BULLETIN of the
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **23** (2001) 163-171

On a Class of Functions whose Derivatives Map the Unit Disc into a Half Plane

DAUD MOHAMAD
Universiti Teknologi MARA, Kampus Bukit Sekilau, 25200 Kuantan, Pahang, Malaysia

Abstract. Let $\mathrm{G}(\alpha, \delta)$ denote the class of functions *f*, $f(0) = f(0) - 1 = 0$ for which $\mathrm{Re}\ e^{i\alpha} f'(z) > \delta$ in $D = \{z : |z| < 1\}$ where $|\alpha| \leq \pi$ and $\cos\ \alpha - \delta > 0$. We discuss some basic properties of the class including representation theorem, extremals and argument of $\mathrm{G}(\alpha, \delta)$.

1. Introduction

We denote $\mathrm{G}(\alpha, \delta)$ the class of normalized analytic functions f in the unit disc D where

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

satisfying $\mathrm{Re}\ e^{i\alpha} f'(z) > \delta$ where $|\alpha| \leq \pi$ and $\cos\ \alpha - \delta > 0$.

Many of the classes $\mathrm{G}(\alpha, \delta)$ have been studied by several researchers such as MacGregors [3] for $\mathrm{G}(0, 0)$, Goel and Mehrok [1] for $\mathrm{G}(\alpha, \delta)(\delta \geq 0)$ and Silverman and Silvia [4] for $\mathrm{G}(\alpha, 0)$. Writing

$$p(z) = \frac{e^{i\alpha} f'(z) - i\ \sin\alpha - \delta}{\cos\alpha - \delta} \qquad (z \in D), \tag{1}$$

clearly $f \in \mathrm{G}(\alpha, \delta)$ if and only if $p \in P$, the class of functions with positive real parts.

Solving (1) for $f'(z)$ yields

$$f'(z) = e^{-i\alpha}(Ap(z) + i\ \sin\alpha + \delta) \qquad (z \in D) \tag{2}$$

where $A = \cos\alpha - \delta$.

2. Representation theorem

We obtain the representation theorem for $G(\alpha,\delta)$, sharing the same approach through Herglotz Representation Theorem for functions in *P*.

Theorem 2.1. *Let* $f \in g(\alpha,\delta)$. *Then for some probability measure* μ *on the unit circle X,*

$$f(z) = \int_X \left[-e^{-i\alpha}(e^{-i\alpha} - 2\delta)z - 2e^{-i\alpha}A\bar{x}\log(1-xz)\right]d\mu(x). \tag{3}$$

Conversely, if f is given by the above equation, then $f \in G(\alpha,\delta)$.

Proof. For some probability measure μ on the circle *X*,

$$p \in P \;\Leftrightarrow\; p(z) = \int \frac{1+xz}{1-xz}\, d\mu(x).$$

Using (2), we have

$$f'(z) = e^{-i\alpha}\left[A\int \frac{1+xz}{1-xz} + i\sin\alpha + \delta\right] d\mu(x)$$

and so

$$f(z) = e^{-i\alpha}\left[\int_0^z \left(\int_X A\left(\frac{1+x\psi}{1-x\psi}\right) + (i\sin\alpha + \delta)\, d\mu(x)\right) d\psi\right]$$

$$= \int_0^z \left[\int_X \frac{1 + (e^{-i2\alpha} - 2\delta e^{-i\alpha})x\psi}{1-x\psi}\, d\mu(x)\right] d\psi \tag{4}$$

$$= \int_0^z \left[\int_X -e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2e^{-i\alpha}A}{1-x\psi}\, d\mu(x)\right] d\psi$$

and the desired representation theorem is obtained by reversing the order of integration and integrating with respect to ψ.

