parallel systems with components having an exponential time to failure distribution. Through simulation, and under time censoring, the coverage probabilities of intervals based on the likelihood ratio statistics and their corrections will be investigated and compared.

2. The model of parallel systems with covariates

Assume that the system is composed of *m*-units connected in parallel, and that the failure time of each unit is distributed as Exponential with mean λ, and the life times of the units $(x_1, x_2, \cdots, x_m)$ are independently distributed. The failure time of the system is equal to the failure time of the longest-lasting component [11], that is $T = \max(x_1, x_2, \cdots, x_m)$, where T denotes the failure time of the system. Assume that the failure time of the system depends on some concommitant information (covariates)

$$Z = (1, z_1, \cdots, z_p)'$$

The dependence of the failure time on the covariates can be studied by incorporating the covariates in the model by setting $\lambda = \exp(\beta' Z)$, where β is a column vector of regression parameter (Lawless [10]), thus obtaining a regression model given by

$$u = \beta' Z + \varepsilon$$

where ε is an error term with density function given by

$$h(\varepsilon) = m \exp(\varepsilon) \exp(-\exp(\varepsilon)) \left[1 - \exp(-\exp(\varepsilon))\right]^{(m-1)}$$

Parallel systems are used in industry to increase the reliability of certain products (Bain [1]). In a medical setting, lungs and kidneys are examples of such parallel systems (Elandt-Johnson and Johnson [7]). Parallel systems are a special kind of nomination samples which consist of independently distributed maxima from subsamples with the same underlying distribution (Kvam and Samaniego [9]).

3. Confidence intervals

Intervals based on the likelihood ratio statistic depend on the asymptotic chi-squared distribution of

$$2\left(l(\hat{\theta}, \hat{\lambda}) - l(\theta, \tilde{\lambda})\right)$$

where θ is the parameter of interest, λ is the nuisance parameter, l is the log-likelihood function, $(\hat{\theta}, \hat{\lambda})$ is the joint maximum likelihood estimator of (θ, λ), $\tilde{\lambda}$ is the maximum likelihood estimator of λ for a fixed value of θ. Therefore the end points of the $100(1-\alpha)\%$ likelihood ratio based confidence intervals are given as the solutions of

$$l(\theta, \tilde{\lambda}) = l(\hat{\theta}, \hat{\lambda}) - \frac{1}{2} X^2_{(1, 1-\alpha)}$$

Moolgavkar and Venzon [12] provide an efficient algorithm to find these end points.

The lower and upper limits of intervals based on the mean and variance corrections to the signed squared roots of the likelihood ratio statistic are defined as the solutions θ_L, θ_U to the equations

$$l(\theta, \tilde{\lambda}) = l(\hat{\theta}, \hat{\lambda}) - \frac{1}{2}(\mu - s\, z_{(1-\alpha/2)})^2$$

$$l(\theta, \tilde{\lambda}) = l(\hat{\theta}, \hat{\lambda}) - \frac{1}{2}(\mu + s\, z_{(1-\alpha/2)})^2$$

where $\theta_L < \hat{\theta}$ and $\theta_u > \hat{\theta}$. In these equations, μ and s^2 denote the mean and the variance of the signed squared root of the likelihood ratio statistic respectively. Approximate formulae for μ and s^2 that depend only on observed log-likelihood derivatives are given in Dicicco [4]. These corrections are given in the appendix.

The lower and upper limits of intervals based on Bartlett correction to the likelihood ratio statistic are defined as the two solutions θ_L, θ_u to

$$l(\theta, \tilde{\lambda}) = l(\hat{\theta}, \hat{\lambda}) - \frac{b}{2} x^2_{(1, 1-\alpha)}$$

where b is the Bartlett correction factor given by $b = \mu^2 + s^2$, (Dicicco [4]).

