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Some Results on Normal Family of Meromorphic Functions

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Abstract. In this paper, we study the normality of a family of meromorphic functions and prove the following theorem: Let $\mathcal{F}$ be a family of meromorphic functions in a domain *D*, *k*, $q(\geq 2)$ be two positive integers, and $H(f, f', \cdots, f^{(k)})$ be a differential polynomial of *f* and $\frac{\Gamma}{\gamma}\Big|_H < k+1$. If the zeros of $f(z)$ are of multiplicity $\geq k+1$ and $(f^{(k)})^q + H(f, f', \cdots, f^{(k)}) \neq 1$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in *D*. This result is related to a problem of Hayman [5].

1. Introduction

Let $f(z)$ be meromorphic in domain *D*, $a_1(z), a_2(z), \cdots,$ analytic in *D*, $n_0, n_1, n_2, \cdots, n_k$ be non-negative integers. Set

$$\begin{aligned} M(f, f', \cdots, f^{(k)}) &= f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k}, \\ \gamma_M &= n_0 + n_1 + n_2 + \cdots + n_k, \\ \Gamma_M &= n_0 + 2n_1 + 3n_2 + \cdots + (k+1)n_k. \end{aligned}$$

$M(f, f', \cdots, f^{(k)})$ is called the differential monomial of *f*, γ_M the degree of $M(f, f', \cdots, f^{(k)})$ and Γ_M the weight of $M(f, f', \cdots, f^{(k)})$.

Let $M_1(f, f', \cdots, f^{(k)}), M_2(f, f', \cdots, f^{(k)}), \cdots, M_n(f, f', \cdots, f^{(k)})$ be differential monomials of *f*. Set

$$\begin{aligned} & H(f, f', \cdots, f^{(k)}) \\ &= a_1(z) M_1(f, f', \cdots, f^{(k)}) + \cdots + a_n(z) M_n(f, f', \cdots, f^{(k)}), \\ \gamma_H &= \max\{\gamma_{M_1}, \gamma_{M_2}, \cdots, \gamma_{M_n}\}, \\ \Gamma_H &= \max\{\Gamma_{M_1}, \Gamma_{M_2}, \cdots, \Gamma_{M_n}\}. \end{aligned}$$

$H(f, f', \cdots, f^{(k)})$ is called the differential polynomial of f, γ_H the degree of $H(f, f', \cdots, f^{(k)})$ and Γ_H the weight of $H(f, f', \cdots, f^{(k)})$. If $\gamma_{M_1} = \gamma_{M_2} = \cdots = \gamma_{M_n} = m$, then $H(f, f', \cdots, f^{(k)})$ is called the homogeneous differential polynomial of degree m. Set

$$\frac{\Gamma}{\gamma}\Big|_H = \max\left\{\frac{\Gamma_{M_1}}{\gamma_{M_1}}, \frac{\Gamma_{M_2}}{\gamma_{M_2}}, \cdots, \frac{\Gamma_{M_n}}{\gamma_{M_n}}\right\}.$$

In [5], Hayman posed the following conjecture.

Hayman conjecture: Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k be a positive integer. If , for any $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 1$, then $\mathcal{F}$ is normal in D.

Gu [3] confirmed the conjecture by proving

Theorem A. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k be a positive integer. If, for any $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)} \neq 1$, then $\mathcal{F}$ is normal in D.*

Yang [11] extended Theorem A by proving

Theorem B. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k be a positive integer, $a_1(z)$, $a_2(z)$, $\cdots$, $a_k(z)$ be analytic functions in the domain D. If, for any $f \in \mathcal{F}$, $f \neq 0$, $f^{(k)}(z) + a_1(z)f^{(k-1)}(z) + a_2(z)f^{(k-2)}(z) + \cdots + a_k(z)f(z) \neq 1$, then $\mathcal{F}$ is normal in D.*

Gu [4] considered the case of homogeneous differential polynomial with constant coefficient and proved that