We note that the extreme points of $G(\alpha, \delta)$ are the unit point masses

$$f_x(z) = -e^{-i\alpha}(e^{-i\alpha} - 2\delta)z - 2e^{-i\alpha}A\bar{x} \log(1 - xz)$$

with $|x| = 1$ and the derivatives of the extreme points for $G(\alpha, \delta)$ are the point masses

$$f_x(z) = \frac{1+(e^{-i2\alpha} - 2\delta e^{-i\alpha})xz}{1- xz}, \quad |x| = 1 .$$

3. Extremal properties

Following Silverman and Silvia [4], we now obtain a coefficient bound for functions in $g(\alpha, \delta)$ and distortion theorems for the derivatives of these functions.

Theorem 3.1. *If* $f \in G(\alpha, \delta)$, *then* $|a_n| \le 2A/n$, $n = 2, 3, 4, \cdots$ *and equality is attained for each n when f is an extreme point of* $G(\alpha, \delta)$.

Proof. Using (4) and since $1/(1-x\psi) = \sum_0^\infty (x\psi)^n$, we can write

$$f(z) = z + 2e^{-i\alpha}A\int_X \sum_{n=2}^{\infty} x^{n-1}\, d\mu(x)\, \frac{z^n}{n} .$$

Now, let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then $a_n = \frac{2e^{-i\alpha}A}{n}\int_X x^{n-1}\, d\mu(x)$ and the result follows immediately.

Our further result will be based on the following theorem.

Theorem 3.2. *Let* $f \in G(\alpha, \delta)$. *Then* f' *maps* $|z| \le r$ *into the disc* D_r *with center* $-e^{-i\alpha}(e^{-i\alpha} - 2\delta) + (2e^{-i\alpha}A)/(1 - r^2)$ *and radius* $2Ar/(1 - r^2)$.

Proof. If a and b are complex numbers with $|b| < 1$, and if $0 < r < 1$, the range of the function $(1 + ar\omega)/(1 + br\omega)$ $(|\omega| \le 1)$ is the disc with center and radius

$$\frac{1 - a\bar{b}r^2}{1 - |b|^2 r^2}, \quad \frac{|a - b|r}{1 - |b|^2 r^2}$$

respectively. By taking $a = (e^{-i2\alpha} - 2\delta\, e^{-i\alpha})xr$ and $b = xr$ where $|x| = 1$, we see that

$$\frac{1 + (e^{-i2\alpha} - 2\delta\, e^{-i\alpha})xz}{1 - xz}$$

maps $|z| \le r$ onto D_r. By convexity, any linear combination of functions of this form also maps D onto D_r. Since for some probability measure μ, we have

$$f'(z) = \int_X \frac{1 + (e^{-i2\alpha} - 2\delta\, e^{-i\alpha})xz}{1 - xz}\, d\mu(x),$$

the stated result now follows.

Theorem 3.3. *If* $f \in \mathrm{G}(\alpha, \delta)$, *then*

$$\frac{1 + r^2(2A(A+\delta)-1) - 2rA}{1 - r^2} \le \operatorname{Re} f'(z) \le \frac{1 + r^2(2A(A+\delta)-1) + 2rA}{1 - r^2} \tag{5}$$

and

$$\frac{-2Ar(1 + r\sqrt{1 - (A+\delta)^2})}{1 - r^2} \le \operatorname{Im} f'(z) \le \frac{2Ar(1 + r\sqrt{1 - (A+\delta)^2})}{1 - r^2}.$$

All bounds are sharp for any extreme point f *of* $\mathrm{G}(\alpha, \delta)$.

Proof. By Theorem 3.2, we can write

$$\left| f'(z) - \left\{ -e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2e^{-i\alpha}A}{1 - r^2} \right\} \right| \le \frac{2Ar}{1 - r^2} \tag{6}$$

so that

$$\frac{-2Ar}{1 - r^2} \le \operatorname{Re}\left\{ f'(z) + e^{-i\alpha}(e^{-i\alpha} - 2\delta) - \frac{2e^{-i\alpha}A}{1 - r^2} \right\} \le \frac{2Ar}{1 - r^2}$$

and also

$$\frac{-2Ar}{1 - r^2} \le \operatorname{Im}\left\{ f'(z) + e^{-i\alpha}(e^{-i\alpha} - 2\delta) - \frac{2e^{-i\alpha}A}{1 - r^2} \right\} \le \frac{2Ar}{1 - r^2}.$$

The results are obtained by simplifying the above inequalities.

