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Cyclic Subgroup Separability of Certain HNN Extensions

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Abstract. In this note, we shall show that certain HNN extensions of cyclic subgroup separable groups and certain HNN extensions of subgroup separable groups are cyclic subgroup separable. We also show that one-relator groups with non-trivial centre are cyclic subgroup separable.

1. Introduction

A group *G* is called cyclic subgroup separable if for each pair of elements $x, y \in G$ such that $x \notin \langle y \rangle$, there exists a normal subgroup *N* of finite index in *G* such that $x \notin \langle y \rangle N$. Clearly a cyclic subgroup separable group is residually finite. The concept of cyclic subgroup separability was introduced by Stebe [13] in 1968 and he used it to prove the residual finiteness of a class of knot groups. More recently, Kim [5] and Kim and Tang [6] gave characterisations for HNN extensions of cyclic subgroup separable groups with cyclic associated subgroups to be again cyclic subgroup separable. They then applied their results to show that HNN extensions of nilpotent groups with cyclic associated subgroups are cyclic subgroup separable.

In this note, we shall show that certain HNN extensions of cyclic subgroup separable groups and certain HNN extensions of subgroup separable groups with finitely generated associated subgroups are cyclic subgroup separable. Our results are then applied to HNN extensions of polycyclic-by-finite groups and free-by-finite groups. Finally we also show that one-relator groups with non-trivial centre are cyclic subgroup separable.

The notation used here is standard. In addition the following will be used for any group *G*.

$N \lhd_f G$ means *N* is a normal subgroup of finite index in *G*.

$Z(G)$ shall denote the centre of *G*.

If *G* is an HNN extension and $x \in G$, then $\| x \|$ shall denote the usual reduced length of *x*.

2. Definition and a criterion

First we give the definition and a criterion (i.e. Theorem 1) for an HNN extension of a cyclic subgroup separable group to be again cyclic subgroup separable

Definition 1. *Let H be a subgroup of a group G. Then G is said to be H-separable if for each element* $x \in G \setminus H$, *there exists* $N \lhd_f G$ *such that* $x \notin HN$.

G is said to be subgroup separable if G is H-separable for every finitely generated subgroup H.

G is cyclic subgroup separable (π_c *for short) if G is H-separable for every cyclic subgroup H.*

For ease of exposition, we shall use the term π_c instead of cyclic subgroup separable from now on.

The following result can be easily derived from Theorem 2.2 of Kim [5].

Theorem 1. *Let* $G = \langle B, t; t^{-1}Ht = K, \phi \rangle$ *be an HNN extension where B is a* π_c *group. Suppose that*

(a) *B is H-separable and K-separable;*
(b) *for each* $M \lhd_f B$, *there exists* $N \lhd_f B$ *such that* $N \subseteq M$ *and* $\phi(N \cap H) = N \cap K$.

Then G is π_c.

3. HNN extensions of subgroup separable groups

Now we use Theorem 1 to show that certain HNN extensions of subgroup separable groups are π_c.

Lemma 1. *Let B be a subgroup separable group and H, K be finitely generated subgroups of Z(B) such that* $H \cap K = 1$. *Then for* $R \lhd_f H$, $S \lhd_f K$, *there exists* $N \lhd_f B$ *such that* $N \cap H = R$ *and* $N \cap K = S$.

Proof. Let $R \lhd_f H$, $S \lhd_f K$ be given. Since $R, S \subseteq Z(B)$, we can form $\overline{B} = B / RS$. Then $\overline{B}$ is residually finite since B is subgroup separable and R and S are finitely generated. Furthermore, the subgroups $\overline{H}$ and $\overline{K}$ are finite in $\overline{B}$. Hence there exists $\overline{N} \lhd_f \overline{B}$ such that $\overline{N} \cap \overline{H} = \overline{1}$ and $\overline{N} \cap \overline{K} = \overline{1}$. Let N be the preimage of $\overline{N}$ in B. Then $N \lhd_f B$ is such that $N \cap H = R$ and $N \cap K = S$.

Theorem 2. *Let* $G = \langle B,\ t;\ t^{-1}Ht = K, \phi \rangle$ *be an HNN extension where B is a subgroup separable group. Suppose that H, K are finitely generated subgroups of* $Z(B)$ *such that* $H \cap K = 1$. *Then G is* π_c.

