BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **23** (2000) 173-179

Harmonic Curvatures in Lorentzian Space

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Abstract. In the Lorentzian space, a regular curve is called time-like, space-like and null curve according to velocity vector's status [3], [6]. In this paper, we obtained the Harmonic curvatures for a regular curve in the Lorentzian space. We also obtained the relationship between the Harmonic curvatures and their own derivations.

1. Introduction

In n-dimensional Euclidean Space, a regular curve is described by its curvatures. If all curvatures of a curve are identically zero, then the curve is a geodesic. If only the first curvature is a non-zero constant and others are all identically zero, then the curve is called a circle. If the first and second curvatures are non-zero constants and others are all identically zero, the curve is called a helix.

A regular curve is called general helix if its first and second curvatures k_1, k_2 are not constant, but $\frac{k_1}{k_2}$ is constant [5].

The $\frac{k_1}{k_2}$ ratio is called First Harmonic Curvature of the curve and is shown H_1. Higher harmonic curvatures of a regular curve are defined as follows.

$$H_i := \begin{cases} \dfrac{k_1}{k_2} & , \quad i = 1 \\ \left\{ V_i\left[H_{i-1}\right] + H_{i-2}k_i \right\} \dfrac{1}{k_{i+1}} & , \quad 1 < i \leq n - 2 \end{cases}$$

where V_i is Frenet-frame and k_i is higher curvature of a regular curve. H_0 is assumed as zero.

Harmonic curvatures have important role in characterizations of General Helices. The most importantly, the subject has not been investigated in Lorentzian Space.

The aim of the present work is to define those Harmonic Curvatures in Lorentzian Space.

2. Preliminaries

2.1. Symmetric bilinear forms

Let V be a real vector space. A bilinear form on V is an R-bilinear function $\langle,\rangle : V \times V \to R$ and we consider only the symmetric case $\langle v, w\rangle = \langle w, v\rangle$ for all $v,\ w \in V$. A symmetric bilinear form $\langle\,,\rangle$ on V is [6];

(a) positive [negative] definite provided $v \neq 0$ implies $\langle v, v\rangle > 0\left[< 0\right]$,
(b) positive [negative] semi definite provided $\langle v, v\rangle \geq 0 \left[\leq 0\right]$
(c) nondegenerate provided $\langle v, w\rangle = 0$ for all $w \in V$ implies $v \neq 0$.

If $\langle\,,\rangle$ is a symmetric bilinear form on $\langle\,,\rangle$ then for any subspace W of V the restriction $\langle\,,\rangle\big|_{W\times W}$ denoted merely by $\langle\,,\rangle\big|_W$, is again symmetric and bilinear. If $\langle\,,\rangle$ is [semi-] definite, so is $\langle\,,\rangle\big|_W$.

The index ν of a symmetric bilinear form $\langle\,,\rangle$ on V is the largest integer that is the dimension of a subspace $W \subset V$ on which $\langle\,,\rangle\big|_W$ is negative definite.

Thus $0 \leq \nu \leq \dim V$ and $\nu = 0$ if and only if $\langle\,,\rangle$ is positive semidefinite [6].

A scalar product $\langle\,,\rangle$ on a vector space V is a nondegenerate symmetric bilinear form on V.

If a symmetric bilinear form $\langle\,,\rangle$ on V is nondegenerate, then it is called a scalar product on V. A scalar product space $\nu \neq 0$ has an orthonormal basis and

$$\left\langle e_i, e_j\right\rangle = \delta_{ij}\,\varepsilon_j \quad \text{where} \quad \varepsilon_j = \left\langle e_j, e_j\right\rangle = \pm 1 .$$

For any orthonormal basis $e_1,\ e_2,\ \cdots,\ e_n$ for V the number of negative signs in the signature $(\varepsilon_1,\ \varepsilon_2, \cdots, \varepsilon_n)$ is the index ν of V.

If V has index $(\nu\ \ 0 \leq \nu \leq n)$, for two vectors v_p and w_p , we can write

$$\left\langle v_p, w_p\right\rangle = -\sum_{i=1}^{\nu} v^i w^i + \sum_{j=\nu+1}^{n} v^j w^j$$

The resulting Semi-Euclidean Space R_ν^n reduces to R^n if $\nu = 0$.

A Lorentz vector space L^n to be a scalar product space of index 1 and dimension ≥ 2 [6].

Fix the notation;

$$\varepsilon_{i-1} = \begin{cases} -1 & \text{for } 0 \le i-1 \le \nu - 1 \\ 1 & \text{for } \nu \le i-1 \le n-1 \end{cases} .$$

3. Curves

A curve in a Lorentzian space L^n is a smooth mapping

$$\gamma : I \to L^n$$

where I is an open interval in the real line R. The interval I has a coordinate system consisting of the identity map u of I. The velocity vector of γ at $t \in I$, is

$$\gamma'(t) = \frac{d\gamma(u)}{du}\Big|_t$$

A curve γ is said to be regular if $\gamma'(t) \neq 0$ for all $t \in I$. A curve γ in a Lorentzian space L^n is said to be space-like if it's velocity vectors γ' are space-like for all $t \in I$, similarly for time-like and null.

