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Jacobi Fields and its Application of Normal Contact Lorentzian Manifolds

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1. Introduction

Let M be a differentiable manifold. If M has a Lorentzian metric g, that is, a symmetric nondegenerate (0,2)-type tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric g is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This variety gives interesting properties on M.

A differentiable manifold has a Lorentzian metric if and only if the manifold has a 1-dimensional distribution. Hence an odd-dimensional manifold can be equipped with a Lorentzian metric.

On the other hand, in some odd-dimensional manifolds, a normal contact (Riemannian) metric structure (that is, a Sasakian structure) can be defined (cf. [3], [10]). If we change the Riemannian metric of the Sasakian structure to a Lorentzian one, we can define a normal contact Lorentzian metric structure. This definition was given at almost starting time of the study of the Sasakian structure and some results were given (see [14]). But more practical study of it has not been given yet. In [8] and [9], we studied the fundamental properties of an odd-dimensional manifold with a normal contact Lorentzian structure. In this paper we continue the study of it.

The purpose of this paper is to consider the normal contact Lorentzian manifold of constant φ -sectional curvature, find the Jacobi field of it and characterize it by means of geodesic spheres. After the preliminaries of Section 2, we consider the Jacobi fields with respect to the structure vector field ξ in Section 3. Section 4 is devoted to the determination of the Jacobi fields with respect to spacelike geodesics. In the final section, we characterize the normal contact Lorentzian manifold of constant φ-sectional curvature by small geodesic spheres. This characterization is given by using the Jacobi fields of Section 4.

2. Normal contact Lorentzian manifolds

Let M be a $(2n+1)$-dimensional $(n \geq 2)$ differentiable manifold of class C^{∞} and g a Lorentzian metric of M.

A non-zero vector X is called *spacelike*, *timelike*, *null* if it satisfies $g(X,X) > 0$, < 0, $= 0$, respectively.

A *normal contact Lorentzian structure* (ϕ, ξ, η, g) of M is given by a $(1,1)$-type skew symmetric tensor field φ, a vector field ξ, a 1-form η and a Lorentzian metric g as

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi\,, \\ &\eta(\phi X) = 0\,,\ \eta(\xi) = 1\,,\ \eta(X) = -g(X,\xi)\,, \\ &(\nabla_X \eta)Y = g(\phi X, Y)\,,\ \nabla_X \xi = -\phi X\,, \\ &(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X\,, \end{aligned} \tag{2.1}$$

where X is a vector field of M and ∇ is the covariant derivative with respect to g.

The curvature tensor field $R(X,Y)$ of M satisfies

$$\begin{aligned} &R(X,Y)\xi = \eta(Y) - \eta(X)Y\,, \\ &R(X,Y)\phi Z = \phi R(X,Y)\phi Z - g(\phi X, Z)Y \\ &-g(X,Z)\phi Y + g(\phi Y, Z)(X + g(Y,Z)\phi X\,. \end{aligned} \tag{2.2}$$

A plane section of the tangent space at a point of M is called a φ-section if it is spanned by vectors X and φX orthogonal to ξ. The sectional curvature of a φ-section is called a φ-sectional curvature. If M has *constant* φ-sectional curvature h, then the curvature tensor satisfies

$$\begin{aligned} R(X,Y)Z \;=\; &\frac{h-3}{4}(g(Y,Z)X \;-\; g(X,Z)Y) \\ &+\;\frac{h+1}{4}(\eta(Y)\eta(Z)X \;-\; \eta(X)\eta(Z)Y \;+\; g(X,Z)\eta(Y)\xi \\ &-\; g(Y,Z)\eta(X)\xi \;+\; g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - \; 2g(\phi X, Y)\phi Z)\,. \end{aligned} \tag{2.3}$$