Proof. Since B is a subgroup separable group and H, K are finitely generated, B is H-separable and K-separable. This proves Theorem 1(a). Let $M \lhd_f B$ be given. Now let $R = M \cap H$ and $S = M \cap K$. Then $R \lhd_f H$ and $S \lhd_f K$. Let $R_1 = R \cap \phi^{-1}(S)$ and $S_1 = S \cap \phi(R)$. Then $R_1 \lhd_f H$, $S_1 \lhd_f K$ and $\phi(R_1) = S_1$. By Lemma 1, there exists $P \lhd_f B$ such that $P \cap H = R_1$ and $P \cap K = S_1$. Let $N = M \cap P$. Then $N \lhd_f B$ and $N \cap H = R_1, N \cap K = S_1$. Since $\phi(R_1) = S_1$, it follows that $\phi(N \cap H) = N \cap K$. This proves Theorem 1(b). Therefore G is π_c.

Lemma 2. *Let B be a subgroup separable group and H be a finitely generated subgroup of B. Let R be a normal subgroup of B and R is of finite index in H. Then there exists* $N \lhd_f B$ *such that* $N \cap H = R$.

Proof. Let $R \lhd B$ where R is of finite index in H be given. Since R is normal in B, we can form $\overline{B} = B / R$. Then $\overline{B}$ is residually finite since B is subgroup separable and R is finitely generated. Furthermore, the subgroup $\overline{H}$ is finite in $\overline{B}$. Hence there exists $\overline{N} \lhd_f \overline{B}$ such that $\overline{N} \cap \overline{H} = \overline{1}$. Let N be the preimage of $\overline{N}$ in B. Then $N \cap H = R$.

Theorem 3. *Let* $G = \langle B, t; t^{-1}Ht = K, \phi \rangle$ *be an HNN extension where B is a subgroup separable group and* $\phi(H \cap K) = H \cap K$. *Suppose that H, K are finitely generated normal subgroups of B such that* $H \cap K \lhd_f H$ *and* $H \cap K \lhd_f K$. *Then G is* π_c.

Proof. Since B is a subgroup separable group and H, K are finitely generated, B is H-separable and K-separable. This proves Theorem 1(a). Let $M \lhd_f B$ be given. Now let $R = M \cap H \cap K$. Then $R \lhd_f H \cap K$. Suppose R is of index r in $H \cap K$. Since $H \cap K$ is finitely generated, there exist only a finite number of subgroups of index r in $H \cap K$. Let R_1 be the intersection of all these subgroups of index r in $H \cap K$. Then $R_1 \subset R$ and R_1 is characteristic and of finite index in $H \cap K$. Since $\phi(H \cap K) = H \cap K$, it follows that $\phi(R_1) = R_1$. Since $R_1 \lhd_f H$ and $R_1 \lhd_f K$, then by Lemma 2, there exist $P_1 \lhd_f B$, $P_2 \lhd_f B$ such that $P_1 \cap H = R_1 = P_2 \cap K$. Let $N = M \cap P_1 \cap P_2$. Then $N \lhd_f B$ and $N \cap H = R_1 = N \cap K$. Since $\phi(R_1) = R_1$, it follows that $\phi(N \cap H) = N \cap K$. This proves Theorem 1(b). Therefore G is π_c.

An easy consequence of Theorem 3 is as follows:

Corollary 1. *Let* $G = \langle B, t; t^{-1} Ht = H, \phi \rangle$ *be an HNN extension where B is a subgroup separable group. Suppose that H is a finitely generated normal subgroup of B. Then G is* π_c.

Proof. Let $H = K$ in Theorem 3 and we are done.

It is well known that polycyclic groups and free groups are subgroup separable (Mal'cev [7], Hall [2]) and finite extensions of subgroup separable groups are again subgroup separable (Romanovski [10], Scott [11]). Hence polycyclic-by-finite groups and free-by-finite groups are subgroup separable. Therefore from Theorems 2, 3 and Corollary 1, we have the following:

Corollary 2. *Let* $G = \langle B, t; t^{-1} Ht = K, \phi \rangle$ *be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group. Suppose that H, K are finitely generated subgroups of B and*

(a) $H, K \subseteq Z(B)$ *such that* $H \cap K = 1$ *or*
(b) $H \lhd B$ *and* $H = K$ *or*
(c) $H \lhd B$, $K \lhd B$ *such that* $H \cap K \lhd_f H$, $H \cap K \lhd_f K$ *and* $\phi(H \cap K) = H \cap K$.

Then G is π_c.

Corollary 3. *Let* $G = \langle B, t; t^{-1}Ht = K, \phi \rangle$ *be an HNN extension where B is a finitely generated abelian group. Suppose that*

(a) $H \cap K = 1$ *or*
(b) $H = K$ *or*
(c) $H \cap K \lhd_f H$, $H \cap K \lhd_f K$ *and* $\phi(H \cap K) = H \cap K$.

Then G is π_c.

4. HNN extensions of π_c groups

In this section, we show that certain HNN extensions of π_c groups with identical associated subgroups are again π_c.

