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One-Parameter Family of Neville-Aitken Algorithm on q-Triangle

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Abstract. Two dimensional polynomial interpolation on triangular region with geometric spacing is considered. Lagrange form and Neville-Aitken algorithm for interpolating polynomials on q-triangle are obtained and the question of inter-generating between these form is studied.

1. Introduction

In [3], Schoenberg discussed various works on one dimensional polynomial interpolation at the points of geometric progression and give a unified version of the problem. Lee and Phillips [2] extend these results to two dimensional case, for a triangular domain where the nodes are not uniformly spaced. Given a positive integer n, let $[n]$ be a q-integer defined by $[n]=\frac{1-q^n}{1-q}$, where $q>0$ and $q\neq 1$. Specifically, they proved that, there exists a unique interpolating polynomial, $P_n(x,y)$ for a function f on the triangular geometric mesh points $\{([i],[j]) : 0 \le i \le j \le n\}$. They also derived forward difference formula (in y and 'diagonal' directions) and Lagrange form for P_n and, obtained Neville-Aitken type algorithm to evaluate the polynomial efficiently.

We now consider the triangular array of points $S=\{([i],[j]) : i, j \ge 0,\ i+j\le n\}$ formed by the lines $x=[i]$ and $y=[j]$. This array of nodes is bounded by the X-axis, the Y-axis and the hyperbola $x+y-(1-q)xy=[n]$. We shall call this region a q-triangle of order n which includes the standard triangle as a special case. In this setting, it has been shown that there is a forward difference formula in x and y directions for the interpolating polynomial of degree at most n, at the nodes of S. In this paper we shall derive a Lagrange form of an interpolating polynomial and discuss a one-parameter family of Neville-Aitken algorithms.

2. Forward difference and Lagrange formulas on the q-triangle

Let $f(x, y)$ be a function defined over the q-triangle. Since the interpolation nodes S lie on lines parallel to coordinates axes, it is appropriate to define forward difference operators along these directions. Let denote $f([i],[j])$ by $f_{i,j}$ and define $D_x^0 f_{i,j} = f_{i,j}$ and $D_y^0 f_{i,j} = f_{i,j}$. For $m = 1, 2, 3, \cdots$, define recursively

$$D_x^m f_{i,j} = D_x^{m-1} f_{i+1,j} - q^{m-1} D_x^{m-1} f_{i,j}$$

and

$$D_y^m f_{i,j} = D_y^{m-1} f_{i,j+1} - q^{m-1} D_y^{m-1} f_{i,j}$$

It follows that for $m = 1, 2, 3, \cdots$ and $n = 0, 1, 2, \cdots$ the mixed q-differences satisfy

$$D_x^m D_y^n f_{i,j} = D_x^{m-1} D_y^n f_{i+1,j} - q^{m-1} D_x^{m-1} D_y^n f_{i,j}$$

We need to extend q-integer to q-real. For any $t \in \boldsymbol{R}$ the q-real t, denoted by $[t]$, is defined by

$$[t] = \begin{cases} \dfrac{1-q^t}{1-q}, & q \neq 1 \text{ and } q > 0 \\ t, & q = 1 \end{cases}$$

Also for any $t \in \boldsymbol{R}$ and $k \in \boldsymbol{Z}$, with $0 \leq k \leq t$, the q-binomial coefficient is defined by

$$\begin{bmatrix} t \\ k \end{bmatrix} = \frac{1}{[k]!} \prod_{\upsilon=0}^{k-1} [t-\upsilon] \quad \text{where} \quad [k]! = [k][k-1] \ldots [1]$$

Using these notations we see that there exists a q-forward difference formula for the interpolating polynomial of degree n on the q-triangle.

Theorem 1. *Let $f(x, y)$ be defined at all points of the q-triangle of order n. For any (x, y) in the q-triangle, let*

$$P_n(x, y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \begin{bmatrix} \bar{x} \\ r-s \end{bmatrix} \begin{bmatrix} \bar{y} \\ s \end{bmatrix} D_x^{r-s} D_y^s f_{0,0} \tag{1}$$

where $x = [\bar{x}]$ and $y = [\bar{y}]$ for some $\bar{x}, \bar{y} \in \boldsymbol{R}$ then the polynomial $P_n(x, y)$ interpolates f at the nodes of S.