We note that if $f \in G(\alpha, \delta)$, then since $f_0'(0) = 1$, we have $\text{Re}\, f'(z) > 0$ for $|z| < \rho$ and some ρ in $(0, 1]$. However if

$$f_o(z) = \frac{1 + (e^{-i2\alpha} - 2\delta\, e^{-i\alpha})z}{1 - z}, \quad (z \in D),$$

then the left side of inequality (5) is sharp so that

$$(1 - r^2)\,\text{Re}\, f_0'(-r) = 1 + r^2(2A(A + \delta) - 1) - 2rA \to 2(\cos\alpha - \delta)(\cos\alpha - 1) \ (r \to 1)$$

and the last expression is negative if $|\alpha| \neq 0$. This shows that $\rho \neq 1$ in general, and it is natural to ask for the best possible value of ρ. We answer this question in the following application of Theorem 3.2

Theorem 3.4. *Let* $f \in g(\alpha, \delta)$ *and put* $\rho = 1/(A + \sqrt{1 - A(2\delta + A)})$. *Then* $0 < \rho \leq 1$ *and* $\text{Re}\, f'(z) \geq 0$ *for r* $|z| < \rho$. *If* $\rho \leq r \leq 1$, *then* $\text{Re}\, f_0'(z) < 0$ *for some z on* $|z| < r$.

Proof. Let $f \in G(\alpha, \delta)$ and define ρ as above. Obviously $\rho > 0$ since $A > 0$, and $1 - A(2\delta + A) = 1 + \delta^2 - \cos\alpha \geq 0$. The inequality $\rho \leq 1$ is equivalent to $A + \sqrt{1 - A(2\delta + A)} \geq 1$ and this is obviously true if $A \geq 1$. If $A < 1$, it is true if and only if $1 - A(2\delta + A) \geq (1 - A)^2$, and thus reduces to the trivially true inequality $\cos\alpha \leq 1$. So in both cases, $\rho \leq 1$.

Now, put $\sigma(x) = (2A(A + \delta) - 1)x^2 - 2x + 1$ for real values of *x*. From (5), we have $(1 - r^2)\,\text{Re}\, f'(z) \geq \sigma(r)$ $(0 \leq |z| = r < 1)$ with equality for each *r* when $f = f_o$ and z is a suitable value on $|z| = r$. To prove the theorem, it is sufficient to show that $\sigma(x)$ is positive on $[0, \rho)$ and non-positive on $[\rho, 1]$.

If $2A(A + \delta) = 1$, so that $\sigma(x)$ is linear in *x*, then $\rho = 1/(2A)$ and it is clear that $\sigma(x)$ is positive on $[0, \rho)$ and non-positive on $[\rho, 1]$. When $2A(A + \delta) \neq 1$, $\sigma(x)$ is quadratic and has zeros

$$x = \frac{A \pm \sqrt{1 - A(2\delta + A)}}{2A(A + \delta) - 1} = \frac{1}{A \mp \sqrt{1 - A(2\delta + A)}}. \tag{7}$$

One of the zeros is ρ. Let the other zero be μ, If $2A(A + \delta) < 1$, then $\mu\rho < 0$ and (7) shows that $\mu < 0$ and $\rho > 0$. Since σ is concave, $\sigma(x)$ is positive on $[0, \rho)$ and non-positive on $[\rho, 1]$. If $2A(A+\delta)>1$, then $\mu, \rho > 0$ since $\mu\rho>0$, $\mu+\rho>0$. Also $\rho<\mu$ by (5). In this case σ is convex so $\sigma(x)$ is positive on $[0,\rho)$ and non-positive on $[\rho,\mu]$. In particular, since $\sigma(1) = 2A(\cos\alpha - 1) \le 0$, $\sigma(x)$ is non-positive on $[\rho, 1]$. This completes the proof.