Lemma 3. *Let H be a retract of a group B. If B is residually finite then B is H-separable.*

Proof. Lemma 2.6 of Kim [3].

The following result can be easily obtained.

Lemma 4. *Let H be a retract of a group B. Then for each* $R \lhd_f H$, *there exists* $N \lhd_f B$ *such that* $N \cap H = R$.

Theorem 4. *Let* $G = \langle B, t;\ t^{-1}Ht = H, \phi \rangle$ *be an HNN extension where B is a* π_c *group. Suppose that H is a finitely generated subgroup of B and H is a retract of B. Then G is* π_c.

Proof. By Lemma 3, B is H-separable. This proves Theorem 1(a). Let $M \lhd_f B$ be given. Let $R = M \cap H$. Then $R \lhd_f H$. As in the proof of Theorem 3, we can find $R_1 \subset R$ such that R_1 is characteristic and of finite index in H. Since $\phi(H) = H$, it follows that $\phi(R_1) = R_1$. By Lemma 4, we can find $P \lhd_f B$ such that $P \cap H = R_1$. Let $N = M \cap P$. Then $N \lhd_f B$ and $N \cap H = R_1$. Since $\phi(R_1) = R_1$, it follows that $\phi(N \cap H) = N \cap H$. This proves Theorem 1(b). Therefore G is π_c.

Theorem 5. *Let* $G = \langle B, t;\ t^{-1}Ht = H, \phi \rangle$ *be an HNN extension where B is a* π_c *group. Suppose that H is a finitely generated normal subgroup of finite index in B. Then G is* π_c.

Proof. Since $H \lhd_f B$, B is H-separable. This proves Theorem 1(a). Let $M \lhd_f B$ be given. Let $R = M \cap H$. Then $R \lhd_f H$. As in the proof of Theorem 3, we can find $R_1 \subset R$ such that R_1 is characteristic and of finite index in H. Since $\phi(H) = H$, it follows that $\phi(R_1) = R_1$. Let $N = R_1$. Then $N \lhd_f B$ and $N \cap H = R_1$. Since $\phi(R_1) = R_1$, it follows that $\phi(N \cap H) = N \cap H$. This proves Theorem 1(b). Therefore G is π_c.

Apply Theorems 4 and 5 to polycyclic-by-finite groups and free-by-finite groups, we have the following:

Corollary 4. *Let* $G = \langle B, t;\ t^{-1}Ht = H, \phi \rangle$ *be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group. Suppose that H is a finitely generated subgroup of B and*

(a) $H \lhd_f B$ *or*

(b) H is a retract of B.

Then G is π_c.

Corollary 5. *Let* $G = \langle B, t; t^{-1}Ht = H, \phi \rangle$ *be an HNN extension where B is a finitely generated abelian group. Suppose that H is a subgroup of finite index in B. Then G is* π_c.

Theorem 6. *Let* $G = \langle B, t; t^{-1}Ht = H, \phi \rangle$ *be an HNN extension where B is a* π_c *group. Suppose that* $\phi(a) = a$ *for all* $a \in H$ *(or* $\phi(a) = a^{-1}$ *for all* $a \in H$ *if H is abelian). Then G is* π_c *if and only if B is H-separable.*

Proof. First suppose *G* is π_c. Then *G* is residually finite and hence *G* is *H*-separable from Theorem 1 of Shirvani [12]. Conversely, if *B* is *H*-separable, then *G* is π_c by Theorem 1.

Next we apply Theorem 6 to get a characterisation for the one-relator groups $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta} \rangle$ to be π_c. We shall need the following lemma.

Lemma 5. *The group* $G = \langle h, t; t^{-1}ht = h^{\delta} \rangle$, $|\delta| \neq 1$, *is not* π_c.

Proof. Theorem 4 of [16].

Corollary 6. *The group* $G = \langle h, t; t^{-1} h^{\gamma} t = h^{\delta} \rangle$ *is* π_c *if and only if* $\gamma = \pm\delta$.

Proof. First suppose that *G* is π_c. Since *G* is residually finite, then by Meskin [8], $|\gamma| = 1$ or $|\delta| = 1$ or $|\gamma| = |\delta|$. Hence by Lemma 5, $|\gamma| = |\delta|$. Conversely, suppose that $|\gamma| = |\delta|$. Then *G* is π_c by Theorem 6.

Note that the characterisation of the group *G* in Corollary 6 was first obtained by Stebe [13].

5. One-relator groups with non-trivial centre

Next we apply Theorem 1 to show that one-relator groups with non-trivial centre are π_c. First we have the following definition and theorem.