The proof of Theorem 1 is given by Yahaya and Phillips [4]. Taking $q=1$ as a special case, (1) does reduces to a forward difference formula for polynomial interpolation on the standard triangle.

We also recognize that the nodes in set S are formed by the systems of $x=[\upsilon]$, $y=[\upsilon]$ and $\gamma(x,y)=[\upsilon]$ where $\gamma(x,y)=x+y-(1-q)xy$ and $\upsilon=0,1,\cdots,n$. So given any point $([i],[j])$ on the triangle, the union of the hyperbolas $\gamma(x,y)=[n-\upsilon]$ for $\upsilon=0,1,\cdots,n-i-j-1$, the straight lines $x=[\upsilon]$ for $\upsilon=0,1,\cdots,i-1$ and $y=[\upsilon]$ for $\upsilon=0,1,\cdots,j-1$ contain all nodes on the triangle except the point $([i],[j])$ itself. It follows that, for $i,\ j\geq 0,\ i+j\leq n$, the polynomial

$$M_{i,j}^{n}(x,y) = \frac{1}{\omega(i,j)}\prod_{v=0}^{i-1}(x-[v])\prod_{v=0}^{j-1}(y-[v])\prod_{v=0}^{n-i-j-1}([n-v]-\gamma(x,y)) \tag{2}$$

where

$$\omega(i,j) = [i]!\,[j]!\,[n-i-j]!\,q^{(i+j)(2n-1-i-j)/2-ij}$$

satisfies the conditions

$$M_{i,j}^{n}([h],[k]) = \begin{cases} 1 & \text{if} \quad ([h],[k]) = ([i],[j]) \\ 0 & \text{at all other nodes in } S \end{cases}$$

We note that, in the above expression (2) for $M_{i,j}^{n}(x,y)$, an empty product (when $i=0$ or $j=0$ or $i+j=n$) is taken to have value 1.

Thus we obtain a Lagrangian form of an interpolating polynomial which uses hyperbolas and two linear systems. This polynomial can be expressed as

$$P(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} M_{i,j}^{n}(x,y)f_{i,j}\,. \tag{3}$$

In this case the degree of $P(x,y)$ is at most $2n$, since the degree of any $M_{i,j}^{n}(x,y)$ is at most $2n-i-j$ and we note that the interpolating polynomial will not be unique. However, letting q tend to 1, the polynomial $P(x,y)$ in (3) reduces to the interpolating polynomial of degree at most n on the standard triangle.

3. Neville-Aitken algorithm

We now construct a Neville-Aitken algorithm for an interpolating polynomial on the q-triangle of order n. For each $m = 1, 2, \cdots, n$, the algorithm generates a one parameter family of polynomials $f_{i,j}^{m}(x, y)$: $i, j \geq 0$, $i + j \leq n - m$, which interpolate $f(x, y)$ on $T_{i,j}^{m}$: $i, j \geq 0$, $i + j \leq n - m$ respectively. Here we have used the notation $T_{i,j}^{m}$ to mean the set of nodes $T_{i,j}^{m} = \{([i+s], [j+t]) : s, t \geq 0, s + t \leq m\}$ for which $T_{0,0}^{n} = S$. These are the q-triangle bounded by the lines $x = [i]$, $y = [j]$ and the hyperbola $\gamma(x, y) = [m+i+j]$.

Lemma 2. *Let* $f_{i,j}^{0}(x, y) = f_{i,j}$, $i, j \geq 0$, $i + j \leq n$. *For* $m = 1, 2, \cdots, n$, *we define* $f_{i,j}^{m}(x, y), 0 \leq i + j \leq n - m$, *recursively by*

$$q^{i+j}[m]\ f_{i,j}^{m} = \{[m+i+j] - \gamma(x, y)\}\ f_{i,j}^{m-1}(x, y) + (x - [i])\{q^{j} - \lambda(1-q)(y-[j])\}$$
$$f_{i+1,j}^{m-1}(x, y) + (y - [j])\{q^{i} - (1-\lambda)(1-q)(x-[i])\}\ f_{i,j+1}^{m-1}(x, y) \qquad (4)$$

where λ *is an arbitrary real number. Then* $f_{i,j}^{m}(x, y)$ *interpolates* $f(x, y)$ *on* $T_{i,j}^{m}$.