3. Jacobi fields with respect to ξ

Generally, a Jacobi field X is defined as follows. Let p be a point of M and u be a unit tangent vector at p. By γ, we denote a geodesic with the initial conditions $\gamma(0) = p$ and $\gamma'(0) = u$. We also denote by u the unit tangent vector field of γ. A *Jacobi field X* along γ is a vector field that satisfies the equation

$$\nabla_u \nabla_u X + R(X, u)u = 0. \tag{3.1}$$

In this section we consider the case $u = \xi$, i.e., the geodesic γ is an integral curve of the vector field ξ. Since M is a normal contact Lorentzian manifold, the curvature tensor $R(X, Y)Z$ satisfies (2.2). Hence, for a Jacobi field orthogonal to ξ along γ, (3.1) reduces to

$$\nabla_\xi \nabla_\xi + X = 0$$

From this equation we obtain

Theorem 3.1. *Let M be a $(2n+1)$-dimensional normal contact Lorentzian manifold with structure $(\varphi, \xi, \eta, g\}$. Let γ be an integral curve of the vector field ξ and X a Jacobi vector field orthogonal to ξ along γ. With respect to a parallel base $\{\xi, E_\alpha(\alpha = 2, \cdots, 2n+1\}$, X can write as*

$$X = \sum_{\alpha=2}^{2n+1} (A_\alpha \sin s + B_\alpha \cos s) E_\alpha$$

where A_α, B_α are constants along γ and s is the arc length parameter of γ.

4. Jacobi fields with respect to spacelike geodesics

Let γ be a geodesic with initial condition $\gamma(0) = p$ and $\gamma'(u) = u$ as in Section 2. In this section we consider the case that u is a unit spacelike vector field.

Let D be the field of planes spanned by φu and $\xi + \eta(u)u$ along γ and let $D^\perp$ denote the orthogonal complement of $D \oplus [u]$.

Lemma 4.1. *D and $D^\perp$ are parallel along γ.*

Proof. By using (2.1), it follows that

$$\nabla_u(\eta(u)) = -g(u, \nabla_u \xi) = g(u, \phi u) = 0.$$

So, $\eta(u)$ is constant along γ. Using this fact, we have

$$\nabla_u(\phi u) = (\nabla_u \phi)u = -\eta(u)u - g(u,u)\xi = -\xi - \eta(u)u,$$

and

$$\nabla_u(\xi + \eta(u)) = -\phi u + \eta(u)\nabla_u u = -\phi u.$$

Hence D is parallel along γ. Therefore $D^{\perp}$ is also parallel along γ.

At the point $p=\gamma(0)$, we consider an orthonormal basis $\{e_1,\cdots,e_{2n+1}\}$ with $u=e_1$ and

$$e_{2n} = \frac{\xi+\eta(u)u}{\sqrt{1+\eta(u)^2}} \ , \quad e_{2n+1} = \frac{\phi u}{\sqrt{1+\eta(u)^2}} .$$

Let $\{E_1,\ \cdots,\ E_{2n+1}\}$ be the base obtained by parallel translation of the basis $\{e_1,\ \cdots,\ e_{2n+1}\}$ along γ. Then, from Lemma 4.1, we have

$$E_{2n} = \frac{\xi+\eta(u)u}{\sqrt{1+\eta(u)^2}}\cosh s + \frac{\phi u}{\sqrt{1+\eta(u)^2}}\sinh s ,$$

$$E_{2n+1} = \frac{\xi+\eta(u)u}{\sqrt{1+\eta(u)^2}}\sinh s + \frac{\phi u}{\sqrt{1+\eta(u)^2}}\cosh s ,$$

where s denotes the arc length from p along γ.