Theorem C. *Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, k, $q \geq 3$ be two positive integers, $H(f, f', \cdots, f^{(k)}) = a_1 M_1(f, f', \cdots, f^{(k)}) + \cdots + a_n M_n(f, f', \cdots, f^{(k)})$ be a homogeneous differential polynomial with constant coefficient of degree q and the degree of $f^{(k)}$ of $M_i(f, f', \cdots, f^{(k)})$ be $\leq q-2$, $(i = 1, 2, \cdots, n)$. If $f \neq 0$ and $(f^{(k)})^q + H(f, f', \cdots, f^{(k)}) \neq 1$ for each $f \in \mathcal{F}$, $\mathcal{F}$ is normal in D.*

In this note, we have proved

Theorem 1. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain D, k,* $q(\geq 2)$ *be two positive integers, and* $H(f, f', \cdots, f^{(k)})$ *be differential polynomial and* $\frac{\Gamma}{\gamma}\big|_H < k+1$. *If the zeros of* $f(z)$ *are of multiplicity* $\geq k+1$ *and* $(f^{(k)})^q + H(f, f', \cdots, f^{(k)}) \neq 1$ *for each* $f \in \mathcal{F}$, *then* $\mathcal{F}$ *is normal in D.*

In 1916, Montel (see [6, 7, 11]) proved the following result.

Theorem D. *A family of meromorphic functions is normal if every function in the family omits three fixed distinct complex values (one of them may be ∞).*

In this note, we improve Theorem D slightly as follows.

Theorem 2. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a domain D. If for any two functions* $f, g \in \mathcal{F}$, $\{z : f(z) = 0, 1, \infty\} = \{z : g(z) = 0, 1, \infty\}$ *(counting multiplicities), then* $\mathcal{F}$ *is normal in D.*

Yang posed the following principle (see [7,10]).

Principle A. Let *P* be the property such that if two entire (or meromorphic) functions *f* and *g* satisfying *P* in the plane will ensure *f* being identical to *g*, then a family of holomorphic (or moromorphic) functions with the property *P* in a domain *D* will in general make this family normal in *D*.

In section 5, we will give an example to show that the above principle is not always true.

2. Some lemmas

For the proof of Theorem1, we need the following lemmas.

Lemma 1([8]). *Let* $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 + \frac{q(z)}{p(z)}$, *where* $a_0, a_1, \cdots, a_n$ *are constants with* $a_n \neq 0$, $q(z)$ *and* $p(z)$ *are two coprime polynomials with* $\deg q(z) < \deg p(z)$, k *be a positive integer. If* $f^{(k)}(z) \neq 1$, *then we have*

(1) $n = k$, *and* $k! a_k = 1$;

(2) $f(z) = \frac{1}{k!} z^k + \cdots + a_0 + \frac{1}{(az+b)^m}$;

(3) *If the zeros of* $f(z)$ *are of order* $\geq k+1$, *then* $m = 1$ *in* (2) *and* $f(z) = \frac{(cz+d)^{k+1}}{az+b}$, *where* $c(\neq 0)$, d *are constants.*

Lemma 2([1]). *Let $f(z)$ be a transcendental meromorphic functions of finite order in the plane. If no zeros of f(z) are simple, then $f'(z)$ assumes every non-zero finite complex value infinitely often.*

Lemma 3([9]). *Let f(z) be a transcendental meromorphic function in the plane and $k(\geq 3)$ be a positive integer. Then for any $\in > 0$, we have*

$$(k-2)\,\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) \leq 2\,\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}}\right) + \in T(r,f) + S(r,f).$$

Lemma 4. *Let $f(z)$ be a meromorphic function with finite order, k, $q(\geq 2)$ be two positive integers. If the zeros of $f(z)$ are of multiplicity $\geq k+1$ and $(f^{(k)}(z))^q \neq 1$, then $f(z) \equiv C$.*

Proof. Obviously we have $f^{(k)}(z) \neq 1$. We proceed in the proof step by step as follows.

Step 1. We prove that $f(z)$ is not a transcendental meromorphic function with finite order.