Proof. First we note that the coefficient of $f_{i,j}^{m-1}(x, y)$ in (4) may be expressed as

$$[m+i+j] - \gamma(x, y) = q^{i+j}[m] - q^{j}(x-[i]) - q^{i}(y-[j]) + (1-q)(x-[i])(y-[j]).$$

Clearly the above result holds for $m = 0$. Suppose that (4) holds for some $m - 1$. Therefore the polynomials $f_{i,j}^{m-1}$, $f_{i+1,j}^{m-1}$ and $f_{i,j+1}^{m-1}$ interpolate $f(x, y)$ on the sets $T_{i,j}^{m-1}$, $T_{i+1,j}^{m-1}$ and $T_{i,j+1}^{m-1}$ respectively. For any integers i, $j \geq 0$, $0 \leq i + j \leq n - m$ consider the function $f_{i,j}^{m}$ at the nodes

$$T_{i,j}^{m} = T_{i,j}^{m-1} \bigcup T_{i+1,j}^{m-1} \bigcup T_{i,j+1}^{m-1} = \{([i+s], [j+t]),\ s, t \geq 0, 0 \leq s + t \leq m\}.$$

We now show that polynomial $f_{i,j}^{m}(x, y)$ interpolates $f(x,y)$ on $T_{i,j}^{m}$. First we see that (see Figure 1), if the node $([h], [k]) \in T_{i,j}^{m-1} \bigcap T_{i+1,j}^{m-1} \bigcap T_{i,j+1}^{m-1}$ then

$$f_{i,j}^{m-1}([h], [k]) = f_{i+1,j}^{m-1}([h], [k]) = f_{i,j+1}^{m-1}([h], [k]) = f_{h,k},$$

and hence

$$q^{i+j}[m]\, f_{i,j}^{m}\,([h],[k]) = \Big\{\, q^{i+j}\{[m]-[h-i]-[k-j]+(1-q)\,[h-i]\,[k-j]\,\}$$
$$+q^{i+j}[h-i]\,\{\,1-\lambda(1-q)\,[k-j]\,\}+q^{i+j}[k-j]\,\{\,1-(1-\lambda)\,(1-q)[h-i]\,\}\,\Big\}\, f_{h,k}\;,$$

which equals to $q^{i+j}[m]\, f_{h,k}$. If the nodes are the extreme points $([i],[j])$, $([i+m]\,,[j])$ and $([i],[j+m])$, then we can check that $f_{i,j}^{m}(x,y)$ interpolates $f(x,y)$ at these points.

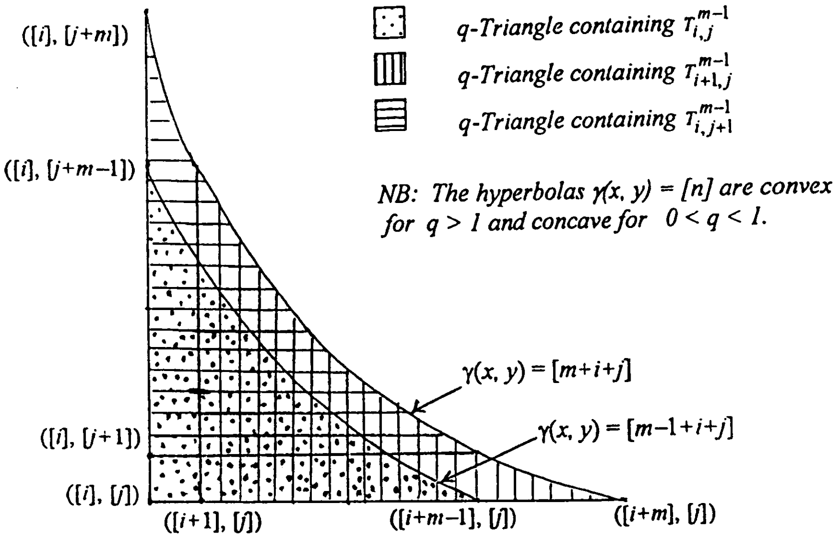

Figure 1. Interpolation nodes $T_{i,j}^{m-1}$, $T_{i+1,j}^{m-1}$ and $T_{i,j+1}^{m-1}$

