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On the Solution of the Nonlinear Wave Equation by the Decomposition Method

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Abstract. We present the Adomian decomposition method for a nonlinear wave equation subject to the initial conditions. Our objective is to obtain an analytic solution of the nonlinear parabolic equation which is calculated in the form of a series with easily computable components. The inhomogeneous equation is effectively solved by employing the phenomena of the self-canceling "noise" terms where sum of components vanishes in the limit. Comparing the methodology with some known techniques shows that the present approach is powerful and reliable. The decomposition method does not require unjustified assumptions, linearization , or perturbation.

1. Introduction

Current developments in mathematical physics, energy problems, and the other areas have given impetus to research on nonlinear partial differential equations and linearization techniques. Unfortunately, such techniques which assume essentially that a nonlinear system is "almost linear" often have little physical justification. It has become vital, not only to mathematics but to the areas of application, that further advances be made.

Recently, we have studied a first order nonlinear wave problem [1] differently by utilizing the Adomian decomposition method. The solution obtained by this method is derived in the form of a power series with easily computable components. The series used is a series of functions rather than terms as in the case in Taylor series. The method is very reliable and effective method that provide the solution in terms of rapid convergent series. This has been justified by [2]. The equation characterizing one-directional nonlinear wave motion is $u_x + uu_t = \phi(x,t)$, with initial condition $u(x,0) = f(x)$. In this paper, we will again make use of the Adomian decomposition method in order to obtain analytic solutions inhomogeneous form of nonlinear partial differential equation has the form

$$u_{xx} - uu_{tt} = \phi(x,t), \tag{1.1}$$

with initial conditions

$$u(0,t) = f(t), \quad u_x(0,t) = g(t). \tag{1.2}$$

The Adomian decomposition method [3-5] has now been applied to a wide class of stochastic operator equations, involving algebraic differential, integro-differential, differential-delay, and partial differential equations and systems. The solution of the simpler mathematical problem may not be a good approximation to the solution of the original problem. “weak” nonlinearity and “small” perturbations are common assumptions. However, nature is nonlinear and stochastic in general. The deterministic case in which randomness can be ignored. Similarly, linearity can be regarded as a limiting case, as can cases where perturbation theory is adequate. The decomposition solution is also a approximation, but one which does not change the problem. Therefore it is often physically more realistic. While the solution obtained by decomposition is generally an infinite series.

2. Derivation of the method

To apply the decomposition method, we write equation (1.1) in an operator form

$$L_x(u(x,t)) = \phi(x,t) + Nu \tag{2.1}$$

with nonlinear term $Nu = uu_{tt}$ and $L_x = \partial^2 / \partial x^2$ are the differential operators. It is clear that L_x^{-1} is the two-fold integration from 0 to *x*, i.e., $L_x^{-1} = \int_0^x (.)dx \int_0^x (.)dx$.

Applying the inverse operator L_x^{-1} to (2.1) yields

$$L_x^{-1} L_x \left(u(x,t)\right) = L_x^{-1}(\phi(x,t)) + L_x^{-1}(Nu)$$

from which it follows that

$$u(x,t) = f(t) + xg(t) + L_x^{-1}(\phi(x,t)) + L_x^{-1}(Nu)\,. \tag{2.2}$$

The decomposition method consist of decomposing the unknown function $u(x,t)$ into a sum of components defined by the decomposition series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{2.3}$$

and the nonlinear term $Nu = uu_{tt}$ be expressed in the A_n Adomian’s polynomials; thus $Nu = \sum_{n=0}^{\infty} A_n$ where the polynomials A_n of course are developed for the particular nonlinearity which are generated by [3-5]:

$$\begin{aligned} A_0 &= u_0 (\partial^2/\partial t^2) u_0 \\ A_1 &= u_0 (\partial^2/\partial t^2) u_1 + u_1 (\partial^2/\partial t^2) u_0 \\ A_2 &= u_0 (\partial^2/\partial t^2) u_2 + u_1 (\partial^2/\partial t^2) u_1 + u_2 (\partial^2/\partial t^2) u_0 \\ &\vdots \end{aligned} \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2) leads to the recursive relationship

$$\begin{aligned} u_0 &= f(t) + xg(t) + L_x^{-1}\phi(x,t), \\ u_1 &= L_x^{-1}(A_0), \\ u_2 &= L_x^{-1}(A_1), \\ &\vdots \\ u_{n+1} &= L_x^{-1}(A_n), n \geq 0. \end{aligned} \tag{2.5}$$

