

Non-Existence of Proper Semi-Invariant Submanifolds of a Lorentzian Para-Sasakian Manifold

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Abstract. The main objective of the paper is to prove that a Lorentzian para-Sasakian manifold does not admit any proper semi-invariant submanifold.

1. Introduction

Recently, in [1], semi-invariant submanifolds of a Sasakian manifold have been defined and studied. It has been shown that a Sasakian manifold always admits a proper semi-invariant submanifold [1]. In [2] Matsumoto introduced the idea of Lorentzian para-contact structure and studied its several properties. Subsequently, in [3] semi-invariant submanifolds of a Lorentzian para-Sasakian (LP-Sasakian) manifold have been studied. In the present paper we prove that there does not exist any proper semi-invariant submanifold of an LP-Sasakian manifold and also such a manifold do not admit proper mixed foliated semi-invariant submanifold. Some interesting results concerning integrability of the distributions which arise naturally on the semi-invariant submanifold have also been established.

2. Preliminaries

Let \bar{M} be an n -dimensional real differentiable manifold of differentiability class C^∞ endowed with a C^∞ -vector valued linear function φ , a C^∞ vector field ξ , 1-form η and Lorentzian metric g of type $(0, 2)$ such that for each point $p \in \bar{M}$, the tensor $g_p : T_p\bar{M} \times T_p\bar{M} \rightarrow R$ is a non-degenerate inner product of signature $(-, +, +, \dots, +)$, where $T_p\bar{M}$ denotes the tangent vector space of \bar{M} at p and R is the real number space, which satisfies

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad (2.1)$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y) \quad (2.2)$$

$$g(X, \xi) = \eta(X) \quad (2.3)$$

for all vector field X, Y tangent to \bar{M} . Such a structure (φ, η, ξ, g) is termed as Lorentzian para-contact structure [2].

Also in a Lorentzian para-contact structure the following relations hold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \text{rank}(\varphi) = n - 1.$$

A Lorentzian para-contact manifold \bar{M} is called Lorentzian para-Sasakian (LP-Sasakian) manifold if [2]

$$(\bar{\nabla}_X \varphi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.4)$$

and from (2.3), we find

$$\bar{\nabla}_X \xi = \varphi X \quad (2.5)$$

for all X, Y tangent to \bar{M} , where $\bar{\nabla}$ is the Riemannian connection with respect to g .

Again if we put

$$\Phi(X, Y) = g(X, \varphi Y), \quad (2.6)$$

then $\Phi(X, Y)$ is a symmetric (0,2) tensor field ([2]), that is,

$$\Phi(X, Y) = \Phi(Y, X). \quad (2.7)$$

A submanifold M of an LP-Sasakian manifold \bar{M} is said to be semi-invariant submanifold if the following conditions are satisfied

- (i) $TM = D \oplus D^\perp \oplus \{\xi\}$, where D, D^\perp are orthogonal differentiable distributions on M and $\{\xi\}$ is the 1-dimensional distribution spanned by ξ ,
- (ii) The distribution D is invariant by φ , that is, $\varphi D_x = D_x$ for each $x \in M$,
- (iii) The distribution D^\perp is anti-invariant under φ , that is, $\varphi D^\perp \subset T_x M^\perp$ for each $x \in M$.

If both the distribution D and D^\perp are non-zero then the semi-invariant submanifold is called a proper semi-invariant submanifold. For any vector bundle H on M [resp., \bar{M}], we denote by $\Gamma(H)$ the module of all differentiable section of H on a neighbourhood co-ordinate on M [resp., \bar{M}].

The equations of Gauss and Weingarten of the immersion of M in \bar{M} are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.8)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N \quad (2.9)$$

for any $X, Y \in \Gamma(TM)$ and $N \in TM^\perp$, where ∇ is the Levi-Civita connection on M , ∇^\perp is the linear connection induced by $\bar{\nabla}$ on the normal bundle TM^\perp , h is the second fundamental form of M and A_N is the fundamental tensor of Weingarten with respect to the normal section N . From (2.8) and (2.9), it follows that

$$g(h(X, Y), N) = g(A_N X, Y) \quad (2.10)$$

for any $X, Y \in \Gamma(TM), N \in \Gamma(TM^\perp)$.