To complete the proof we consider the rest of the nodes, which are on the hyperbola $\gamma(x,y)=[m+i+j]$ or one of the straight lines $x=[i]$ and $y=[j]$. On the hyperbola $\gamma(x,y)=[m+i+j]$, at the nodes $([h],[k])$ such that $h<i+m\,, k<j+m$, we have $f_{i+1,j}^{m-1}([h],[k]) = f_{i,j+1}^{m-1}([h],[k]) = f_{h,k}$ and thus

$$[m]\;\; f_{i,j}^{m}([h],[k]) = \Big\{\,\{[h-i]-\lambda(1-q)[h-i][k-j]\}+\{[k-j]-(1-\lambda)\,(1-q)[h-i][k-j]\}\,\Big\}\, f_{h,k}$$
$$=\Big\{\,[h-i]+[k-j]-(1-q)[h-i][k-j]\,\Big\}\, f_{h,k} \;=\; [m]\;\; f_{h,k}\;,$$

where $h,\ k \geq 0,\ h+k = m+i+j$. It follows similarly that $f_{i,j}^{m}(x, y)$ interpolates $f(x, y)$ on the line $x = [i]$, with $j < k < j+m$ and on the line $y = [j]$, with $i < h < i+m$. Thus, by induction, the formula is true for all $m,\ 0 \leq m \leq n$.

We see that $f_{0,0}^{n}(x, y)$ in (4) and $P(x, y)$ in (3) are two interpolating polynomials on the same q-triangle and their degrees are at most $2n$. In fact some of the Lagrange coefficients $M_{i,j}^{n}$ are of degree precisely n. However none of the Neville-Aitken algorithms of the form (4) generate the interpolating polynomial defined in (3). This is shown in the following counter example.

Example 1. Consider the two interpolating polynomials $P(x, y)$ and $f_{0,0}^{1}(x, y)$ defined by (3) and (4) respectively on a q-triangle of order 1. From (3) we have

$$\begin{aligned} P(x, y) &= M_{0,0}^{1}(x, y) f_{0,0} + M_{1,0}^{1}(x, y) f_{1,0} + M_{0,1}^{1}(x, y) f_{0,1} \\ &= \{1-\gamma(x, y)\} f_{0,0} + x\, f_{1,0} + y\, f_{0,1} \end{aligned}$$

Now let us consider the recurrence relation (4). We have

$$f_{0,0}^{1}(x, y) = \{1-\gamma(x, y)\}\, f_{0,0} + x\{1-\lambda(1-q)\, y\}\, f_{1,0} + y\{1-(1-\lambda)(1-q)\, x\}\, f_{0,1}\,.$$

Hence

$$P(x, y) - f_{0,0}^{1}(x, y) = (1-q)\{\lambda\, f_{1,0} + (1-\lambda)\, f_{0,1}\}\, xy$$

which is identically zero only for $q = 1$.

2. Generalised Neville-Aitken algorithm

Having shown that none of the Neville-Aitken algorithms of the form (4) generate the interpolating polynomial defined in (3), it is interesting to explore whether there exists some other Neville-Aitken algorithm which generates the interpolating polynomial defined in (3).