Suppose that $f(z)$ is a transcendental meromorphic function with finite order. If $k=1$, then by Lemma 2, we know that $f'(z)$ assumes 1 infinitely often, which contradicts $f'(z) \neq 1$.

If $k \geq 2$, then by Lemma 3, we have

$$\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) \leq 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k+1)}}\right) + \in T(r,f) + S(r,f). \tag{2.1}$$

From Milloux's inequality, we have

$$T(r,f) \leq \overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right) - N\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f). \tag{2.2}$$

Thus we obtain from (2.1) and (2.2) that

$$T(r,f) \leq 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)}-1}\right) + \in T(r,f) + S(r,f). \tag{2.3}$$

Considering the zeros of $f(z)$ are of order $\geq k+1>2$, and setting $\in = \frac{1}{6}$ in (2.3), we obtain that

$$T(r,f) \leq 6N\left(r, \frac{1}{f^{(k)}-1}\right) + S(r,f). \tag{2.4}$$

Obviously, (2.4) contradicts $f^{(k)}(z) \neq 1$.

Step 2. We prove that $f(z)$ is not a rational function $\frac{q(z)}{p(z)}$, where $q(z)$ and $p(z)$ are comprime polynomials with $\deg p(z) > 0$.

Suppose that $f(z)$ is a rational function $\frac{q(z)}{p(z)}$, where $q(z)$ and $p(z)$ are comprime polynomials with $\deg p(z) > 0$, then by Lemma 1, we have $f(z) = \frac{q(z)}{p(z)} = \frac{(cz+d)^{k+1}}{az+b}$.
Hence $f^{(k)}(z) = 1 + \frac{(-1)^k k! a^k}{(az+b)^{k+1}}$, which contradicts that $(f^{(k)})^q \neq 1$.

Thus by Step 1 and Step 2 we know that $f(z)$ is a polynomial. In the following, we prove that $f(z)$ is a constant.

If $f(z)$ is not a constant, then $f(z)$ is a polynomial with $\deg f(z) \geq k+1$, hence $f^{(k)}(z)$ is a polynomial with $\deg f^{(k)}(z) \geq 1$. Therefore $f^{(k)} = 1$ has solutions, which contradicts $f^{(k)}(z) \neq 1$. Thus we deduce that $f(z) \equiv C$. The proof of the lemma is complete.

Lemma 5([2, 8]). *Let* F *possesses the property that every function* $f \in \mathsf{F}$ *has only zeros of order at least k. If* F *is not normal at a point* z_0, *then for* $0 \leq \alpha < k$, *there exist a sequence of functions* $f_n \in \mathsf{F}$, *a sequence of complex numbers* $z_n \to z_0$ *and a sequence of positive numbers* $\rho_n \to 0$, *such that* $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ *converges locally uniformly to a non-constant function* $g(\zeta)$ *on* C, *and moreover, g is of order at most two, and g has only zeros of order at least k.*

3. Proof of Theorem 1

Without lose of generality we assume that $D = \{|z| < 1\}$. Suppose that F is not normal at point 0. Then by Lemma 5, for $\alpha = k$, there exist

(a) a sequence of functions $f_j \in \mathcal{F}$;

(b) a sequence of complex numbers $z_j \to 0, |z_j| < r < 1$;

(c) a sequence of positive numbers $\rho_j \to 0$

such that $g_j(\zeta) = \rho_j^{-k} f_j(z_j + \rho_j \zeta)$ converges locally uniformly to a non-constant function $g(\zeta)$. Moreover, $g(\zeta)$ is of order at most 2 and only zeros of multiplicity at least $k+1$.

If $(g^{(k)}(\zeta))^q \neq 1$, then by Lemma 4 we deduce that $g(\zeta)$ is a constant, a contradiction. Hence there exists ζ_0 such that $(g^{(k)}(\zeta_0))^q = 1$. Obviously, $g(\zeta_0) \neq \infty$. Hence there exists $\delta > 0$ such that $g(\zeta)$ is analytic on $D_{2\delta} = \{\zeta : |\zeta - \zeta_0| < 2\delta\}$. Thus $g_j^{(i)}(\zeta)$ $(i = 0, 1, 2, 3, \cdots, k)$ are analytic on $D_\delta = \{\zeta : |\zeta - \zeta_0| < \delta\}$ for large j and $g_j^{(i)}(\zeta)$ converges uniformly to $g^{(i)}(\zeta)$ ($i = 0, 1, 2, \cdots, k$) on $\overline{D}_\delta = \{\zeta : |\zeta - \zeta_0| \leq \delta\}$.