If γ is a space-like or time-like curve,we can reparametrize it such that $\langle \gamma'(t), \gamma'(t) \rangle = \varepsilon_0$ (where $\varepsilon_0 = +1$ if γ is space-like and $\varepsilon_0 = -1$ if γ is time-like respectively). In this case γ is said to be unit speed or it has arc length parametrization [3], [6].

Definition 3.1. *Let γ be a curve in L^n, parametrized by it own arc length. Denoting the Frenet vector fields of this curve $V_1(s), V_2(s), \cdots, V_r(s)$. Assuming;*

$$\frac{dV_i(s)}{ds} = \sum_{j=1}^{r} k_{ij}(s) V_j(s) \quad , \quad i, j = 1, \cdots, r$$

the functions defined by the equality

$$k_{ij}(s) = \varepsilon_{j-1} < \frac{dV_i(s)}{ds}, V_j(s) >$$

are called the higher ordered curvatures of the curve γ. Where

$$\varepsilon_{j-1} k_{ij}(s) = -\varepsilon_{i-1} k_{ji}(s) \qquad [1]$$

Theorem 3.1. ([1], [3]) *Let* $\gamma \subset L^n$ *be a regular curve coordinate neighborhood* (I,γ) *and* $\{V_1(s), V_2(s), \cdots, V_r(s)\}$ *be the Frenet r-frame at* $\gamma(s)$ *with* $s \in I$ *. Then;*

(a) $V_1'(s) = k_{12}(s)V_2(s)$

(b) $V_i'(s) = \varepsilon_{i-2}\,\varepsilon_{i-1}\,k_{i(i-1)}(s)V_{i-1}(s) + k_{i(i+1)}(s)V_{i+1}(s)$

(c) $V_r'(s) = \varepsilon_{r-2}\,\varepsilon_{r-1}\,k_{r(r-1)}(s)V_{r-1}(s)$.

We can write matrix representation as follows

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \\ . \\ . \\ . \\ V_{r-2}' \\ V_{r-1}' \\ V_r' \end{bmatrix} = \begin{bmatrix} 0 & k_{12} & 0 & . & . & . & 0 & 0 \\ \varepsilon_0\varepsilon_1 k_{21} & 0 & k_{23} & . & . & . & 0 & 0 \\ 0 & \varepsilon_1\varepsilon_2 k_{32} & 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & k_{(r-2)(r-1)} & 0 \\ 0 & 0 & 0 & . & . & . & 0 & k_{(r-1)r} \\ 0 & 0 & 0 & . & . & . & \varepsilon_{i-2}\varepsilon_{i-1}k_{r(r-1)} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ . \\ . \\ . \\ V_{r-2} \\ V_{r-1} \\ V_r \end{bmatrix}$$

for $n = 3$ in the matrix representation, we get

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} = \begin{bmatrix} 0 & k_{12} & 0 \\ -\varepsilon_0\varepsilon_1 k_{12} & 0 & k_{23} \\ 0 & -\varepsilon_1\varepsilon_2 k_{23} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}.$$

In particular when the curve is “time-like” we get

$$\langle V_1, V_1 \rangle = \varepsilon_0 = -1 \ , \ \langle V_2, V_2 \rangle = \varepsilon_1 = +1 \ , \ \langle V_3, V_3 \rangle = \varepsilon_2 = +1$$

therefore we obtain,

$$\begin{bmatrix} V_1' \\ V_2' \\ V_3' \end{bmatrix} = \begin{bmatrix} 0 & k_{12} & 0 \\ k_{12} & 0 & k_{23} \\ 0 & -k_{23} & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

4. Harmonic curvatures

Definition 4.1. *Let γ be a time-like curve in L^n and V_1 be the first Frenet vector field of γ. $X \in \chi(L^n)$ being a constant unit vector field, if*

$$\langle V_1, X \rangle = \cosh\ \varphi \quad \textit{(constant)}$$

then γ is called an general helix (inclined curves) in L^n. φ is called slope angle and the space $Sp\{X\}$ is called slope axis.

Definition 4.2. *Let $\gamma : I \to L^n$ be a general helix, parametrized by its arc length. Let X be an unit and constant vector field of L^n and let $\{V_1, V_2, \cdots, V_r\}$ be Frenet r-frame at the point of $\gamma(s)$ of γ. If we consider the angle between γ' and X as φ;*

$$H_j : I \to R$$
$$\langle V_{j+2}, X \rangle = H_j \cosh\ \varphi$$

then the value of the H_j function at the point of $\gamma(s)$ is called as the j-th harmonic curvature according to X at the point of $\gamma(s)$ of γ.