4. Simulation study

The cases of parallel systems based on two and three components are considered, that is $m = 2$ and $m = 3$. The sample sizes are taken to be $20, 30, \cdots, 60$. The values of β_0 and β_1 are set initially equal to zero. The covariate values are taken on an equally spaced lattice $(0.0, 0.2, \cdots, 1.8)$. Complete and time censored samples were considered, where the proportions of censored cases are taken to be 0.1, 0.3 and 0.5 to span the light, moderate, and severe censorships. The nominal level of the confidence intervals is taken to be 0.05. There were 5000 samples for each simulation run.

The simulation program is written in FORTRAN in double precision. The maximum likelihood estimators were found using the Newton-Raphson method. The necessary and sufficient conditions for the existence of the maximum likelihood estimator are given in Hamada and Tse [8] and Burridge [3].

5. Results

The results are given in Tables 1, 2, 3 and 4. The tables provide values of the lower (L), upper (U) and total (T) error probabilities (multiplied by 10^4) of confidence intervals based on the likelihood ratio statistic (LR), the Bartlett corrected likelihood ratio statistic (BLR) and the mean and variance correction to the ratio statistic (SLR).

6. Findings and conclusions

Intervals based on the likelihood ratio statistic (LR) generally attain the nominal level, except when the censoring is heavy, in which case they tend to be anticonservative.

Intervals based on Bartlett corrected likelihood ratio statistic (BLR) attain the nominal level; however, for the slope parameter they tend to be anticonservative under moderate to heavy censoring.

Intervals based on mean and variance corrected likelihood ratio statistic (SLR) do attain the nominal level in all situations considered.

All intervals tend to be symmetric regardless of the censoring proportion. As the censoring proportion increases all intervals tend to have less coverage probability. LR intervals appear to perform as well as the corrected ones, and when the censoring is heavy and the sample size is small, LR intervals tend to be anticonservative, in which case SLR intervals give very accurate results.

In conclusion LR intervals perform satisfactorily and are preferred. When the sample size is small and heavily censored, the mean and variance correction appear to give better results.

Table 1. Simulated values of the error probabilities of confidence intervals for the intercept parameter when $m = 2$

	LR			BLR			SLR		
	L	U	T	L	U	T	L	U	T
				Complete samples					
20	0222	0252	0474	0216	0248	0464	0200	0280	0480
30	0242	0238	0480	0242	0238	0480	0220	0258	0478
40	0228	0266	0494	0228	0264	0492	0214	0284	0498
50	0262	0266	0528	0262	0266	0528	0244	0280	0524
60	0228	0320	0548	0226	0318	0544	0216	0342	0558
				$Cp = 0.1$					
20	0254	0260	0514	0258	0260	0518	0228	0286	0514
30	0280	0234	0514	0284	0238	0522	0252	0268	0520
40	0256	0280	0536	0258	0282	0540	0240	0298	0538
50	0258	0270	0528	0258	0270	0528	0240	0288	0528
60	0242	0324	0566	0244	0324	0568	0228	0340	0568
				$Cp = 0.3$					
20	0260	0280	0540	0250	0286	0536	0210	0306	0516
30	0268	0256	0524	0268	0260	0528	0236	0294	0530
40	0254	0286	0540	0254	0292	0546	0226	0322	0548
50	0258	0266	0524	0258	0268	0526	0242	0288	0530
60	0244	0316	0560	0244	0318	0562	0218	0346	0564
				$Cp = 0.5$					
20	0304	0272	0576	0258	0274	0532	0200	0294	0494
30	0270	0264	0534	0258	0272	0530	0192	0294	0486
40	0272	0316	0588	0266	0320	0586	0226	0342	0568
50	0266	0268	0534	0258	0268	0526	0224	0284	0508
60	0246	0304	0550	0244	0304	0548	0216	0334	0550

LR : likelihood ratio intervals
BLR : Bartlett corrected likelihood ratio intervals.
SLR : Mean and variance corrected likelihood ratio intervals
Cp : Censoring proportion.

