BULLETIN *of the*
MALAYSIAN
MATHEMATICAL
SCIENCES
SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **24** (2001) 111-128

Functional Integral Solution of the Complex Diffusion Equation with Complex Quadratic Potential in a Classical Path Space

ZAINAL ABDUL AZIZ
Jabatan Matematik, Fakulti Sains, Universiti Teknologi Malaysia, 81310 UTM Skudai, Johor, Malaysia
e-mail: zaa@mel.fs.utm.my

Abstract. Based on our result of constructing the real integral solution of the complex diffusion equation, we successfully generate, via the pull back operation in the space of classical paths Γ^{kla}, the solution of the complex diffusion equation with complex quadratic potential as a Feynmannian path integral with built-in Feynmannian measure in Γ^{kla}.

1. Motivation

This article extends the framework set up by Shaharir [17, 19, 20] to formulate the Feynman integral. Intrinsically, Shaharir's framework aspired to generalise the work done by Albeverio and Hoegh-Krohn [2,3] and to generate a path integral over the Riemannian manifold which possesses the qualitative characteristics of the original Feynman integral, specifically the concept of 'sum over all the classical paths'. The central part of our work encompasses the effort to derive a 'Feynmannian' integral (lending Shaharir's [17] connotation) or simply a Feynman integral type from the path integral solution ψ of the complex diffusion equation in the space of n-tuples set of complex number $\boldsymbol{C}^{\boldsymbol{n}}$:

$$\psi_t = \alpha\Delta\psi + \beta V(\boldsymbol{q})\psi \tag{1.1}$$

$\alpha, \beta \in \boldsymbol{C}$, complex numbers and $V(\boldsymbol{q})$ a complex quadratic potential. Subsript t represents partial differentiation with respect to t and Δ is the Laplacean. Essentially this is done via a refinement of the complex Hilbert space

$H = \{\gamma \in L^{2,2}([\mathbf{0}, t]; \boldsymbol{C}^{\boldsymbol{n}}),\ \gamma(t) = \mathbf{0}$ almost everywhere, and for any γ and η in the $L^{2,2}$

$$\langle \gamma | \eta \rangle = \int_{\mathbf{0}}^{t} \left(\dot{\gamma}(s) \cdot \dot{\overline{\eta}}(s) + \gamma(s) \cdot \overline{\eta}(s)\right) ds\} \tag{1.2}$$

into

$$\Gamma^{kla} = \{ \text{ classical paths over } \boldsymbol{C}^{\boldsymbol{n}} \text{ with the inner product as in } H \}$$

$$= \{ \gamma \colon \mathbf{m}\ddot{\gamma} = -\nabla V(\gamma),\ \gamma(0) = \boldsymbol{q}_{\mathbf{0}},\ \gamma(t) = \boldsymbol{q} \text{ and } V \text{ a potential } \} \qquad (1.3)$$

$L^{2,2}$ is a space of twice differentiable function and each derivative is a Lebesgue square integrable.

This 'Feynmannian' integral is found to be more in agreement with the qualitative aspects of the original Feynman path integral. This article clearly shows the relation between this integral with the original Feynman path integral and the real integral *via* path integral solution for the complex diffusion equation with complex quadratic potential in $\boldsymbol{C}^{\boldsymbol{n}}$.

2. Principal results

In the following we give the necessary results to produce our main conclusion in Theorem 5.

Lemma 1. *(Separable complex Hilbert space, H)*

H is a separable complex Hilbert space.

Proof. We'll show that H is a direct sum of two separable complex Hilbert spaces H_1 and H_2 (i.e. $H_1 \oplus H_2$), where

$$H_1 = \Big\{ \gamma \in L^{2,2}([0,t];\ \boldsymbol{C}^{\boldsymbol{n}}),\ \gamma(t) = \mathbf{0} \text{ almost everywhere, and for each pair } \gamma_1,\ \gamma_2,$$
$$\left\langle \gamma_1 \,|\, \gamma_2 \right\rangle_1 = \int_0^t \gamma_1(s) \cdot \bar{\gamma}_2(s)\, ds \Big\},$$

and

$$H_2 = \Big\{ \eta \in L^{2,2}([0,t];\ \boldsymbol{C}^{\boldsymbol{n}}),\ \eta(t) = \mathbf{0} \text{ almost everywhere, and for each pair } \eta_1,\ \eta_2,$$
$$\left\langle \eta_1 \,|\, \eta_2 \right\rangle_2 = \int_0^t \dot{\eta}_1(s) \cdot \dot{\bar{\eta}}_2(s)\, ds \Big\}.$$