We assume that M is of constant φ-sectional curvature. Then, from (2.3), it follows that

$$\begin{aligned}
&g(T(u,E_i)E_j,u)=\frac{1}{4}(h-3)+(h+1)\eta(u)^2)\delta_{ij} ,\\
&g(R(u,E_i)u,E_{2n})=0 ,\\
&g(R(u,E_i)u,E_{2n+1})=0 ,\\
&g(R(u,\xi)u,\phi u)=0 ,\\
&g(R(u,\phi u)u,\phi u)=(-h-(h+1)\eta(u^2)(1+\eta(u)^2) ,\\
&g(R(u,E_{2n})u,E_{2n})=-1-(h+1)(1+\eta)(u)^2)\sinh^2 s ,\\
&g(R(u,E_{2n+1})u,E_{2n+1})=-1-(h+1)(1+\eta)(u)^2)\cosh^2 s ,\\
&g(R(u,E_{2n})u,E_{2n+1})=-(h+1)(1+\eta)(u)^2)\sinh s\cosh s .
\end{aligned} \tag{4.1}$$

Let X be a Jacobi field orthogonal to γ and put

$$X=\sum_{i=2}^{2n-1} f_i E_i + f_{2n}E_{2n} + f_{2n+1}E_{2n+1} .$$

Then, from (3.1) and (4.1), we have

Proposition 4.1. *The coefficients* f_i, f_{2n}, f_{2n+1} *satisfy the following differential equations.*

$$f_n'' + \frac{1}{4}(h-3)+(h+1)\eta(u^2)f_i = 0, \tag{4.3}$$

$$f_{2n}'' - (1+(h+1)(1+\eta)(u)^2)\sinh^2 s)f_{2n} \\ -((h+1)(1+\eta(u)^2)\sinh s\cosh s)f_{2n+1} = 0, \tag{4.4}$$

$$f_{2n+1}'' - (1+(h+1)(1+\eta)(u)^2)\cosh^2 s)f_{2n+1} \\ -((h+1)(1+\eta(u)^2)\sinh s\cosh s)f_{2n} = 0. \tag{4.5}$$

From (4.3), we easily have

Proposition 4.2. *The coefficients* f_i *are given as follows:*
If $A=0$, *then* $f_i = a_i s + b_i$;
If $A>0$, *then* $f_i = a_i \sin\sqrt{A}\,s + b_i \cos\sqrt{A}\,s$;
If $A<0$, *then* $f_i = a_i \sinh\sqrt{-A}\,s + b_i \cosh\sqrt{-A}\,s$,
where $A := \frac{1}{4}((h-3) + (h+1)\eta(u^2)$ *and* a_i, b_i *are constants along* γ.

Next, we consider the differential equations with respect to f_{2n}'' and f_{2n+1}''. From (4.4) and (4.5), we have

$$f_{2n}'' - f_{2n} \\ -(h+1)(1+\eta(u)^2)\sinh s(f_{2n}\sinh s + f_{2n+1}\cosh s) = 0, \tag{4.6}$$

and

$$f_{2n+1}'' - f_{2n+1} \\ +(h+1)(1+\eta(u)^2)\cosh s(f_{2n}\sinh s + f_{2n+1}\cosh s) = 0. \tag{4.7}$$

Therefore, it follows that

$$f_{2n}''\cosh s + f_{2n+1}''\sinh s = f_{2n}\cosh s + f_{2n+1}\sinh s \tag{4.8}$$

and

$$f_{2n}''\sinh s + f_{2n+1}''\cosh s = f_{2n}\sinh s + f_{2n+1}\cosh s \\ -(h+1)(1+\eta(u)^2)(f_{2n}\sinh s + f_{2n+1}\cosh s). \tag{4.9}$$

If we put

$$F_{2n} = f_{2n}\sinh s + f_{2n+1}\cosh s,$$
$$F_{2n+1} = f_{2n}\cosh s + f_{2n+1}\sinh s,$$

then we have

$$F''_{2n} = f''_{2n}\sinh s + f'_{2n}\cosh s + f''_{2n+1}\cosh s + f'_{2n+1}\sinh s + F'_{2n+1}, \qquad (4.10)$$

$$F''_{2n+1} = f''_{2n}\cosh s + f''_{2n+1}\sinh s + 2F'_{2n} + F'_{2n+1}. \qquad (4.11)$$

