BULLETIN of the
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) 23 (2000) 99-106

CR-Submanifolds of a Lorentzian Para-Sasakian Manifold

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Abstract. The purpose of the present paper is to study CR-submanifolds of a Lorentzian para-Sasakian manifold and obtain their basic properties. The integrability conditions of the distributions on CR-submanifold are obtained. D-parallel normal section as well as totally umbilic CR-submanifolds have been studied.

1. Introduction

In 1978, Bejancu introduced the notion of CR-submanifold of a Kaehler manifold [1]. Since then several papers on CR-submanifolds of Sasakian manifolds have been studied by Kobayashi [2], J.S. Pak [3], Yano and Kon [4] and others. On the other hand, Matsumoto [5] introduced the idea of Lorentzian paracontact structure and studied its several properties. Bhagwat Prasad [6], S. Prasad and Ram Hit Ojha [7] studied submanifold of Lorentzian para-Sasakian manifolds. Semi-invariant and semi-invariant product submanifolds of a Lorentzian para-Sasakian manifold have been studied by Kalpana and Guha [8]. In the present paper we study CR-submanifolds and CR-structure of a CR-submanifold of Lorentzian para-Sasakian manifolds. CR-submanifolds have good interaction with other parts of mathematics and substantial applications to (Pseudo) - conformal mapping and relativity [9], [10].

2. Preliminaries

Let $\overline{M}$ be an n-dimensional real differentiable manifold of differentiability class C^∞-endowed with a C^∞-vector valued linear function φ, a C^∞-vector field ξ, 1-form η and Lorentzian metric g of type $(0,2)$ such that for each point $p \in \overline{M}$, the tensor $g_p : T_p\overline{M} \times T_p\overline{M} \to R$ is a non-degenerate inner product of signature $(-,+,+,\cdots,+)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and R is the real number space, which satisfies

$$\varphi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \xi) = \eta(X) \tag{2.3}$$

for all vector fields X, Y tangent to $\overline{M}$. Such a structure (φ, η, ξ, g) is termed as Lorentzian para-contact structure [5].

Also in a Lorentzian para-contact structure the following relations hold:

$$\varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \text{rank}\,(\varphi) = n - 1,$$

A Lorentzian para-contact manifold $\overline{M}$ is called Lorentzian para-Sasakian (LP-Sasakian manifold if [5].

$$(\overline{\nabla}_X \varphi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.4}$$

and from (2.3), we find

$$\overline{\nabla}_X \xi = \varphi X \tag{2.5}$$

for all X, Y tangent to $\overline{M}$, where $\overline{\nabla}$ is the Riemannian connection with respect to g.

Again if we put

$$\phi(X, Y) = g(X, \varphi Y) \tag{2.6}$$

then $\phi(X,Y)$ is a symmetric $(0,2)$ tensor field [5] that is

$$\phi(X, Y) = \Phi(Y, X) \tag{2.7}$$

Definition. *An m-dimensional Riemannian submanifold M of a Lorentzian paraSasakian manifold $\overline{M}$ is called a CR-submanifold if ξ is tangent to M and there exists on M a differentiable distribution $D : x \to D_x \subset T_x(M)$ such that*

(i) *D_x is invariant under φ i.e., $\varphi D_x \subset D_x$ for each $x \in M$,*

(ii) *The orthogonal complementary distribution $D^{\perp} : x \to D_x^{\perp} \subset T_x(M)$ of the distribution D on M is totally real i.e., $\varphi\, D_x^{\perp}(M) \subset T_x^{\perp}(M)$ where $T_x(M)$ and $T_x^{\perp}(M)$ are the tangent space and normal space of M at $x(\in M)$.*

D (resp. $D^{\perp}$) is called the horizontal (resp. vertical) distribution. The pair $(D, D^{\perp})$ is called ξ-horizontal (resp. ξ-vertical) if $\xi_x \in D_x$ (resp. $\xi_x \in D_x^{\perp}$) for each $x \in M$.

Let us denote the orthogonal complement of $\varphi D^{\perp}$ in $T^{\perp}(M)$ by μ. Then we have,

$$TM = D \oplus D^{\perp}, \quad T^{\perp}M = \varphi D^{\perp} \oplus \mu .$$

It is obvious that $\varphi\mu = \mu$.

We denote by g the metric tensor field of $\overline{M}$ as well as that induced on M. Let $\overline{\nabla}$ (resp.∇) be the covariant differentiation with respect to the Levi-Civita connection on $\overline{M}$ (resp. M). The Gauss and Weingarten formulas for M are respectively given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X N = -A_N X + \nabla_X{}^{\perp} N \tag{2.8}$$

for $X, Y \in T(M), N \in T^{\perp}(M)$, where h(resp. A) is the second fundamental form (resp. tensor) of M in $\overline{M}$ and $\nabla^{\perp}$ denotes the operator of the normal connection.