We next obtain a distortion theorem for $\mathsf{G}(\alpha,\delta)$.

Theorem 3.5. *If* $f \in \mathsf{G}(\alpha,\delta)$, *then*

$$\left| f'(z) \right| \le C(r) + \frac{2Ar}{1-r^2}$$

where

$$C(r) = \sqrt{\frac{4Ar^2}{1-r^2}\left(\frac{A}{1-r^2}+\delta\right)+1} \tag{8}$$

and the bound is sharp for any extreme point f *of* $\mathsf{G}(\alpha,\delta)$.

Proof. Let $\Gamma(r) = -e^{-i\alpha}(e^{-i\alpha}-2\delta) + \dfrac{2e^{-i\alpha}A}{1-r^2}$. By using (6) we have

$$\begin{aligned} \left| f'(z) \right| &\le \left|\Gamma(r)\right| + \frac{2Ar}{1-r^2} \\ &= C(r) + \frac{2Ar}{1-r^2} \end{aligned}$$

as required.

4. Argument of $f'(z)$

We see that if $\delta \ge 0$, then f' is non-zero throughout *D*, and has continuous argument. But if $\delta<0$, and if f_o is any extreme function of $\mathsf{G}(\alpha,\delta)$, then at some point of *D*, f_0' has a zero and hence no argument. So to obtain result for argument of f', we restrict the values of $|z|$ considered in the case $\delta<0$. We will also use the following property for argument: for a given α in $[-\pi, \pi]$ and as *x* varies in some interval $[0, c]$, so that $e^{i\alpha}+x \ne 0$, $\phi_\alpha(x)$ is the continuous argument of $e^{i\alpha}+x$, for which $\phi_\alpha(0)=\alpha$. We have

$$\phi_\alpha(x) = \begin{cases} \tan^{-1}\left(\dfrac{\sin\alpha}{\cos\alpha + x}\right) & , \text{ if } x+\cos\alpha > 0 \\ \pi + \tan^{-1}\left(\dfrac{\sin\alpha}{\cos\alpha + x}\right) & , \text{ if } x+\cos\alpha < 0 \\ \pi/2 & , \text{ if } x+\cos\alpha = 0 \end{cases}$$

when $0 < \alpha < \pi$, and similar formulae for the case $-\pi < \alpha < 0$, $\alpha = 0$, $\pm\pi$.

Theorem 4.1. *Let* $f \in \mathrm{G}(\alpha, \delta)$, *and put* $x(r) = 2Ar^2 / (1-r^2)$ $(0 \le r < 1)$. *Let*

$$r_o = \begin{cases} 1 & , \quad \delta \ge 0 \\ \dfrac{1}{\sqrt{1-4A\delta}} & , \quad \delta < 0. \end{cases}$$

Then, for $0 < |z| = r < r_o$, *and for suitable determination of argument*

$$\left|\arg f'(z) + \alpha - \phi_\alpha(x(r))\right| \le \sin^{-1}\frac{2Ar}{(1-r^2)C(r)} \tag{9}$$

where $\phi_\alpha(x)$ *is defined on* $[0, x(r_o))$ *as above and* $C(r)$ *is given by* (8).

Proof. We restrict the value of $|z| = r$ by the condition

$$\left|\frac{2A}{1-r^2} + 2\delta - e^{-i\alpha}\right| > \frac{2Ar}{1-r^2}$$

to ensure that $f'(z) \ne 0$. Squaring both sides and simplifying, we have

$$\frac{4A\delta}{1-r^2} - 4A\delta + 1 > 0.$$

The inequality holds for all r in $[0,1)$ if $\delta \ge 0$ and for $0 \le r < 1 / \sqrt{1-4\delta A}$ if $\delta < 0$. This establishes the restriction on $|z|$. By using (6) and Theorem 3.5, we deduce that

$$\left|\arg f'(z) - \arg\Gamma(r)\right| \le \sin^{-1}\frac{2Ar}{(1-r^2)C(r)} \tag{10}$$

and also

$$\arg\Gamma(r) = \arg\left[-e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2e^{-i\alpha}A}{1 - r^2}\right]$$
$$= -\alpha + \arg\left[e^{i\alpha} + \frac{2Ar^2}{1 - r^2}\right].$$

Put $x(r) = 2Ar^2/(1-r^2)$, then $\arg\Gamma(r) = -\alpha + \phi_\alpha(x(r))$ and the desired result follows using (10).