The facts that $H_i, i = 1, 2$ are complex vector spaces with an inner product $<\cdot|\cdot>_i$ are trivial. The norms $\|\cdot\|_i$ in H_i, which are determined by the above inner products, are given respectively by

$$\|\gamma\|_1 = \left(\int_0^t |\gamma(s)|^2 ds\right)^{1/2}, \|\eta\|_2 = \left(\int_0^t |\dot{\eta}(s)|^2 ds\right)^{1/2}.$$

Refering to Weierstrass's theorem (refer Kufner *et al.* [10]) every path $\gamma, \eta \in L^{2,2}([0,t]; \boldsymbol{C}^n)$ can be approximated in the normed space by a polynomial sequence, i.e.

$$\gamma_n(s) = \sum_{|j| \le n} a_j s^j, \ \gamma_n(t) = 0 \text{ and } a_{n+1} = a_{n+2} = \cdots = 0,$$

$$\eta_n(s) = \sum_{|j| \le n} b_j s^j, \eta_n(t) = 0 \text{ and } b_{n+1} = b_{n+2} = \cdots = 0,$$

$$n = 0,1,2,\cdots, a_j, \ b_j \in \boldsymbol{C}, \text{ such that } \|\gamma - \gamma_n\|_1 < \varepsilon, \ \|\eta - \eta_n\|_2 < \varepsilon.$$

Consequently,

$$\|\gamma - \gamma_n\|_1^2 = \int_0^t |\gamma - \gamma_n|^2 ds \ , \ \|\eta - \eta_n\|_2^2 = \int_0^t \left|\dot{\eta} - \left(\frac{j}{s}\right)\eta_n\right|^2 ds \ ,$$

show that γ and $\dot{\eta}$ are respectively limits of the sequences $\{\gamma_n\}, \{\eta_n\}$ in $L^{2,2}$. Via the completion of $L^{2,2}$, the sequences $\{\gamma_n\}, \{\eta_n\}$ converge separately to γ and $\dot{\eta}$, and as a result

$$\lim_n \|\gamma - \gamma_n\|_1^2 = \int_0^t |\gamma - \gamma_n|^2 ds, \ \lim_n \|\eta - \eta_n\|_2^2 = \int_0^t \left|\dot{\eta} - \left(\frac{1}{2}\right)\eta_n\right|^2 ds \ .$$

These outcomes imply Cauchy sequences in $L^{2,2}$ (i.e. converge in the mean square) and subequently $\gamma \in H_1, \dot{\eta} \in H_2$, and H_1, H_2 are complete. These conclusions guarantee that both H_1, H_2 are complex Hilbert spaces.

In order to show that H_1, H_2 are separable, we consider the sets D_1, D_2 of all paths γ, η which originate from H_1, H_2 respectively. With reference to Naimark's theorem [12] (i.e. for each coefficient a_j of the polynomial γ_n, there exists a rational number coefficient u_j of the polynomial Γ_n such that

$\left|a_j - u_j\right| < (\varepsilon / xd^{|j|})$, $\varepsilon > 0$, $\left|s^j\right| < d^{|j|}$, d a positive number, and x the number of coefficients a_j), there exist in the sets D_1, D_2 other polynomials Γ_n and Π_n, where u_j and v_j are the respective rational coefficients, such that

$$\|\gamma_n - \Gamma_n\|_1 < \varepsilon, \quad \|\eta_n - \Pi_n\|_2 < \varepsilon.$$

Moreover $\gamma_n(t) = \eta_n(t) = 0$, and accordingly $u_{n+1} = u_{n+2} = \cdots = 0, v_{n+1} = v_{n+2} = \cdots = 0$ for some integer n, $n = 0, 1, 2, \cdots$. As a consequence, all the polynomials Γ_n, Π_n ; $n = 0, 1, \cdots$ with the rational coefficients u_j and v_j, of the sets D_1, D_2 are countable. These are due to the fact that the sets can be numerated by the natural numbers n.

