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Fixed Point Approximation of Weakly Commuting Mappings in Banach Space

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Abstract. A fixed point theorem for a pair of weakly commuting mappings using Mann iterative process is presented. In the process the results due to Rhoades, Khan–Imdad and Ghosh are improved.

1. Introduction

The Mann iterative process (cf.[8]) associated with a self mapping T of a Banach space X is described as follows:

For $x_0 \varepsilon X$ we define $x_{n+1} = (1-c_n)x_n + c_n Tx_n$ for $n > 0$, where $\{c_n\}$ satisfies

(i) $c_0 = 1$
(ii) $0 \leq c_n < 1$ for $n > 0$
(iii) $\sum c_n$ diverges.

The above scheme has been used by many authors e.g. Bose and Mukherji [1], Das *et al.* [2], Dotson and Senter [3], Emmanuelle [4], Ishikawa [6], Massa [9] and Rhoades [11] etc.

Using Mann iterative process described above Rhoades [11] has proved the following generalization of a theorem of Pal and Maiti [10].

Theorem 1. *Let T be a self mapping of a Banach space X such that*

$$\| x - Tx \| + \| y - Ty \| \leq \alpha \| x - y \|$$

holds for all x,y in X where $1 < \alpha < 2$. *Let* $\{x_n\}$ *be a sequence of Mann iterates associated with T and with* $\{c_n\}$ *satisfying* (*i*) *and* (*ii*) *and* $\lim c_n = h > 0$ *instead of* (*iii*). *If* $\{x_n\}$ *converges then it converges to the fixed point of T.*

The intent of this note is to present yet another extension of Theorem 1 which in turn generalizes earlier results due to Ghosh [5] and Khan-Imdad [7].

2. Main result

Before presenting our result, we recall that a pair of self-mapping $\{G,T\}$ of a normed linear space X is said to be weakly commuting (cf. [12]) if

$$\| GTx - TGx \| \leq \| Tx - Gx \| \quad \text{for all} \quad x \,\varepsilon\, X.$$

Clearly a commuting pair is weakly commuting but the converse need not be true in general.
Now we prove the following:

Theorem 2. *Let G and T be two self mappings of a Banach space X such that the inequality*

$$\| Gx - Tx \| + \| Gy - Ty \| \leq \alpha \| Gx - Gy \| + \beta \left[\| Gx - Ty \| + \| Gy - Tx \| \right]$$

holds for all x,y in X, where $\beta < h$, $(1-\beta/h) < \alpha < 2(1-\beta/h)$, $\lim c_n = h > 0$.
Then the sequence $\{Gx_n\}$ *defined by*

$$Gx_{n+1} = (1-c_n)\, Gx_n + c_n Tx_n \tag{2.2}$$

converges to a point $p \,\varepsilon\, X$ *where* $\{c_n\}$ *is a decreasing sequence and enjoys the properties of Theorem 1. Also if* $\{G,T\}$ *is a weakly commuting pair and G is continuous at* $\{p\}$ *then p is a coincidence point of G and T. Further if* $G^2 p = Gp$ *then G and T have a common fixed point.*

Proof. In order to prove that $\{Gx_n\}$ is convergent, we show that it is a Cauchy sequence. Consider for $n > 0$

$$\begin{aligned} &\| Gx_{n+1} - Gx_n \| + \| Gx_n - Gx_{n-1} \| \qquad (2.3) \\ &\quad = c_n \| Gx_n - Tx_n \| + c_{n-1} \| Gx_{n-1} - Tx_{n-1} \| \\ &\quad \leq \alpha \| Gx_n - Gx_{n-1} \| + \beta \left[\| Gx_n - Tx_{n-1} \| + \| Gx_{n-1} - Tx_n \| \right] \end{aligned}$$

Now using (2.2) we have

$$\| Gx_n - Tx_{n-1} \| = \| Gx_n - 1/c_{n-1}\{ Gx_n - (1-c_{n-1})\, Gx_{n-1}\} \| \tag{2.4}$$
$$= \frac{1-c_{n-1}}{c_{n-1}} \| Gx_n - Gx_{n-1} \| \le \frac{(1-h)}{h} \| Gx_n - Gx_{n-1} \|$$

and

$$\|Gx_{n-1} - Tx_n\| = \| Gx_{n-1} - 1/c_n \{ Gx_{n+1} - (1-c_n)\, Gx_n \} \| \tag{2.5}$$
$$= \left\| \frac{1}{c_n} \{ Gx_{n+1} - Gx_n) + (Gx_n - Gx_{n-1}) \right\|$$
$$\le \frac{1}{h} \| (Gx_{n+1} - Gx_n) + (Gx_n - Gx_{n-1}) \|$$

Substituting these values from (2.4) and (2.5) in (2.3) it follows that

$$\| Gx_{n+1} - Gx_n \| \le k \| Gx_n - Gx_{n-1} \| \le \cdots \le k^n \| Gx_1 - Gx_0 \|$$

where $k = (\alpha - 1 + \beta / h)/(1 - \beta / h) < 1$ which implies that $\{Gx_n\}$ is a Cauchy sequence and so converges to some point p in X

Now consider

$$\| Tx_n - p \| \le \| Tx_n - Gx_n \| + \| Gx_n - p \|$$
$$= \frac{1}{c_n} \| \{ Gx_{n+1} - Gx_n)) \| + \| (Gx_n - p) \|$$

On letting $n \to \infty$, it yields that $\{Tx_n\}$ also converges to p.

Since G is continuous at $\{p\}$, the sequences $\{G^2 x_n\}$ and $\{GTx_n\}$ converge to Gp. Also since G and T are weakly commuting and so

$$\| TGx_n - GTx_n \| \le \| Gx_n - Tx_n \|$$

which on letting $n \to \infty$, yields $\| TGx_n - Gp \| \to 0$. This implies that $\{TGx_n\}$ also converges to Gp.

To prove $\{TGx_n\}$ converges to Tp, consider

$$\begin{aligned}\|TGx_n - Tp\| &\le \left\|TGx_n - G^2x_n\right\| + \left\|G^2x_n - Gp\right\| + \|Gp - Tp\| \\ &\le \left\|G^2x_n - Gp\right\| + \alpha\left\|G^2x_n - Gp\right\| \\ &\quad + \beta\left[\|TGx_n - Gp\| + \left\|G^2x_n - Tp\right\|\right]\end{aligned}$$

On letting $n \to \infty$, it yields

$$\|Gp - Tp\| \le \beta\|Gp - Tp\|$$

which is a contradiction. This implies that

$$Gp = Tp = p'(say)$$

Thus, we have shown that p is the coincidence point G and T.
Further if we assume $G^2p = Gp$ and since G and T weakly commute,

$$\|Tp' - Gp'\| = \|TGp - GTp\| \le \|Gp - Tp\| = 0$$

and so

$$Tp' = Gp' = G^2p = Gp = p'$$

so that

$$p'(= Gp = Tp)$$

is a common fixed point of G and T. This completes the proof.

Remark 1. As remarked in Ghosh [5], we do not assume the convergence of the sequence $\{Gx_n\}$ but rather it is a consequence of condition (2.1)

Remark 2. If we set $\beta = 0$ in Theorem 2, then we get a sharpened form of Theorem 2 of Khan *et al.* [7] as it involves weak commutativity instead of commutativity.

Remark 3. By setting $\beta = 0$ and $G = I$ (the identity mapping) we get a Theorem of Ghosh [5] which refines the cited Theorem 1 due to Rhoades [11]. Setting $\alpha = 0$ we get yet another result.

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