From (2.8) we have

$$g(h(X,Y),N) = g(A_N X,Y) \tag{2.9}$$

for $X, Y \in T(M)$ and $N \in T^{\perp}(M)$.

For $X \in T(M)$, we put

$$X = PX + QX \tag{2.10}$$

where $PX \in D$ and $QX \in D^{\perp}$.

For $N \in T^{\perp}(M)$, we put

$$\varphi N = BN + CN \tag{2.11}$$

where BN (resp. CN) denotes the tangential (resp. normal) component of φN.

3. Integrability of distributions on a CR-submanifold

First we prove the following lemmas.

Lemma 3.1. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$. Then we have*

$$P\nabla_X \varphi PY - PA_{\varphi QY}X = \varphi P\nabla_X Y + \eta(Y)PX + 2\eta(X)\eta(Y)P\xi + g(X,Y)P\xi \tag{3.1}$$

$$Q\nabla_X \varphi PY - QA_{\varphi QY}X = Bh(X,Y) + \eta(Y)QX + 2\eta(X)\eta(Y)Q\xi + g(X,Y)Q\xi \tag{3.2}$$

for $X, Y \in T(M)$.

Proof. It follows immediately from (2.4) by equating horizontal, vertical and normal components.

Lemma 3.2. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$. Then we have*

$$\varphi P[Y,Z] = A_{\varphi Y}Z - A_{\varphi Z}Y + \eta(Y)Z - \eta(Z)Y, \text{ for } Y, Z \in D^{\perp}$$

Proof. For $Y, Z \in D^{\perp}$, we have

$$\overline{\nabla}_Y \varphi Z = (\overline{\nabla}_Y \varphi)Z + \varphi\overline{\nabla}_Y Z$$

$$\Rightarrow -A_{\varphi Z}Y + \nabla_Y{}^{\perp}\varphi Z = g(Y,Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi + \varphi(\nabla_Y Z + h(Y,Z)), \quad \text{by (2.4)}$$

$$\Rightarrow -A_{\varphi Z}Y + \nabla_Y{}^{\perp}\varphi Z = g(Y,Z)\xi + \eta(Z)Y + 2\eta(Y)\eta(Z)\xi + \varphi P\nabla_Y Z + \varphi Q\nabla_Y Z + Bh(Y,Z) + Ch(Y,Z).$$

$$\Rightarrow -A_{\varphi Z}Y + \varphi Q\nabla_Y Z + Ch(Y,Z) = g(Y,Z)\xi + \eta(Z)Y + 2\eta(Z)\eta(Y)\xi + \varphi P\nabla_Y Z + \varphi Q\nabla_Y Z + Bh(Y,Z) + Ch(Y,Z), \quad \text{by (3.3)}$$

$$\Rightarrow \varphi P\nabla_Y Z = -A_{\varphi Z}Y - g(Y,Z)\xi - \eta(Z)Y - 2\eta(Y)(Z)\xi - Bh(Y,Z)$$

Therefore, $\varphi P[Y,Z] = A_{\varphi Y}Z - A_{\varphi Z}Y + \eta(Y)Z - \eta(Z)Y$, for $Y, Z \in D^{\perp}$.

Thus the lemma follows.

Now we have

Theorem 3.1. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable iff*

$$A_{\varphi Y}Z - A_{\varphi Z}Y = \eta(Z)Y - \eta(Y)Z, \quad \text{for } Y, Z \in D^{\perp}.$$

Corollary. *Let M be a ξ-horizontal CR-submanifold of an LP-Sasakian manifold $\overline{M}$. The distribution $D^{\perp}$ is integrable iff $A_{\varphi Y}Z = A_{\varphi z}Y$, for $Y, Z \in D^{\perp}$.*

The following theorem states the integrability condition of distribution D.

Theorem 3.2. *Let M be a CR-submanifold of an LP-Sasakian manifold $\overline{M}$. The distribution D is integrable iff $h(X, \varphi Y) = h(Y, \varphi X)$, for X, $Y \in D$.*

Proof. From (3.3) it follows that

$$h(X, \varphi Y) = \varphi Q \nabla_X Y + Ch(X, Y) \tag{3.4}$$

for X, $Y \in D$.

As h is symmetrical form, (4.1) implies

$$\varphi Q[X, Y] = h(X, \varphi Y) - h(Y, \varphi X)$$

Thus D is integrable iff $h(X, \varphi Y) = h(Y, \varphi X)$, for X, $Y \in D$.

4. Parallel horizontal distributions of CR-submanifolds

Definition. *The horizontal distribution D is said to be parallel with respect to the connection ∇ on M if $\nabla_X Y \in D$ for all vector fields $X, Y \in D$.*

Proposition 4.1. *Let M be a ξ-horizontal CR-submanifold of an LP-Sasakian manifold $\overline{M}$. Then the distribution D is parallel if and only if*

$$h(X, \varphi Y) = h(\varphi X, Y) = \varphi h(X, Y) \tag{4.1}$$

for all $X, Y \in D$.