As a results the terms of the series u_0, u_1, u_2,… are identified and the exact solution may be entirely determined by using the approximation

$$u(x,t) = \lim_{n\to\infty} \Phi_n, \tag{2.6}$$

where $\Phi_n = \sum_{k=0}^{n-1} u_k$. It is useful to note that the recursive relationship is constructed on the basis that the zeroth component $u_0\,(x,t)$ is defined by all terms that arise from the initial condition and from integrating the source term. The remaining components $u_x(x,t),\ n \geq 1,$ can be completely determined such that each term is computed by using the previous term. Accordingly, considering few terms only, the relation Eq. (2.5) gives $u_0(x,t),\ u_1(x,t),\ u_2(x,t),\ \cdots$ and so on.

Adomian and Rach [6] and Wazwaz [7] have investigate the phenomena of the self-canceling "noise" terms where some of the terms of series vanishes in the limit. An important observation was made that "noise" terms appear for nonhomogenous cases only. Further, it was formally justified that if terms in u_0 are canceled by terms in u_1, even though u_1 includes further terms, then the remaining non-cancel terms in u_0 constitute the exact solution of the equation.

3. Examples

Example 1. Let us consider a nonhomogenous nonlinear wave equation. The equation of the form

$$u_{xx} - uu_{tt} = 2 - 2(t^2 + x^2), \tag{3.1}$$

the initial conditions posed are

$$\begin{aligned} u(x,0) &= x^2, \\ u(0,t) &= t^2, u_x(0,t) = 0. \end{aligned} \tag{3.2}$$

Using (2.5) to determine the individual terms of the decomposition, we find

$$u_0 = x^2 + t^2 - x^2t^2 - \frac{1}{6}x^4, \tag{3.3}$$

and

$$\begin{aligned} u_1 &= L_x^{-1}(A_0) \\ &= x^2t^2 + \frac{1}{6}x^4 - \frac{1}{3}x^4t^2 - \frac{7}{90}x^6 + \frac{2}{15}x^6t^2 + \frac{1}{16}x^8, \end{aligned} \tag{3.4}$$

$$\begin{aligned} u_2 &= L_t^{-1}(A_1) \\ &= \frac{1}{3}x^4t^2 + \frac{7}{90}x^6 - \frac{2}{15}x^6t^2 - \frac{1}{16}x^8 - \cdots, \end{aligned} \tag{3.5}$$

and so on for other components. It can be easily observed that the self-canceling "noise" terms appear between various components. Canceling the third term in u_0 and the first term in u_1, the fourth term in u_0 and the first term in u_1, in keeping the non canceled terms in u_0 yields the exact solution of (3.1) given by

$$u(x, t) = x^2 + t^2. \tag{3.6}$$

This can be verified through substitution.

Example 2. Next we will solve the equation of the form

$$u_{xx} - uu_{tt} = -xe^{-2t}, \tag{3.7}$$

with initial conditions

$$u(0,t) = 0, \text{ and } u_x(0, t) = -e^{-t}. \tag{3.8}$$

To determine the components of the decomposition series of the solution $u(x,t)$ of (2.3), we again use (2.5), hence we yield

$$u_0 = xe^{-t} - \frac{x^4}{4!}e^{-2t}, \tag{3.9}$$

and

$$\begin{aligned} u_1 &= L_x^{-1}(A_0) \\ &= \frac{x^4}{4!}e^{-2t} - 5\frac{x^7}{4!42}e^{-3t} + 4\frac{x^{10}}{(4!)^2 90}e^{-4t}, \end{aligned} \tag{3.10}$$

It is obvious that the self-canceling “noise” terms appear between various components. Canceling the second term in u_0 and the first terms in u_1, keeping the non noise terms in u_0 yields the exact solution of (3.7) given by

$$u(x,\ t) = xe^{-t} \tag{3.11}$$

which is easily verified.

It is worth noting that other noise terms between other components of $u(x,t)$ will be canceled, as the second term of the u_1 will be canceled by other noise terms of the other components u_i, $i \geq 2$. This examples confirm our belief that a fast convergent solution may be achieved by observing the self canceling “noise” terms since the sum of these “noise” terms will vanish in the limit.

In closing, the methods avoids the difficulties and massive computational work by determining the analytic solution. The solution is very rapidly convergent by utilizing the Adomian’s decomposition method. The convergence can be made faster if the noise terms appear as discussed in [3-5]. The decomposition method provides a reliable technique that requires less work if compared with the traditional techniques and the method does not also require unjustified assumptions, linearization, or perturbation.

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