We obtain another result for argument of $G(\alpha,\delta)$, features $\arg(f'(z) + k)$ for some real k that satisfy $f'(z) + k \neq 0$ for $z \in D$ and for all $f \in G(\alpha, \delta)$. When $|\alpha| = \pi/2$, such a choice is impossible, for if f_o is an extreme function in $G(\alpha, \delta)$, then $f_0'(z) + k$ maps D onto either $\operatorname{Im} w > \delta$ or $\operatorname{Im} w < -\delta$ and since $\delta < 0$ both these half planes contain 0. If $|\alpha| \neq \pi/2$, any choice of k with $k\cos\alpha + \delta > 0$ ensures that $f_0'(z) + k \neq 0$ for $z \in D$, $f \in G(\alpha,\delta)$.

In the statement of the following theorem, for a given $\alpha \in [-\pi, \pi]$, and as x varies in some interval $[0, c)$, so that $(k+1)e^{i\alpha} + x \neq 0$, $\psi_\alpha(\alpha)$ is the continuous argument of $(k+1)e^{i\alpha} + x$ for which $\psi_\alpha(0)$ is principal.

Theorem 4.2. *Let* $f \in G(\alpha, \delta)$, *where* $|\alpha| \neq \pi/2$. *Put* $x(r) = 2A/(1 - r^2)\ (0 \leq r < 1)$ *and let k be a real number such that* $k\cos\alpha + \delta > 0$. *Then*

$$\left|\arg(f'(z) + k) + \alpha - \psi_\alpha(x(r))\right| \leq \sin^{-1}\frac{2Ar}{(1 - r^2)C_1(r)}$$

where $\psi_\alpha(\alpha)$ *is defined on* $[0,\infty)$ *as above, and*

$$C_1(r) = \sqrt{\frac{4Ar^2}{1 - r^2}\left(\frac{A}{1 - r^2} + k\cos\alpha + \delta\right) + (k+1)^2}\,. \tag{11}$$

Proof. Let $|\alpha| \neq \pi/2$, and let k satisfy $k\cos\alpha + \delta > 0$. We have, using (6),

$$\left| f'(z) + k - (\Gamma(r) + k)\right| \leq \frac{2Ar}{1 - r^2}$$

where

$$\Gamma(r) = -e^{-i\alpha}(e^{-i\alpha} - 2\delta) + \frac{2e^{-i\alpha}A}{1-r^2} = 1 + \frac{2Ar^2}{1-r^2}e^{-i\alpha}.$$

Hence

$$\left|\arg(f'(z) + k) - \arg(\Gamma(r) + k)\right| \le \sin^{-1}\frac{2Ar}{(1-r^2)C_1(r)} \tag{12}$$

where $C_1(r) = \left|\Gamma(r) + k\right|$ and is written as in (11). Now

$$\arg(\Gamma(r) + k) = -\alpha + \arg\left[2\delta - e^{-i\alpha} + \frac{2A}{1-r^2} + ke^{i\alpha}\right] = -\alpha + \psi_\alpha(x(r))$$

and the proof is complete by using (12).

References

1. R.M. Goel and B.S. Mehrok, A class of univalent functions, *J. Austral. Maths Soc.* (Series A) **35** (1983), 1-17.
2. A.W. Goodman, *Univalent Functions*, Vol. I and II, Mariner Publishing Co. Inc. Tampa, Florida, 1983.
3. T.H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.* **104** (1962), 532-537.
4. H. Silverman and E.M. Silvia, On α-close to convex function, *Publ. Math. Debrecen*, **49** (1996), 532-537.