Proof. As every parallel distribution is involutive, we have from Theorem (3.2)

$$h(X,\varphi Y) = h(\varphi X, Y) \tag{4.2}$$

for all $X, Y \in D$. Now for $X, Y \in D$, we have from (3.3) and (4.2)

$$h(X,\varphi Y) = Ch(X, Y) \tag{4.3}$$

Also, for $X, Y \in D$, using D-parallelness, we have

$$\nabla_X \varphi Y \in D,\ \nabla_Y \varphi X \in D.$$

Then from (3.2) we have $Bh(X,Y) = 0,\ \forall X,\ Y \in D.$ Now, from $\varphi h(X,Y) = Bh(X,Y) + Ch(X,Y)$, it follows that

$$\varphi h(X,Y) = Ch(X,Y) = h(X,\varphi Y),\ \forall X,\ Y \in D.$$

Conversely using (4.1) in (3.3) we get $\nabla_X Y \in D$ for $X, Y \in D$, which shows that D is parallel.

Definition. *A CR-submanifold M of an LP-Sasakian manifold $\overline{M}$ is said to be mixed totally geodesic if $h(X,Y) = 0$ for $X \in D$ and $Y \in D^{\perp}$.*

It follows immediately that a CR-submanifold is mixed totally geodesic if and only if $A_N X \in D$ for each $X \in D$ and $N \in T^{\perp}(M)$.

Definition. *A normal vector field $N \neq 0$ is said to be D-parallel normal section if $\nabla_X^{\perp} N = 0$ for each $X \in D$.*

Let $X \in D$ and $N \in \varphi D^{\perp}$. For a mixed totally geodesic ξ-vertical CR-submanifold M of LP-Sasakian manifold $\overline{M}$, we have

$$(\overline{\nabla}_X \varphi)N = 0, \text{ by (2.4)}$$

$$h(X,\varphi N) = 0, \text{ by hypothesis}$$

and $A_N X \in D$.

Now,
$$(\overline{\nabla}_X\varphi)N = \overline{\nabla}_X\varphi N - \varphi\overline{\nabla}_X N$$
$$\Rightarrow \overline{\nabla}_X\varphi N = \varphi\overline{\nabla}_X N$$
$$\Rightarrow \nabla_X(\varphi N) + h(X,\varphi N) = \varphi(-A_N X + \nabla_X{}^{\perp}N)$$
$$\Rightarrow \nabla_X(\varphi N) = -\varphi A_N X + \varphi\nabla_X{}^{\perp}N \tag{4.4}$$

As $A_N X \in D$, and $\varphi A_N X \in D$ by (4.4), it follows that

$$\nabla_X{}^{\perp}N = 0 \text{ iff } \nabla_X(\varphi N) \in D.$$

Thus we have the following proposition.

Proposition 4.2. *Let M be a mixed totally geodesic ξ-vertical CR-submanifold of an LP-Sasakian manifold $\overline{M}$. Then the normal section $N \in \varphi D^{\perp}$ is D-parallel iff $\nabla_X(\varphi N) \in D$, for $X \in D$.*

5. The CR-structure of a CR-submanifold

A complex distribution H on M (i.e. $H \subset TM \otimes_R C$) is said to define a Cauchy-Riemann structure [11] if it satisfies the following conditions:

i) $H \cap \overline{H} = \{0\}$, where $\overline{H}$ is the conjugated distribution of H;
ii) H is involutive, i.e., for any $A, B \in H, [A, B] \in H$.

Theorem 5.1. *Each CR-submanifold M of an LP-Sasakian manifold is a Cauchy-Riemann manifold.*

Proof. Let $P : TM \to D$ and $Q : TM \to D^{\perp}$ be the projection operators. Then each vector X can be expressed by $X = PX + QX$.

We put $H = \{X - i\varphi X \mid X \in D\}$.

Let $A, B \in H$; then $A = X - i\varphi X, B = Y - i\varphi Y$, for certain $X, Y \in D$.

Since $\overline{M}$ is normal, we have

$$N\varphi(X,Y) + 2d\eta(X,Y)\xi = 0$$

where $N\varphi(X,Y)$ denotes the Nijenhuis tensor.

Then we get

$$[\varphi X, \varphi Y] - [X, Y] - \varphi P\{[\varphi X, Y] + [X, \varphi Y]\} = 0,$$
$$Q\{[\varphi X, Y] + [X, \varphi Y]\} = 0.$$

Replacing X by φX, we obtain

$$[\varphi X, \varphi Y] - [X, Y] \in D.$$

On the other hand we may write

$$[A, B] = [X, Y] - [\varphi X, \varphi Y] - i[\varphi X, Y] - i[X, \varphi Y]$$
$$= [X, Y] - [\varphi X, \varphi Y] - i\varphi\{[X, Y] - [\varphi X, \varphi Y]\} \in H.$$

This completes the proof.

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