From (4.8), we obtain

$$f''_{2n}\cosh s + f''_{2n+1}\sinh s = F_{2n+1}$$

By substituting this equation into (4.11), it follows that

$$F''_{2n+1} = 2F'_{2n}.$$

Similarily, from (4.9) and (4.10), we have

$$F''_{2n} = 2F'_{2n+1} - (h+1)(1+\eta(u)^2)F_{2n}.$$

If we put $G = F'_{2n}$, this equation reduces to

$$G'' + ((h-3) + (h+1)\eta(u)^2)G = 0.$$

Therefore, the differential equations with respect to f''_{2n} and f''_{2n+1} reduce to

$$F''_{2n+1} = 2G,$$
$$G = F'_{2n},$$
$$G'' + ((h-3) + (h+1)\eta(u)^2)G = 0.$$

From these equations we easily obtain

Proposition 4.3. *The coefficients f_{2n} and f_{2n+1} are given as follows:*
If $A = 0$, then

$$f_{2n} = \left(\frac{1}{3}as^3 + bs^2 + cs + d\right)\cosh s - \left(\frac{1}{2}as^2 + bs + c\right)\sinh s,$$

$$f_{2n+1} = \left(\frac{1}{3}as^3 + bs^2 + cs + d\right)\cosh s - \left(\frac{1}{2}as^2 + bs + c\right)\sinh s,$$

If $A>0$, *then*

$$f_{2n}=\left(-\frac{a}{2A}\sin\sqrt{4A}s-\frac{a}{2A}\cos\sqrt{4A}s+2cs+d\right)\cosh s$$
$$-\left(-\frac{a}{\sqrt{4A}}\cos\sqrt{4A}s+\frac{a}{\sqrt{4A}}\sin\sqrt{4A}s+c\right)\sinh s,$$
$$f_{2n+1}=\left(-\frac{a}{\sqrt{4A}}\cos\sqrt{4A}s+\frac{a}{\sqrt{4A}}\sin\sqrt{4A}s+c\right)\cosh s,$$
$$-\left(-\frac{a}{2A}\sin\sqrt{4A}s+\frac{a}{2A}\cos\sqrt{4A}s+2cs+d\right)\sinh s;$$

If $A<0$, *then*

$$f_{2n}=\left(-\frac{a}{2A}\sinh\sqrt{-4A}s-\frac{a}{2A}\cosh\sqrt{-4A}s+2cs+d\right)\cosh s$$
$$-\left(-\frac{a}{\sqrt{-4A}}\cosh\sqrt{4A}s+\frac{a}{\sqrt{-4A}}\sinh\sqrt{-4A}s+c\right)\sinh s,$$
$$f_{2n+1}=\left(-\frac{a}{\sqrt{-4A}}\cosh\sqrt{-4A}s+\frac{a}{\sqrt{-4A}}\sinh\sqrt{-4A}s+c\right)\cosh s,$$
$$-\left(\frac{a}{2A}\sinh\sqrt{-4A}s+\frac{a}{2A}\cosh\sqrt{-4A}s+2cs+d\right)\sinh s,$$

where $A:=\frac{1}{4}((h-3)+(h+1)\eta(u)^2)$ *and a, b, c, d are constants along γ.*

Putting Proposition 4.2 and Proposition 4.3 together, we obtain the following theorem.