The sets D_1, D_2 can be shown as follows to be dense everywhere in H_1, H_2 respectively. Let us consider any paths $\gamma \in H_1$ and $\eta \in H_2$, where Γ_n and Π_n respectively represent the polynomials with rational coefficients u_j and v_j. For any $\varepsilon > 0$ and since $\gamma, \eta \in L^2$, then there exist $\gamma_n \in D_1$, $\eta_n \in D_2$, such that $\|\gamma - \gamma_n\|_1 < \varepsilon$, $\|\eta - \eta_n\|_2 < \varepsilon$. (*via* Weierstrass's theorem) and $\|\gamma_n - \Gamma_n\|_1 < \varepsilon$, $\|\eta_n - \Pi_n\|_2 < \varepsilon$. (*via* Naimark's theorem).
As a result, we obtain

$$\|y - \Gamma_n\|_1 \leq \|\gamma - \gamma_n\|_1 + \|\gamma_n - \Gamma_n\|_1 = 2\varepsilon,$$

$$\|\eta - \Pi_n\|_2 \leq \|\eta - \eta_n\|_2 + \|\eta_n - \Pi_n\|_2 = 2\varepsilon,$$

which indicate that D_1 and D_2 are respectively dense everywhere in H_1, H_2, and thus demonstrate that H_1, H_2 are separable.

The above results show that H_1, H_2 are separable complex Hilbert spaces. Finally we proceed by referring to a theorem by Prugovecki [14] which states that: if $H_1, H_2, \cdots$ represent a finite or infinite sequence of countable separable Hilbert spaces, then the direct sum $\oplus_n H_n = H$ corresponds to a separable Hilbert spaces. As a consequence of this theorem, the abovementioned direct sum of $H_1 \oplus H_2$ will form a separable complex Hilbert space with an inner product $<\cdot|\cdot>$, as stated in the proposition of Lemma 1.

The form of the inner product (1.2) on the $L^{2,2}(G)$ space, G a domain in R^n, has been briefly mentioned in Rektorys [15] and Griffel [8].

Corollary 2. *(construction of* Γ^{kla}*)*

If

$$\Gamma^{kla} = \left\{ \begin{array}{l} \sigma \in L^{2,2}([0,t]\ ;\ \boldsymbol{C^n}), m\ddot{\sigma} = -\nabla(\gamma - \boldsymbol{q}),\ \textit{where m mass of the} \\ \textit{particle which is being influenced by a potential} \\ \boldsymbol{V(z)} = (1/2)\boldsymbol{z^T}\Omega\boldsymbol{\bar{z}} + \boldsymbol{L}\cdot\boldsymbol{z} + P, \Omega\ \textit{Hermitean matrice},\ \boldsymbol{L}\ \textit{vector} \\ P\ \textit{complex constant ;}\ \sigma(t) = \boldsymbol{0}\ \textit{almost everywhere,} \\ \textit{and for any}\ \sigma\ \textit{and}\ \beta\ \textit{in the set} \\ \langle \sigma | \rho \rangle = \dfrac{p}{t}\displaystyle\int_0^t \left(\dot{\sigma}(s) \cdot \dot{\bar{\rho}}(s) + \sigma(s)^T \left(\frac{\Omega}{m}\right) \bar{\rho}(s) \right) ds,\ p > 0, \end{array} \right\}$$

then Γ^{kla} *is a closed subspace of* H *.*

The proof of Corollary 2 is given after the following remarks.

Remarks 3. (form of the classical path σ of interest)

The substitution $\sigma = \gamma - \boldsymbol{q}$ is used to obtain Γ^{kla} instead of the space

$$\left\{ \begin{array}{l} \gamma \in L^{2,2}([0,t]\ ;\ \boldsymbol{C^n}), \ddot{\gamma} = -\nabla V(\gamma), \gamma(0) = \boldsymbol{q_0}, \gamma(t) = \boldsymbol{q}\ \text{almost everywhere}, \\ \text{and for any pair of } \gamma \text{ and } \eta\ \langle \gamma | \eta \rangle = \dfrac{p}{t}\displaystyle\int_0^t \left(\dot{\gamma}(s) \cdot \dot{\bar{\eta}}(s) + \gamma(s)^T \left(\frac{\Omega}{m}\right) \bar{\eta}(s) \right) ds. \end{array} \right\}$$