Table 2. Simulated values of the error probabilities of confidence intervals for the slope parameter when $m = 2$

	LR			BLR			SLR		
	L	U	T	L	U	T	L	U	T
				Complete samples					
20	0258	0222	0480	0256	0220	0476	0256	0220	0476
30	0234	0246	0480	0230	0244	0474	0228	0242	0470
40	0236	0244	0480	0236	0242	0478	0236	0242	0478
50	0224	0288	0512	0222	0284	0506	0222	0284	0506
60	0258	0240	0498	0254	0240	0494	0254	0240	0494
				Cp = 0.1					
20	0276	0252	0528	0274	0254	0528	0272	0248	0520
30	0246	0284	0530	0284	0238	0536	0238	0282	0520
40	0252	0258	0510	0256	0258	0514	0254	0258	0512
50	0238	0308	0546	0238	0308	0546	0238	0308	0546
60	0284	0248	0532	0286	0248	0534	0284	0248	0532
				Cp = 0.3					
20	0298	0276	0574	0302	0272	0574	0286	0242	0528
30	0264	0286	0550	0266	0288	0554	0252	0274	0526
40	0298	0242	0540	0302	0246	0548	0294	0236	0530
50	0236	0276	0512	0240	0276	0516	0234	0270	0504
60	0316	0262	0578	0318	0264	0582	0314	0256	0570
				Cp = 0.5					
20	0312	0306	0618	0286	0294	0580	0246	0248	0494
30	0282	0294	0576	0276	0282	0558	0262	0256	0518
40	0298	0276	0574	0296	0272	0568	0272	0250	0522
50	0266	0258	0524	0264	0258	0522	0256	0244	0500
60	0308	0258	0566	0306	0252	0558	0294	0240	0534

LR : likelihood ratio intervals
BLR : Bartlett corrected likelihood ratio intervals.
SLR : Mean and variance corrected likelihood ratio intervals
Cp : Censoring proportion.

Table 3. Simulated values of the error probabilities of confidence intervals for the slope parameter when $m = 3$

	LR			BLR			SLR		
	L	U	T	L	U	T	L	U	T
				Complete samples					
20	0232	0240	0472	0230	0240	0470	0210	0258	0468
30	0242	0244	0486	0242	0244	0486	0224	0256	0480
40	0226	0254	0480	0226	0252	0478	0220	0276	0496
50	0268	0268	0536	0266	0266	0532	0258	0278	0536
60	0228	0304	0532	0228	0304	0532	0216	0324	0540
				$Cp = 0.1$					
20	0246	0252	0498	0250	0252	0502	0230	0268	0498
30	0266	0238	0504	0274	0238	0512	0256	0252	0508
40	0258	0260	0518	0264	0258	0522	0244	0278	0522
50	0252	0256	0508	0252	0256	0508	0248	0268	0516
60	0242	0306	0548	0242	0306	0548	0230	0324	0554
				$Cp = 0.3$					
20	0260	0268	0528	0256	0270	0526	0212	0284	0496
30	0272	0246	0518	0270	0248	0518	0236	0264	0500
40	0256	0284	0540	0254	0284	0538	0228	0304	0532
50	0254	0272	0526	0256	0278	0534	0236	0286	0522
60	0236	0328	0564	0238	0330	0568	0222	0344	0566
				$Cp = 0.5$					
20	0304	0264	0568	0264	0264	0528	0208	0284	0492
30	0270	0270	0540	0258	0270	0528	0214	0288	0502
40	0276	0308	0584	0270	0310	0580	0234	0318	0552
50	0268	0258	0526	0264	0258	0522	0230	0280	0510
60	0250	0302	0552	0248	0302	0550	0222	0320	0542

LR : likelihood ratio intervals
BLR : Bartlett corrected likelihood ratio intervals.
SLR : Mean and variance corrected likelihood ratio intervals
Cp : Censoring proportion.