As

$$\begin{aligned}
&(g_j^{(k)}(\zeta))^q - 1 \\
&= ((f_j^{(k)}(z_j + \rho_j\zeta))^q + H(f_j(z_j + \rho_j\zeta), \cdots, f_j^{(k)}(z_j + \rho_j\zeta)) - 1 \\
&- H(f_j(z_j + \rho_j\zeta), \cdots, f_j^{(k)}(z_j + \rho_j\zeta)),
\end{aligned}$$

and

$$\begin{aligned}
&H(f_j(z_j + \rho_j\zeta), \cdots, f_j^{(k)}(z_j + \rho_j\zeta)) \\
&= \sum_{i=1}^{n} a_i(z_j + \rho_j\zeta) M_i(f_j(z_j + \rho_j\zeta), \cdots, f_j^{(k)}(z_j + \rho_j\zeta)) \\
&= \sum_{i=1}^{n} a_i(z_j + \rho_j\zeta) \rho_j^{(k+1)\gamma M_j - \Gamma_{M_j}} M_i(g_j(\zeta), \cdots, g_j^{(k)}(\zeta)).
\end{aligned}$$

Considering $a_i(z)$ are analytic on D $(i = 1, 2, \cdots, n)$, we have

$$|a_i(z_j + \rho_j\zeta)| \leq M\left(\frac{1+r}{2}, a_i(z)\right) < \infty, \ (i = 1, 2, \cdots, n),$$

for sufficiently large j.

Hence we deduce from $\frac{\Gamma}{\gamma}\Big|_H < (k+1)$ that

$$\sum_{i=1}^{n} a_i(z_j+\rho_j\zeta)\rho_j^{(k+1)\gamma_{M_j}-\Gamma_{M_j}} \; M_i(g_j(\zeta),\cdots,g_j^{(k)}(\zeta))$$

converges uniformly to 0 on $D_{\frac{1}{2}\delta}=\left\{\zeta : \left|\zeta-\zeta_0\right| < \frac{1}{2}\delta\right\}$.

Thus we know that

$$\left(g_j^{(k)}(\zeta)\right)^q + \sum_{i=1}^{n} a_i(z_j+\rho_j\zeta)\,\rho_j^{(k+1)\gamma_{M_j}-\Gamma_{M_j}} M_i(g_j(\zeta),\,\cdots,\,g_j^{(k)}(\zeta))-1$$

converges uniformly to $(g^{(k)})^q(\zeta)-1$ on $D_{\frac{1}{2}\delta}=\left\{\zeta : \left|\zeta-\zeta_0\right| < \frac{1}{2}\delta\right\}$.

Since

$$\begin{aligned}\left(g_j^{(k)}(\zeta)\right)^q &+ \sum_{i=1}^{n} a_i(z_j+\rho_j\zeta)\,\rho_j^{(k+1)\gamma_{M_j}-\Gamma_{M_j}} M_i(g_j(\zeta),\,\cdots,\,g_j^{(k)}(\zeta))-1 \\ &= (f_j^{(k)}(z_j+\rho_j\zeta))^q + H(f_j(z_j+\rho_j\zeta),\,\cdots,\,f_j^{(k)}(z_j+\rho_j\zeta))-1\neq 0\end{aligned}$$

Hence, by Hurwitz's theorem we deduce that $(g^{(k)}(\zeta))^q \equiv 1$ on $D_{\frac{1}{2}\delta}=\left\{\zeta : \left|\zeta-\zeta_0\right| < \frac{1}{2}\delta\right\}$, thus we have

$$(g^{(k)}(\zeta))^q \equiv 1, \quad \text{for all} \;\; \zeta\in \mathbb{C}.$$

Next we can easily obtain that $g(\zeta)$ is a constant, a contradiction. The proof of the theorem is complete.