This implies that Γ^{kla} corresponds to the space of classical paths $\sigma = \gamma - \boldsymbol{q}$. The path σ, mentioned in the above Corollary 2, is formed from the consideration of the complex quadratic potential

$$V(\boldsymbol{q}) = \left(\frac{1}{2}\right)\boldsymbol{q^T}\,\Omega\boldsymbol{q} + \boldsymbol{L}\cdot\boldsymbol{q} + P$$

together with the Euler-Lagrange equation

$$\ddot{\gamma} = -\left(\frac{1}{m}\right)(\Omega\gamma + \boldsymbol{L}),\ \gamma(0) = \boldsymbol{q_0},\ \gamma(t) = \boldsymbol{q},\ \text{ i.e. the path } \sigma = \gamma - \boldsymbol{q}$$

$$\Rightarrow \ddot{\sigma} = -\left(\frac{\mathbf{1}}{\mathbf{m}}\right)\left(\Omega(\sigma + \boldsymbol{q}) + \boldsymbol{L}\right),\ \sigma(0) = \boldsymbol{q_0} - \boldsymbol{q},\ \sigma(t) = \mathbf{0}$$

Proof of Corollary 2

It can be seen that $\Gamma^{kla} \subset L^{2,2}$ is a vector subspace of $L^{2,2}$, since $\sigma, \ \rho \in \Gamma^{kla}$ implies that $a\sigma + b\rho$ is an element of Γ^{kla} for all scalars $a, b \in \boldsymbol{C}$. Moreover $\Gamma^{kla} \subset L^{2,2}$ forms a closed subspace of $L^{2,2}$ (accordingly of H): if Γ^{kla} is a vector subspace and $\{\sigma_n\} \subset \Gamma^{kla}$, then $\sigma_n \to \sigma$ implies $\sigma \in \Gamma^{kla}$.
With reference to the proof of Lemma 1, evidently we have in $\Gamma^{kla} \subset L^{2,2}$

$$\left\langle \sigma - \sigma_n \middle| \sigma - \sigma_n \right\rangle_{\Gamma^{kla}} = \left\| \sigma - \sigma_n \right\|^2_{\Gamma^{kla}},$$

and

$$\left\| \sigma - \sigma_n \right\|^2_{\Gamma^{kla}} = \frac{p}{t}\int_0^t \left(\left(\frac{\Omega}{m} \right) \left| \sigma - \sigma_n \right|^2 + \left| \dot{\sigma} - \left(\frac{j}{s} \right) \sigma_n \right|^2 \right) ds.$$

These show that the convergence of $\sigma_n \to \sigma$ signifies that the simultaneous convergence of the sequence $\{\sigma_n\}$ and the corresponding sequence of derivatives $\{\dot{\sigma}_n\}$ in Γ^{kla} to their respective limits σ_n and $\dot{\sigma}_n$. Consequently, via the completeness of $L^{2,2}$,

$$\lim_n \frac{p}{t}\int_0^t \left(\left(\frac{\Omega}{m} \right) \left| \sigma - \sigma_n \right|^2 + \left| \dot{\sigma} - \left(\frac{j}{s} \right) \sigma_n \right|^2 \right) ds = 0,$$

this result implies Cauchy sequence in $L^{2,2}$ and thus for $\sigma, \dot{\sigma} \in \Gamma^{kla}$, we have a complete Γ^{kla}. This implies that Γ^{kla} represents a closed subspace of H.

Lemma 4. (*use of the pull-back operation in* Γ^{kla})

1. *Let* $C_0[(0,t)\,;\,\boldsymbol{C^n}]$ *be a set of twice differentiable smooth mapping from* $(0,t)$ *to unitary space* $\boldsymbol{C^n}$ *such that* $m\ddot{\sigma} = -\nabla W(\sigma)$, $W(\sigma) = V(\gamma - \boldsymbol{q})$. *m stands for the particle's mass which is being influenced by the quadratic potential* $V(\boldsymbol{z}) = \left(\frac{1}{2}\right)\boldsymbol{z^T}\Omega\,\bar{\boldsymbol{z}} + \boldsymbol{L}\cdot\boldsymbol{z} + P$, Ω *a nonsingular Hermitean matrix (if* $\Omega \neq 0$*),* $\boldsymbol{L}$ *vector,* P *a complex constant;* $\sigma(t) = \boldsymbol{0}$, *and the inner product in* $\boldsymbol{C^n}$ *is given by* $g_1 : g_1(a,b) = a \cdot \bar{b}$.
   Let

$$ev_s : C_0[(0,\ t)\,;\,\boldsymbol{C^n}] \to \boldsymbol{C^n}, \quad \sigma \mapsto ev_s(\sigma) = \sigma(s),$$

represents the evaluation map at s. As a result $<,>_1$ *is an inner product on* $C_0[(0,t);\, \boldsymbol{C^n}]$ *and is defined by*

$$\langle \sigma, \rho \rangle_1 \;=\; g_1(\sigma(s),\; \rho(s)) \;=\; \sigma(s) \,\cdot\, \overline{\rho}(s).$$