Theorem 4.1. *Let M be a* $(2n+1)$*-dimensional normal contact Lorentzian manifold with structure* (φ,ξ,η,g). *Assume that M is of constant φ-sectional curvature h. Let γ be a spacelike geodesic and X a Jacobi field orthogonal to γ. If we put*

$$X=\sum_{i=2}^{2n-1} f_i E_i + f_{2n}E_{2n} + f_{2n+1}E_{2n+1},$$

with respect to a parallel base $\{\gamma', E_i, E_{2n}, E_{2n+1}\}$ *mentioned in this section, then the coefficients* f, f_{2n}, f_{2n+1} *are given as follows:*

If $A = 0$, *then*

$$f_i = a_i s + b,$$
$$f_{2n} = \left(\frac{1}{3}as^3 + bs^2 + cs + d\right)\cosh s - \left(\frac{1}{2}as^2 + bs + c\right)\sinh s,$$
$$f_{2n+1} = \left(\frac{1}{2}as^2 + bs + c\right)\cosh s - \left(\frac{1}{3}as^3 + bs^2 + cs + d\right)\sinh s;$$

If $A > 0$, *then*

$$f_i = a_i \sin\sqrt{A}s + b_i \cos\sqrt{A}s,$$
$$f_{2n} = \left(-\frac{a}{2A}\sin\sqrt{4A}s - \frac{a}{2A}\cos\sqrt{4A}s + 2cs + d\right)\cosh s$$
$$-\left(-\frac{a}{\sqrt{4A}}\cos\sqrt{4A}s + \frac{a}{\sqrt{4A}}\sin\sqrt{4A}s + c\right)\sinh s,$$
$$f_{2n+1} = \left(-\frac{a}{\sqrt{4A}}\cos\sqrt{4A}s - \frac{a}{\sqrt{4A}}\sin\sqrt{4A}s + c\right)\cosh s$$
$$-\left(-\frac{a}{2A}\sin\sqrt{4A}s + \frac{a}{\sqrt{2A}}\cos\sqrt{4A}s + c\right)\sinh s;$$

If $A < 0$, *then*

$$f_i = a_i \sinh\sqrt{-A}s + b_i \cosh\sqrt{-A}s,$$
$$f_{2n} = \left(-\frac{a}{2A}\sinh\sqrt{-4A}s - \frac{a}{2A}\cosh\sqrt{-4A}s - 2cs + d\right)\cosh s$$
$$-\left(\frac{a}{\sqrt{-4A}}\cosh\sqrt{-4A}s + \frac{a}{\sqrt{-4A}}\sinh\sqrt{-4A}s + c\right)\sinh s,$$
$$f_{2n+1} = \left(\frac{a}{\sqrt{-4A}}\cosh\sqrt{-4A}s + \frac{a}{\sqrt{-4A}}\sinh\sqrt{-4A}s + c\right)\cosh s$$
$$-\left(\frac{a}{2A}\sinh\sqrt{-4A}s + \frac{a}{\sqrt{2A}}\cosh\sqrt{-4A}s - 2cs + d\right)\sinh s;$$

where $A := \frac{1}{4}((h-3) + (h+1)\eta(u)^2)$ *are constants along* γ *and* s *is the arc length parameter* γ

Remark. The Jacobi field with respect to timelike geodesics can be obtained by a similar consideration as in this section (cf. [7]).

5. Geodesic spheres

In this section, we characterize a normal contact Lorentzian manifold of constant φ-sectional curvature by using some extrinsic property of geodesic spheres. For this purpose we use the Jacobi fields mentioned in the previous section.

For a point $a \in M$, let U be a normal neighbourhood about p. In U, we consider a *geodesic sphere* $S(r)$ with center p and radius $r(>0)$. Let γ be a spacelike geodesic starting at p and contained in U, and u its unit tangent vector field.

We denote the shape operator at a point $\gamma(s)$ of the geodesic sphere by A. If A satisfies the equation

$$AX = aX + bg(\xi, X)\xi + cg(\phi u, X)\phi u + dg(\xi, X)\phi u + dg(\phi u, X)\xi \tag{5.1}$$

at a point $\gamma(s)$, where a, b, c, d are functions, then $S(r)$ is called *quasi-umbilical* with respect to the plane of ξ and φX at the point (where we assume that $\eta(u) = 0$).