We denote by the same symbol g both metrics on \bar{M} and M .

3. Non-existence of proper semi-invariant submanifolds

We first prove a lemma.

Lemma 3.1. *On an LP-Sasakian manifold \bar{M} , the distribution T determined by η is involutive.*

Proof. Let $X, Y \in \Gamma(T)$. Here $\eta(X) = 0, \eta(Y) = 0$ and consequently in view of (2.3), (2.5), (2.6) and (2.7), it follows that $\eta([X, Y]) = 0$, which completes the proof of the lemma.

The above lemma provides the proof of the following:

Theorem 3.1. *The distribution $D \oplus D^\perp$ of a semi-invariant submanifold of an LP-Sasakian manifold is always integrable.*

Now we prove the main theorem of the paper.

Theorem 3.2. For a semi-invariant submanifold M of an LP-Sasakian manifold \bar{M} , $\dim D^\perp = 0$. Consequently, an LP-Sasakian manifold \bar{M} does not admit any proper semi-invariant submanifold.

To prove the theorem we state the following:

Lemma 3.2. [3]. Let M be a semi-invariant submanifold of an LP-Sasakian manifold \bar{M} . Then we have $A_{\phi X}Y + A_{\phi Y}X = 0$, for all $X, Y \in \Gamma(D^\perp)$.

Proof of the Main Theorem. Let $X, Y \in \Gamma(D^\perp)$. Hence $\phi X, \phi Y \in \Gamma(TM^\perp)$. In view of (2.3), (2.9) and (2.5), we get

$$\eta(A_{\phi Y}X) = -g(\bar{\nabla}_X(\phi Y), \xi) = g(\phi Y, \bar{\nabla}_X \xi) = g(\phi Y, \phi X) = g(X, Y); \quad X, Y \in \Gamma(D^\perp).$$

Interchanging X and Y in this equation and then adding both the equations, in view of Lemma 3.2, we have

$$2g(X, Y) = \eta(A_{\phi X}Y + A_{\phi Y}X) = 0; \quad \text{for all } X, Y \in \Gamma(D^\perp).$$

Hence $\dim D^\perp = 0$. This completes the proof of the theorem.

Theorem 3.1 and Theorem 3.2 lead to

Theorem 3.3. If M is a semi-invariant submanifold of an LP-Sasakian manifold, then

- (i) the distribution D is integrable;
- (ii) $\nabla_X(\phi Y) - \nabla_Y(\phi X) = \phi([X, Y])$;
- (iii) $h(X, \phi Y) = h(\phi X, Y)$; $X, Y \in \Gamma(D)$.

Proof. Since in view of Theorem 3.2, $\dim D^\perp = 0$; taking into account of Theorem 3.1, the distribution D is integrable. From (2.4) we get

$$(\bar{\nabla}_X \phi)(Y) - (\bar{\nabla}_Y \phi)(X) = 0; \quad X, Y \in \Gamma(D). \quad (3.1)$$

Using (2.8) in (3.1), we have

$$0 = (\bar{\nabla}_X \varphi)(Y) - (\bar{\nabla}_Y \varphi)(X) = \nabla_X(\varphi Y) - \nabla_Y(\varphi X) - \varphi([X, Y]) \\ + h(X, \varphi Y) - h(\varphi X, Y); \quad X, Y \in \Gamma(D),$$

which on equating tangential and normal parts, yeilds (ii) and (iii) respectively.

4. Non-existence of proper mixed foliated semi-invariant submanifolds

In [4], a semi-invariant submanifold is said to be foliated if $D \oplus \{\xi\}$ is integrable and $h(Z + \xi, X) = 0$ for all $Z \in D$ and $X \in D^\perp$.

Here we prove

Theorem 4.1. *LP-Sasakian manifolds do not admit proper mixed foliated semi-invariant submanifolds.*

Proof. From (2.5) and (2.8) we have

$$\varphi X = \nabla_X \xi + h(X, \xi), \quad X \in TM.$$

If $X \in D^\perp$ then $\nabla_X \xi = 0$ and $h(X, \xi) = \varphi X$. Moreover, if M is mixed foliated then the above equation yields $\varphi X = 0$, $X \in D^\perp$. Thus $D^\perp = \{0\}$ and M cannot be a proper mixed foliated semi-invariant submanifold.

References

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