$g_1' :\, \boldsymbol{C^n} \times \boldsymbol{C^n} \to \boldsymbol{C},\; (\sigma, \rho) \;\mapsto\; \sigma \cdot \rho$ *is the 'pull-back' by the mapping* ev_s,

$$<,>_1 \,=\, {ev_s}^{*} g_1.$$

2. *The inner product* $<,>_2$ *on* $C_0[(0,t);\, \boldsymbol{C^n}]$, *which is defined by* $\langle \sigma, \rho \rangle_2 \,=\, \dot{\sigma}(s) \,\cdot\, \dot{\overline{\rho}}(s)$, *is the pull-back* g_2 *by the mapping*

$$dev_s :\, C_0[(0,\, t);\; \boldsymbol{C^n}] \,\to\, \boldsymbol{C^n},\;\; \sigma \;\mapsto\; dev_s(\sigma) \;=\; \dot{\sigma}(s).$$

3. *The inner product on* Γ^{kla}, *which is defined in the Corollary* 2, *is a pull-back of the metric* g (*i.e.* $g \,=\, g_1 \oplus g_2$)

$$g \;\equiv\; (ev_{s_0,a} + \; dev_{s_0,a}) \,:\; \Gamma^{kla} \;\to\; \boldsymbol{C^n},$$

for the constant a *and fixed* s_0,

$$g \;\equiv\; (ev_{s_0,a} + \; dev_{s_0,a})(a) \;=\;\; a\,(\sigma(s_0) \;+\; \dot{\sigma}(s)),$$

and also the inner product g *is the pull-back* $<,>$, *a metric on* Γ^{kla}, *by the mapping*

$$F \;:\; \boldsymbol{C^n} \;\to\; \Gamma^{kla},\;\; z \;\mapsto F(z) \,=\, \sigma,$$

σ *as in Remarks* 3.

Proof.

1. Since ev_s is linear and via the definition of the Frechet derivative (see Abraham *et al.* [1], Lang [11]),

$$D(ev_s)(\rho_0)(\eta) \;=\; \eta(s)$$

for any $\rho_0 \in C_0[(0,t);\, \boldsymbol{C}^n]$ and $\eta \in T_{\rho_0} C_0[(0,t);\, \boldsymbol{C}^n]$, the tangent space of $C_0[(0,t);\, \boldsymbol{C}^n]$ at ρ_0. From the definition of the pull-back (see Abraham *et al.* [1])

$$\langle \sigma, \lambda \rangle_1 = ev_s^*(g_1)(\sigma, \lambda);\ \sigma,\ \lambda \in C_0[(0,\ t);\ \boldsymbol{C}^n]$$
$$\langle \sigma, \lambda \rangle_1 = g_1(ev_s(\rho_0))(T_{\rho_0} ev_s(\sigma), T_{\rho_0} ev_s(\lambda))$$
$$= g_1(\rho_0(s))(\sigma(s), \lambda(s)) = \sigma(s) \cdot \overline{\lambda}(s)$$

2. Since dev_s is linear,

$$T_{\rho_0} dev_s(\eta) = (dev_s(\rho_0),\ D(dev_s(\rho_0)(\eta)) = (dev_s(\rho_0),\ \dot{\eta}(s))$$

and similarly as part 1 above mentioned,

$$\langle \sigma, \lambda \rangle_2 = dev_s^* g_2(\sigma, \lambda) = g_2(dev_s(\rho_0))(T_{\rho_0} dev_s(\sigma), T_{\rho_0} dev_s(\lambda))$$
$$= g_2(dev_s(\rho_0))(\dot{\sigma}(s),\ \dot{\lambda}(s)) = \dot{\sigma}(s) \cdot \dot{\overline{\lambda}}(s).$$

3. (a) The first part: We have to show that

$$\langle \sigma, \lambda \rangle_2 = g(\sigma, \lambda)$$
$$= dev_{s_0,a}{}^* g_2(\sigma, \lambda) + ev_{s_0,a}{}^* g_1(\sigma, \lambda)$$
$$= a^2(\dot{\sigma}(s_0) \cdot \dot{\overline{\lambda}}(s_0) + \sigma(s_0) \cdot \overline{\lambda}(s_0));\quad s_0 \text{ is fixed} \tag{1.4}$$