Table 4. Simulated values of the error probabilities of confidence intervals for the slope parameter when $m = 3$

	LR			BLR			SLR		
	L	U	T	L	U	T	L	U	T
				Complete samples					
20	0246	0224	0470	0242	0222	0464	0242	0220	0462
30	0234	0248	0482	0234	0246	0480	0234	0246	0480
40	0246	0252	0498	0244	0248	0492	0244	0246	0490
50	0222	0288	0510	0222	0286	0508	0222	0286	0508
60	0270	0236	0506	0266	0234	0500	0268	0234	0502
				$Cp =$ 0.1					
20	0270	0236	0506	0278	0240	0518	0268	0234	0502
30	0238	0274	0512	0240	0274	0514	0234	0270	0504
40	0248	0266	0514	0248	0266	0514	0248	0264	0512
50	0228	0292	0520	0230	0292	0522	0228	0290	0518
60	0286	0238	0524	0288	0240	0528	0288	0240	0528
				$Cp = 0.3$					
20	0300	0268	0568	0300	0266	0566	0292	0242	0534
30	0252	0288	0540	0250	0286	0536	0242	0274	0516
40	0290	0256	0546	0290	0256	0546	0282	0246	0528
50	0244	0270	0514	0244	0272	0516	0238	0270	0508
60	0312	0268	0580	0312	0268	0580	0310	0260	0570
				$Cp = 0.5$					
20	0314	0314	0628	0276	0292	0568	0236	0264	0500
30	0280	0290	0570	0266	0276	0542	0256	0256	0512
40	0296	0264	0560	0290	0256	0546	0270	0244	0514
50	0266	0262	0528	0266	0260	0526	0250	0244	0494
60	0292	0248	0540	0292	0244	0536	0284	0230	0514

LR : likelihood ratio intervals
BLR : Bartlett corrected likelihood ratio intervals.
SLR : Mean and variance corrected likelihood ratio intervals
Cp : Censoring proportion.

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Keywords: Bartlett correction, confidence intervals, likelihood ratio tests, mean and variance correction, parallel systems.

Appendix

Computation of μ and s^2

Let the parameter of interest be θ_1, and denote the nuisance parameters by $\theta_2, \cdots, \theta_p$. The mean μ and the variance s^2 of signed squared root of the likelihood ratio statistic is given by (Dicicco [4])

$$\mu = \frac{1}{6} I_{abc} I^{al} \left(2 I^{bc} + J^{bc}\right) / \left(I^{ll}\right)^{(1/2)}$$

$$s^2 = 1 + \frac{1}{2} I_{abcd} \left(I^{ab} I^{cd} - J^{ab} J^{cd}\right) + (1/12)\, I_{abc}\, I_{def}$$

$$\left\{3\left(I^{ab} I^{cd} I^{ef} - J^{ab} J^{cd} J^{ef}\right) + 2\left(I^{ad} I^{be} I^{cf} - J^{ad} J^{be} J^{cf}\right)\right\} - \mu^2$$

where summation is taken over every index appearing as a subscript and superscript, and where

$$I_{ab} = \left[-\frac{\partial^2 l}{\partial\hat{\theta}_a \partial\hat{\theta}_b}\right], \qquad I_{abc} = \left[\frac{\partial^3 l}{\partial\hat{\theta}_a \partial\hat{\theta}_b \partial\hat{\theta}_c}\right]$$

$$I_{abcd} = \left[\frac{\partial^4 l}{\partial\hat{\theta}_a \partial\hat{\theta}_b \partial\hat{\theta}_c \partial\hat{\theta}_d}\right], \qquad a, b, c, d = 1, \cdots, p$$

$J^{ab} = I^{ab} - \left[I^{al} I^{bl} / I^{ll}\right]$, I^{ab} is the (a,b) element of I^{-1}

Higher order derivatives of the log-likelihood

Let $w_i = y_i - z'\beta$, the likelihood function is given by

$$l(\beta) = \sum_{i=1}^{n} \left\{\delta_i \log\left(g(w_i)\right) + (1-\delta_i) \log\left(1 - G(W_i)\right)\right\},$$

where G is the distribution function of u, and δ_i is the censoring indicator. The first order partial derivatives of the log-likelihood functions are given by

$$\frac{\partial l(\beta)}{\partial \beta_j} = -\sum_{i=1}^{n} \left\{ Z_{ji} \left(-\delta_i \frac{d}{dw_i} \log(g(w_i))\right) + (1-\delta_i)\, \lambda(w_i)\right\}, \qquad j = 0, 1$$