Let X be a Jacobi field along γ orthogonal to u. Then the equation

$$R(u, X)u = A^2 X - (\nabla_u A)X \tag{5.2}$$

is well-known [4]. Let $\{e_1, \cdots, e_{2n+1}\}$ be a parallel orthonormal base along γ with $e_1 = u$. We denote by $A_\alpha\,(\alpha = 2, \cdots, 2n+1)$, the Jacobi vector fields along γ determined by the intitial conditions

$$X_\alpha(0) = 0, \quad X'_\alpha(0) = e_\alpha .$$

Let F be an $2n \times 2n$ matrix given by the components of X_i with respect to the frame $\{e_2, \cdots, e_{2n+1}\}$, that is, $\mathsf{F}e_\alpha = X_\alpha$. Then we have

$$A = -\mathsf{F}\,\mathsf{F}^{-1} . \tag{5.3}$$

Theorem 5.1. *Let M be a* $(2n+1)$*-dimensional* $(n \ge 2)$ *normal contact Lorentzian manifold. If for every point p and direction u orthogonal to ξ at p, the geodesic spheres in some normal neighbourhood of p are quasi-umbilical with respect to the plane ξ and φu, then M is of constant φ-sectional curvature. The converse also holds.*

Proof. Assume that geodesic spheres are quasi-umbilical. Then from (5.1) it follows that

$$A^2 \phi u = ((a+b)^2 - d^2)\phi u + (2a - b + c)d\xi$$

and

$$(\nabla_u A)\phi u = (a' + c' - 2d)\phi u + (-b - c + d')\xi .$$

Substituting these equations into (5.2), we obtain

$$R(u,\phi u)u = ((a+c)^2 - d^2 - a' - c')\phi u + ((2a-b+c)d + b + c - d')\xi .$$

But, from (2.2), we have $g(R(u,\phi u)u,\xi)=0$. So $R(u,\phi u)u$ is collinear with φu along γ and hence, we conclude that M is of constant φ-sectional curvature. (This fact is easily proved as for the Riemannian case (see [12])).

Conversely, we assume that M is of constant φ-sectional curvature.

Let γ be a geodesic in a normal neighbourhood U. Assume that the tangent vector field u of γ is orthogonal to ξ. Along γ, we have an orthonormal parallel base $\{E_1,\cdots,E_{2n+1}\}$ as

$$\begin{aligned} E_1 &= \gamma' = u , \\ E_{2n} &= \xi \cosh s + \phi u \sinh s , \\ E_{2n+1} &= \xi \sinh s + \phi u \cosh s . \end{aligned}$$

Then the Jacobi field

$$X = \sum_{i=2}^{2n-1} f_i E_i + f_{2n} E_{2n} + f_{2n+1} E_{2n+1}$$

orthogonal to ξ satisfies

$$\begin{aligned} & f_i'' + \frac{1}{4}(h-3) f_i = 0 , \\ & f_{2n}'' - (1 + (h+1)\sinh^2 s) f_{2n} - ((h+1)\sinh s \cosh s) f_{2n+1} = 0 \\ & f_{2n+1}'' - (1 + (h+1)\cosh^2 s) f_{2n+1} + ((h+1)\sinh s \cosh s) f_{2n} = 0 , \end{aligned}$$

by virtue of Proposition 4.1.

Therefore, for the Jacobi fields $X_\alpha = \sum_{\beta 2}^{2n+1} f_{\beta\alpha} E_\beta \, (\alpha,\beta = 2,\cdots,2n+1)$, with initial conditions

$$X_\alpha(0) = 0 \quad , \quad X'_\alpha(0) = E_\alpha(0) ,$$

we have $(f_{\beta\alpha})$ as a matrix is of the form

$$\mathsf{F} = \begin{bmatrix} fI & 0 & 0 \\ 0 & f_{n-1n-1} & f_{n-1n} \\ 0 & f_{nn-1} & f_{nn} \end{bmatrix}$$

Hence $A = -\digamma \digamma^{-1}$ has the same form and it follows that the geodesic spheres are quasi-umbilical with respect to the plane of ξ and φu.

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