This result is obtained with reference to the fact that for any constant of motion T (energy for a conservative system as similar to this case), we have

$$T(\sigma(s_0)) = \frac{1}{t}\int_0^t T(\sigma(s))\, ds$$

where $T(\sigma(s)) = \left(\frac{m}{2}\right) |\dot{\sigma}(s)|^2 + V(\sigma(s))$, m is the particle's mass with potential V. Accordingly for σ and λ in Γ^{kla},

$$T(\sigma(s_0)) + T(\lambda(s_0)) = \frac{1}{t}\int_0^t (T(\sigma(s) + T(\lambda(s))\, ds,$$

i.e.,

$$\left(\frac{m}{2}\right)\left(\left|\dot{\sigma}(s_0)\right|^2 + \left|\dot{\lambda}(s_0)\right|^2\right) + \left(V(\lambda(s_0)) + V(\sigma(s_0))\right)$$
$$= \frac{1}{t}\int_0^t \left(\frac{m}{2}\right)\left(\left|\dot{\sigma}(s)\right|^2 + \left|\dot{\lambda}(s)\right|^2 + (V(\lambda(s)) + V(\sigma(s))\right) ds.$$

As a consequence of this, for the case of a generalised complex harmonic oscillator, $V(\boldsymbol{q}) = \left(\frac{1}{2}\right)\boldsymbol{q}^T\,\Omega\,\boldsymbol{q}$, we have

$$\left(\frac{m}{2}\right)\left|\dot{\sigma}(s_0) + \dot{\lambda}(s_0)\right|^2 - m\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0) + \left(\frac{1}{2}\right)(\sigma(s_0) + \lambda(s_0))^T\,\Omega\,(\bar{\sigma}(s_0)$$
$$+ \bar{\lambda}(s_0)) - \sigma(s_0)^T\,\Omega\,\bar{\lambda}(s_0)$$
$$= \frac{1}{t}\int_0^t \left(\left(\frac{m}{2}\right)\left|\dot{\sigma}(s) + \dot{\lambda}(s)\right|^2 - m\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s) + \left(\frac{1}{2}\right)(\sigma(s) + \lambda(s))^T\,\Omega\,(\bar{\sigma}(s_0) + \bar{\lambda}(s_0))\right.$$
$$\left. - \sigma(s)^T\,\Omega\,\bar{\lambda}(s)\right) ds$$

or

$$\left(\frac{m}{2}\right)\left|\dot{\gamma}(s_0)\right|^2 + \left(\frac{1}{2}\right)(\gamma(s_0))^T\,\Omega\bar{\gamma}(s_0) - \left\{m\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0) + \sigma(s_0)^T\,\Omega\bar{\lambda}(s_0)\right\}$$
$$= \frac{1}{t}\int_0^t \left\{\left(\frac{m}{2}\right)\left|\dot{\gamma}(s)\right|^2 + \left(\frac{1}{2}\right)(\gamma(s))^T\Omega\bar{\gamma}(s) - (m\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s) + \sigma(s)^T\Omega\bar{\lambda}(s))\right\} ds,$$

where $\sigma, \lambda, \gamma \in \Gamma^{kla}$.

Since $\gamma = \alpha + \lambda \in \Gamma^{kla}$, therefore $T(\sigma(s_0)) = \frac{1}{t}\int_0^t T(\gamma(s))\,ds$, and accordingly we have the result

$$\left\{\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0) + \sigma(s_0)^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s_0)\right\} = \frac{1}{t}\int_0^t \left\{\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s) + \sigma(s)^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s)\right\} ds,$$

or referring to (1.4), we have instead

$$a^2\left\{\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0)+\sigma(s_0)^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s_0)\right\}$$

$$=\frac{a^2}{t}\int_0^t\left\{\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s)+\sigma(s)^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s)\right\}ds=\langle\sigma,\lambda\rangle_1,$$

for the case of a generalised complex harmonic oscillator.
For the complex affine potential $V(\boldsymbol{q}) = \boldsymbol{L}\cdot\boldsymbol{q}+P$,