4. Proof of Theorem 2

Taking $z_0 \in D$, we separate two cases:

Case 1. There exists $f\in \mathcal{F}$ satisfying $f(z_0)\neq 0,\,1,\,\infty$, then there exists a neighbourhood U of z_0 such that $f(z)\neq 0,\,1,\,\infty$. Hence we deduce from $\{z : f(z) = 0,1,\infty\} = \{z : g(z) = 0,1,\infty\}$ (counting multiplicities) that for any $g(z)\in\mathcal{F}$, $g(z)\neq 0,1,\infty$, for $z\in U$. Thus $\mathcal{F}$ is normal at z_0 by Montel's criterion.

Case 2. There exists $f \in \mathrm{F}$ such that $f(z_0) = 0, 1$ or ∞. Without loss of generality we assume that $f(z_0) = 1$, then there exists $r > 0$ such that $f(z) \neq 0, 1, \infty$ for $z \in D_r^o = \{z : 0 < |z - z_0| < r\}$. Hence for any $g(z) \in \mathrm{F}$, $g(z) \neq 0, 1, \infty$ for $z \in D_r^o$. Thus F is normal in D_r^o. Now we claim that F is normal at z_0. Taking $f_n \in \mathrm{F}$, then there exists subsequence f_{n_k} such that $f_{n_k} \to h(z)$ as $k \to \infty$, for $z \in \{|z - z_0| = \frac{r}{2}\}$. If $h(z) \not\equiv \infty$, then there exist $M > 0$ and $l \in N$ such that $|f_{n_k}(z)| \le M$ for $k > l$, $z \in \{|z - z_0| = \frac{r}{2}\}$. Thus by Maximum modulus theorem we obtain that $|f_{n_k}(z)| \le M$ for $k > l$, $z \in \{|z - z_0| \le \frac{r}{2}\}$. Hence there exists subsequence of $\{f_{n_k}\}$ which converges to $h(z)$ in $\{z : |z - z_0| \le \frac{r}{2}\}$. If $h(z) \equiv \infty$, then exists subsequence f_{n_k} such that $f_{n_k} \to \infty$ as $k \to \infty$, for $z \in \{|z - z_0| = \frac{r}{2}\}$ and $f_{n_k}(z) \neq 0$ for $z \in \{|z - z_0| \le \frac{r}{2}\}$. Hence there exist $M > 0$ and $l \in N$ such that $|f_{n_k}(z)| \ge M$ for $k > l$, $z \in \{|z - z_0| = \frac{r}{2}\}$. Thus by Minimum modulus theorem we obtain that $|f_{n_k}(z)| \ge M$ for $k > l$, $z \in \{|z - z_0| \le \frac{r}{2}\}$. Hence there exists subsequence of $\{f_{n_k}\}$ which converges to ∞ in $\{z : |z - z_0| \le \frac{r}{2}\}$. Therefore F is normal at z_0. Thus we have F is normal in *D*.

5. An counterexample to principle A

Let P be the property such that two meromorphic functions *f* and *g* with the property if and only if $f'(z) - g'(z) \neq 1, 2, 3$ and $f(0) = g(0)$. Obviously, if two meromorphic functions *f* and g satisfy the property *P*, then we can easily deduce that $f(z) \equiv g(z)$. But there is not exist a related normal criterion.

Example 1. Let

$$\mathrm{F} = \{4nz, n = 1, 2, 3, \cdots,\}, \qquad D = \{z : |z| < 1\}.$$

Obviously, any two meromorphic functions $f_n(z) = 4nz$, $f_m(z) = 4mz \in \mathrm{F}$ we know that $f_n'(z) - f_m'(z) \neq 1, 2, 3$ and $f_n(0) = f_m(0)$, that is, $f_n(z)$ and $f_m(z)$ have the property *P*. But F is not normal in *D*.

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