Let $f_{i,j}^{0}(x, y) = f_{i,j}$, where $i, j \geq 0$ and $i+j \leq n$. For $m = 1, 2, \cdots, n$, we define $f_{i,j}^{m}(x, y)$, $0 \leq i+j \leq n-m$, recursively by

$$f_{i,j}^{m}(x, y) = c_{i,j}^{m}(x, y) f_{i,j}^{m-1}(x, y) + d_{i,j}^{m}(x, y) f_{i+1,j}^{m-1}(x, y) + e_{i,j}^{m}(x, y) f_{i,j+1}^{m-1}(x, y) \tag{5}$$

where

$$c_{i,j}^{m}(x, y) + d_{i,j}^{m}(x, y) + e_{i,j}^{m}(x, y) = 1. \tag{6}$$

We shall call (5) a generalised Neville-Aitken algorithm. It includes the class of algorithms given in (4) as a special case. We observe that the recurrence relation (5) cannot give (3). For let $P(x, y)$ and $f^1_{0,0}(x, y)$ be the two interpolating polynomials on a q-triangle of order 1 defined by (3) and (5) respectively. Following the argument used in Example 1, we see that $P(x, y) \neq f^1_{0,0}(x, y)$.

The following example shows that, even if we relax the condition (6) so that it holds only for points in $T^m_{i,j}$ and not for all x and y, we still cannot find a Neville-Aitken algorithm of the form (5) which generates $P(x, y)$ in (3).

Example 2. Consider the polynomial in (3) which interpolates $f(x, y)$ on $T^2_{0,0}$,

$$P(x,y) = \frac{1}{[2]}([2]-\gamma(x,y))(1-\gamma(x,y))f_{0,0} + \frac{1}{q}x([2]-\gamma(x,y))\ f_{1,0} + \frac{1}{q}\ y(\ [2]-\gamma(x,y))\ f_{0,1} + xy\, f_{1,1} + \frac{1}{q[2]}x(x-1)\, f_{2,0} + \frac{1}{q[2]}y(y-1)\, f_{0,2}\,.$$

Suppose that the polynomial can be expressed in the form of (5) such that the condition (6) holds on $T^2_{0,0}$. So for some coefficient functions $c^2_{0,0}(x,y)$, $d^2_{0,0}(x,y)$ and $e^2_{0,0}(x,y)$ we can write

$$P(x,y) = c^2_{0,0}(x,y)\ P^{0,0}(x,y) + d^2_{0,0}(x,y)\, P^{1,0}(x,y) + e^2_{0,0}(x,y)\, P^{0,1}(x,y)$$

where

$$P^{0,0} = (1-\gamma(x,y))\ f_{0,0} + xf_{1,0} + y\ f_{0,1},$$

$$P^{1,0} = \frac{[2]-\gamma(x,y)}{q} f_{1,0} + \frac{x-1}{q} f_{2,0} + y\ f_{1,1}$$

and

$$P^{0,1} = \frac{1}{q}([2]\ -\ \gamma(x,y))\ f_{0,1} + \ x\ f_{1,1} + \frac{1}{q}(y-1)\ f_{0,2}$$

are the interpolating polynomials on $T^1_{0,0}$, $T^1_{1,0}$ and $T^1_{0,1}$ respectively. However on comparing the coefficients of $f_{0,0}, f_{2,0}$ and $f_{0,2}$, we obtain

$$c^2_{0,0}(x,y) = \frac{1}{[2]}([2]-\gamma\ (x,y))\,,\ d^2_{0,0}(x,y) = \frac{1}{[2]}x \ \text{ and } \ e^2_{0,0}(x,y) = \frac{1}{[2]}y$$

on $T^2_{0,0}$. This implies that on $T^2_{0,0}$

$$c^2_{0,0}(x,y) + d^2_{0,0}(x,y) + e^2_{0,0}(x,y) = \frac{[2]+(1-q)xy}{[2]} \neq 1 \quad \text{unless } q=1.$$

Now, given a generalised Neville-Aitken algorithm (5) which generates the polynomial $f^n_{0,0}(x,y)=\tilde{P}(x,y)$ say, we can always define the corresponding Lagrange coefficients $a^n_{i,j}(x,y)$ for $\tilde{P}(x,y)$ as follows.