Definition 2. [14] *A group G is said to be weakly potent if for each element x of infinite order in G, we can find a positive integer r such that for each positive integer n, there exists* $N \triangleleft_f G$ *such that xN has order exactly rn in G/N.*

Theorem 7. *Let* $G = \langle B, t; t^{-1}ht = k \rangle$ *be an HNN extension where B is a* π_c *and weakly potent group and* $\langle h \rangle, \langle k \rangle$ *are infinite cyclic subgroups such that* $\langle h \rangle \cap \langle k \rangle \neq 1$. *Then G is* π_c *if and only if* $h^{\delta} = k^{\pm\delta}$ *for some* $\delta > 0$.

Proof. Suppose $h^{\delta} = k^{\pm\delta}$ for some $\delta > 0$. Since B is π_c, then B is $\langle h \rangle$-separable and $\langle k \rangle$-separable. This proves Theorem 1(a). Let $M \lhd_f B$ be given. Suppose $M \cap \langle h \rangle = \langle h^s \rangle$ and $M \cap \langle k \rangle = \langle k^t \rangle$ for some integers *s, t*. Since *B* is weakly potent, we can find $N \lhd_f B$ such that $N \subseteq M$ and $N \cap \langle h \rangle = \langle h^{rst\delta} \rangle$, $N \cap \langle k \rangle = \langle k^{rst\delta} \rangle$ for some positive integer *r*. Since $h^{\delta} = k^{\pm\delta}$, it follows that $t^{-1}(N \cap \langle h \rangle)t = N \cap \langle k \rangle$. This proves Theorem 1(b). Therefore *G* is π_c.

Conversely, suppose *G* is π_c. Since $\langle h \rangle \cap \langle k \rangle \neq 1$, then $h^{\delta} = k^{\gamma}$ for some integer $\gamma > 0$. Since the subgroup $\langle h, t; t^{-1}h^{\gamma}t = h^{\delta} \rangle$ of *G* is π_c, then by Corollary 6, $\gamma = \pm\delta$.

Note that Theorem 7 can also be obtained directly from Lemma 2.5 and Theorem 2.9 of Kim and Tang [6].

Since polycyclic-by-finite groups and free-by-finite groups are subgroup separable (see above) and weakly potent (Wehrfritz [15], Kim [4]) we obtain, from Theorem 7, the following:

Corollary 7. *Let* $G = \langle B, t; t^{-1}ht = k \rangle$ *be an HNN extension where B is a polycyclic-by-finite group or free-by-finite group and* $\langle h \rangle, \langle k \rangle$ *are infinite cyclic subgroups such that* $\langle h \rangle \cap \langle k \rangle \neq 1$. *Then G is* π_c *if and only if* $h^{\delta} = k^{\pm\delta}$ *for some* $\delta > 0$.

Although it is known that any one-relator group with non-trivial centre is subgroup separable (Baumslag and Taylor [1]), here we apply Theorem 7 to show that any one-relator group with non-trivial centre is π_c.

Theorem 8. *Let G be a one-relator group with non-trivial centre. Then G is* π_c.

Proof. First suppose that the abelianisation of *G* is not free abelian of rank two. Then by Pietrowski [9, Theorem 1], *G* has a presentation of the form

$$\langle a_1, a_2, \cdots, a_m;\ a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \cdots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$$

where *m*, $p_i, q_i \geq 2$ and $(p_i, q_i) = 1$ for $i > j$. Clearly *G* is a tree product of infinite cyclic groups and hence *G* is π_c by Corollary 3 of [18].

Now suppose that the abelianisation of G is free abelian of rank two. Again by Pietrowski [9, Theorem 3], G has a presentation of the form

$$\left\langle t, a_1, a_2, \cdots, a_m;\ t^{-1}a_1t = a_1^{p_1} = a_2^{q_1}, a_2^{p_2} = a_3^{q_2}, \cdots, a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \right\rangle$$

where m, p_i, $q_i \geq 2$ and $(p_i, q_j) = 1$ for $i > j$ such that $p_1\ p_2 \cdots p_{m-1} = q_1\ q_2 \cdots q_{m-1}$. Then $G = \langle B, t; t^{-1}\ a_1t = a_m \rangle$ is an HNN extension where $B = \langle a_1, a_2, \cdots, a_m;\ a_1^{p_1} = a_2^{q_1} = a_2^{q_1},\ a_2^{p_2} = a_3^{q_2}, \cdots,\ a_{m-1}^{p_{m-1}} = a_m^{q_{m-1}} \rangle$ and $a_1^{\delta} = a_m^{\delta}$ where $\delta = p_1 p_2 \cdots p_{m-1} = q_1 q_2 \cdots q_{m-1}$. Now B is weakly potent by Corollary 3 of [17] and B is π_c by Corollary 3 of [18]. Therefore G is π_c by Theorem 7.

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