where $\lambda(w_i) = g(w_i)/(1-G(w_i))$. The second order partial derivatives are given by

$$\frac{\partial^2 l(\beta)}{\partial\beta_j \partial\beta_k} = -\sum_{i=1}^{n} \left\{ z_{ji} z_{ki} \left(-\delta_i \frac{d^2}{dw_i^2} \text{Log}(g(w_i) \right) \right.$$

$$\left. + (1-\delta_i) \left(\frac{d}{dw_i} \left\{ \text{Log}(g(w_i)) \right\} \lambda(w_i) + \lambda(w_i)^2 \right) \right\} \qquad j,k = 0,1,$$

the third order derivatives of the log-likelihood are given by

$$\frac{\partial^3 l(B)}{\partial\beta_j \partial\beta_k \partial\beta_l} = \sum_{i=1}^{n} \left\{ z_{ji} z_{ki} z_{li} (-\delta_i) \frac{d^3}{dw_i^3} \log(g(w_i)) + (1-\delta_i) \frac{d}{dw_i} k(w_i) \right\}.$$

$$j,k,l = 0,1,$$

and the fourth order derivatives are given by

$$\frac{\partial^4 l(B)}{\partial\beta_j \partial\beta_k \partial\beta_\ell \partial\beta_m} = -\sum_{i=1}^{n} \left\{ z_{ji} z_{ki} z_{li} z_{mi} \left(-\delta_i \frac{d^4}{dw_i^4} \log(g(w_i)) \right) + (1-\delta_i) \frac{d^2}{dw_i^2} k(w_i) \right\}$$

$$j,k,l,m = 0,1,$$

where

$$k(w_i) = \lambda(w_i) \frac{d}{dw_i} \log(g(w_i) + \lambda(w_i)^2)$$

$$\frac{d}{dw_i} k(w_i) = k(w_i) \frac{d}{dw_i} \log(g(w_i) + \lambda(w_i)) \left(\frac{d}{dw_i^2} \log(g(w_i)) \right) + 2k(w_i)\lambda(w_i)$$

$$+ 2k(w_i)\lambda(w_i)$$

$$\frac{d^2}{dw_i^2} k(w_i) = \frac{d}{dw_i} k(w_i) \frac{d}{dw_i} \log(g(w_i) + 2k(w_i)) \left(\frac{d}{dw_1^2} \log(g(w_i)) \right)$$

$$+ \frac{d^3}{dw_i^3} \log(g(w_i)) \lambda(w_i) + 2 \left\{ g(w_i)^2 + \lambda(w_i) \frac{d}{dw_i} k(w_i) \right\}$$

and with $v_1 = \exp(w_i)$, $v_2 = \exp(-\exp(w_i))$ we have

$$\frac{d}{dw_i}\log(g(w_i)) = 1 - v_1 + (m-1)\frac{v_1}{-1+v_2} ,$$

$$\frac{d^2}{dw_i^2}\log(g(w_i)) = \frac{v_1(v_2 - v_2^2 + v_1 v_2 - m + m\,v_2 - m\,v_1 v_2)}{(-1+v_2)^2} ,$$

$$\frac{d^3}{dw_i^3}\log(g(w_i)) = -v_1 + (m-1)\left\{\frac{v_1^2 - v_1(-1+v_1)^2}{1-v_2} + \frac{3v_1^2(-1+v_1)}{(-1+v_2)^2} + \frac{2v_1^3}{(-1+v_2)^3}\right\} ,$$

and

$$\frac{d^4}{dw_i^4}\log(g(w_i)) = -v_1 + (m-1)\left\{\frac{-6\,v_1^4}{(-1+v_2)^4} + \frac{12\,v_1^3(-1+v_1)}{(1-v_2)^3} + \frac{v_1(1-v_1)^3 + v_1^2(-2+v_1) + 2v_1^2(-1+v_1)}{(-1+v_2)} + \frac{4v_1^3 - 7\,v_1^2\,(-1+v_1)^2}{(-1+v_2)^2}\right\}$$