Let $a^0_{0,0}(x,y) = 1$ and for $m=n-1, \cdots, 0$ define $a^{n-m}_{i,j}(x,y), i, j \geq 0, i+j \leq n-m$, recursively by

$$\begin{aligned} a^{n-m+1}_{i,j}(x,y) &= c^m_{i,j}(x,y)\, a^{n-m}_{i,j}(x,y) + d^m_{i-1,j}(x,y)\, a^{n-m}_{i-1,j}(x,y) \\ &\quad + e^m_{i,j-1}(x,y)\, a^{n-m}_{i,j-1}(x,y) \end{aligned} \tag{7}$$

where $a^m_{i,j}(x,y)=0$ if $i,\ j<0$ or $i+j>m$. Then we shall see that $\tilde{P}(x,y)$ can be written in terms of both $f^m_{i,j}(x,y)$ and $a^{n-m}_{i,j}(x,y)$ for any m satisfying $0 \leq m \leq n$.

Theorem 3. *Let $\tilde{P}(x,y)$ be the interpolating polynomial on a q-triangle of order n generated by the generalised Neville-Aitken algorithm. Then, for $m = 0, 1, \cdots, n$,*

$$\tilde{P}(x,y) = \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} f^m_{i,j}(x,y)\, a^{n-m}_{i,j}(x,y) \tag{8}$$

Proof. The formula is true for $m=n$ since $a^0_{0,0}(x,y)=1$ and $f^n_{0,0}(x,y) = \tilde{P}(x,y)$ is the polynomial generated by (5) and interpolates f on $T^n_{0,0}$. Suppose the formula is true for some $m>0$. We shall show that it is also true for $m-1$. On applying (5) to $f^m_{i,j}(x,y)$ in equation (8) we see that

$$\begin{aligned} \tilde{P}(x,y) &= \sum_{j=0}^{n-m} \sum_{i=0}^{n-m-j} c^m_{i,j}(x,y)\, a^{n-m}_{i,j}(x,y)\, f^{m-1}_{i,j}(x,y) \\ &\quad + \sum_{j=0}^{n-m} \sum_{h=1}^{n-m-j+1} d^m_{h-1,j}(x,y)\, a^{n-m}_{h-1,j}(x,y)\, f^{m-1}_{h,j}(x,y) \\ &\quad + \sum_{k=1}^{n-m+1} \sum_{i=0}^{n-m-k+1} e^m_{i,k-1}(x,y)\, a^{n-m}_{i,k-1}(x,y)\, f^{m-1}_{i,k}(x,y) \end{aligned}$$

where we have written $h = i+1$ and $k = j+1$ in the last two double summations. Thus

$$\begin{aligned}\tilde{P}(x, y) &= \sum_{j=0}^{n-m+1} \sum_{i=0}^{n-m-j+1} c_{i,j}^{m}(x, y)\, a_{i,j}^{n-m}(x, y)\, f_{i,j}^{m-1}(x, y) \\ &+ \sum_{j=0}^{n-m+1} \sum_{h=0}^{n-m-j+1} d_{h-1,j}^{m}(x, y)\, a_{h-1,j}^{n-m}(x, y)\, f_{h,j}^{m-1}(x, y) \\ &+ \sum_{k=0}^{n-m+1} \sum_{i=0}^{n-m-k+1} e_{i,k-1}^{m}(x, y)\, a_{i,k-1}^{n-m}(x, y)\, f_{i,k}^{m-1}(x, y)\end{aligned}$$

where the added terms in each double summation are all zero. This follows, since by definition $a_{i,j}^{r}(x, y) = 0$ if $i,\ j < 0$ or $i + j > r$. Finally, on using (5) we obtain

$$\tilde{P}(x, y) = \sum_{j=0}^{n-m+1} \sum_{i=0}^{n-m+1-j} f_{i,j}^{m-1}(x, y)\, a_{i,j}^{n-m+1}(x, y).$$

Therefore by induction the formula is true for all $m = 0, 1, \cdots, n$. In particular, for $m = 0$, the interpolating polynomial in Theorem 3 reduces to

$$\tilde{P}(x, y) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} f_{i,j}^{0}\, a_{i,j}^{n}(x, y)$$

and thus $a_{i,j}^{n}(x, y)$, $i,\ j \ge 0$, $i + j \le n$, are the Lagrange coefficients for $\tilde{P}(x, y)$.

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