Theorem 4.1. *Let γ be a general helix (inclined curve) in L^n, parametrized by its arc length, k_{ij} (all $k_{ij} \neq 0$) be the higher ordered curvatures. When H_j is the j-th harmonic curvature at $\gamma(s)$ of γ.*
Then

$$H_1 = \varepsilon_0 \varepsilon_1 \frac{k_{12}}{k_{23}}$$
$$H_i = \left(H'_{i-1} + \varepsilon_{i-1}\varepsilon_i k_{i(i+)} H_{i-2}\right) \frac{1}{k_{(i+1)(i+2)}} \quad , \quad 2 \le i \le n-2$$

Proof. If we take the derivative of the equation

$$\langle V_1, X \rangle = \cosh\ \varphi \quad \text{(constant)}$$

we get

$$\langle V_1', X \rangle = 0$$

or

$$\left\langle k_{12} V_2 , X \right\rangle = 0 \Rightarrow \left\langle V_2 , X \right\rangle = 0$$

from this again take the derivative

$$\left\langle V_2' , X \right\rangle = 0$$

from this replacing the V_2''s value we get,

$$\left\langle -\varepsilon_0 \varepsilon_1 k_{12} V_1 + k_{23} V_3 , X \right\rangle = 0$$

or

$$-\varepsilon_0 \varepsilon_1 k_{12} \left\langle V_1 , X \right\rangle + k_{23} \left\langle V_3 , X \right\rangle = 0$$

or

$$-\varepsilon_0 \varepsilon_1 k_{12} \cosh \varphi + k_{23} H_1 \cosh \varphi = 0$$

thus we have

$$H_1 = \varepsilon_0 \varepsilon_1 \frac{k_{12}}{k_{23}} .$$

From Definition 4.1. we can write,

$$\left\langle V_{i+1} , X \right\rangle = H_{i-1} \cosh \varphi . \tag{1}$$

If we take derivations of (1)

$$\left\langle V_{i+1}' , X \right\rangle = H_{i-1}' \cosh \varphi$$

using the value of V_{i+1}' again from this. We get,

$$\left\langle \varepsilon_{i-1} \varepsilon_i k_{(i+1)i} V_i + k_{(i+1)(i+2)} V_{i+2} , X \right\rangle = H_{i-1}' \cosh \varphi$$

or

$$\varepsilon_{i-1} \varepsilon_i k_{(i+1)i} \left\langle V_i , X \right\rangle + k_{(i+1)(i+2)} \left\langle V_{i+2} , X \right\rangle = H_{i-1}' \cosh \varphi$$

or

$$\varepsilon_{i-1} \varepsilon_i k_{(i+1)i} H_{i-2} \cosh \varphi + k_{(i+1)(i+2)} H_i \cosh \varphi = H_{i-1}' \cosh \varphi .$$

Thus we have

$$H_i = (H'_{i-1} + \varepsilon_{(i-1)}\varepsilon_i k_{i(i+1)} H_{i-2})\frac{1}{k_{(i+1)(i+2)}}$$

this completes the proof of theorem.

Finally we obtain the relationship between H'_i and H_i values in the matrix form

$$\begin{bmatrix} H'_1 \\ H'_2 \\ H'_3 \\ \vdots \\ H'_{n-4} \\ H'_{n-3} \\ H'_{n-2} \end{bmatrix} = \begin{bmatrix} 0 & k_{34} & 0 & \cdots & 0 & 0 & 0 \\ -\varepsilon_2\varepsilon_3 k_{34} & 0 & k_{45} & \cdots & 0 & 0 & 0 \\ 0 & -\varepsilon_3\varepsilon_4 k_{45} & 0 & \cdots & 0 & 0 & 0 \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ . & . & . & \cdots & . & . & . \\ 0 & 0 & 0 & \cdots & 0 & k_{(n-2)(n-1)} & 0 \\ 0 & 0 & 0 & \cdots & -\varepsilon_{(n-3)}\varepsilon_{(n-2)}k_{(n-1)n} & 0 & k_{(n-1)n} \\ 0 & 0 & 0 & \cdots & 0 & -\varepsilon_{(n-2)}\varepsilon_{(n-1)}k_{(n-1)n} & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ . \\ . \\ . \\ H_{n-4} \\ H_{n-3} \\ H_{n-2} \end{bmatrix}$$

for $n = 3$ in the matrix representation, we get

$$\begin{bmatrix} H'_1 \\ H'_2 \\ H'_3 \end{bmatrix} = \begin{bmatrix} 0 & k_{34} & 0 \\ -\varepsilon_2\varepsilon_3 k_{34} & 0 & k_{45} \\ 0 & -\varepsilon_3\varepsilon_4 k_{45} & 0 \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}$$

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