$$\left(\frac{m}{2}\right)\left|\dot{\sigma}(s_0)+\dot{\lambda}(s_0)\right|^2-m\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0)+\boldsymbol{L}\cdot(\sigma(\mathrm{s}_0)+\lambda(\mathrm{s}_0))+C$$

$$=\frac{1}{t}\int_0^t\left\{\left(\frac{m}{2}\right)\left|\dot{\sigma}(s)+\dot{\lambda}(s)\right|^2-m\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s)+\boldsymbol{L}\cdot(\sigma(\mathrm{s})+\lambda(\mathrm{s}))+C\right\}ds,$$

where C is a constant, or this can be expressed as

$$\left(\frac{m}{2}\right)\left|\dot{\gamma}(s_0)\right|^2-m\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0)+\boldsymbol{L}\cdot\gamma(\mathrm{s}_0)+C$$

$$=\frac{1}{t}\int_0^t\left\{\left(\frac{m}{2}\right)\left|\dot{\gamma}(s)\right|^2-m\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s)+\boldsymbol{L}\cdot\gamma(\mathrm{s})+C\right\}ds,$$

$\sigma,\lambda,\gamma\in\Gamma^{kla}$ and therefore $\sigma+\lambda=\gamma\in\Gamma^{kla}$. As a result, for complex affine potential, we obtain (again referring to (1.4))

$$a^2\left\{\dot{\sigma}(s_0)\cdot\dot{\bar{\lambda}}(s_0)\right\}=\frac{a^2}{t}\int_0^t\left\{\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s)\right\}ds=\langle\sigma,\lambda\rangle_2 .$$

Consequently for the generalised complex quadratic potential, $V(\boldsymbol{q})=\left(\frac{1}{2}\right)\boldsymbol{q}^{\boldsymbol{T}}\,\Omega\,\bar{\boldsymbol{q}}+\boldsymbol{L}\cdot\boldsymbol{q}+P$, we have the result

$$\langle\sigma,\lambda\rangle=\frac{a^2}{t}\int_0^t\left\{\dot{\sigma}(s)\cdot\dot{\bar{\lambda}}(s)+\sigma^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s)\right\}ds$$

as stated in Lemma 1 and Corollary 2.

(b) The second part: this part shows that the inner product g can be constructed from the pull-back $<,>$ a metric on Γ^{kla} by the mapping F. According to Shaharir and Zainal [20], for the generalised complex quadratic potential $V(\boldsymbol{q}) = \left(\frac{1}{2}\right)\boldsymbol{q}^{\boldsymbol{T}}\, \Omega\, \bar{\boldsymbol{q}} + \boldsymbol{L}\cdot\boldsymbol{q} + P$, Ω a nonsingular Hermitian matrix (if $\Omega \neq 0$) of size n, it can be written explicitly that

$$F(\mathbf{z})(s) = \sigma(s) = \begin{cases} \dfrac{-1}{2m}(s^2 - t^2)\boldsymbol{L} + (s-t)\underline{\ell}; \quad \underline{\ell} = \dfrac{1}{t}\left(\boldsymbol{q} - \boldsymbol{q_0} + \dfrac{t^2}{2m}\boldsymbol{L}\right),\ t > 0;\ \text{if } \Omega = 0 \\ \sin^{-1}\left[\sqrt{\Omega}t\right]\left(\sin\left[\sqrt{\Omega}s\right]\right)\boldsymbol{X} - \sin\left[\sqrt{\Omega}(s-t)\right]\boldsymbol{Y} - \boldsymbol{X}), \\ \text{if } \Omega \neq 0 \text{ and nonsingular,} \\ \left(\sqrt{\Omega}\right)^2 = \Omega,\ \boldsymbol{X} = \boldsymbol{q} + \left(\sqrt{\Omega}\right)^{-1}\boldsymbol{L},\ \boldsymbol{Y} = \boldsymbol{q_0} + \left(\sqrt{\Omega}\right)^{-1}\boldsymbol{L}\ ;\ \text{whereas} \\ t \in (0, t_{\min}),\ t_{\min} = \min\limits_{j}\left[\dfrac{\pi}{\sqrt{\lambda_j}}\right] \quad \text{for } \alpha\beta < 0 \quad \text{where} \\ \left|\sin\left[\sqrt{\Omega}t\right]\right| = \prod\limits_{j=1}^{n}\sin\left[\sqrt{\lambda_j}\, t\right], \quad \text{or } t > 0 \quad \text{for } \alpha\beta > 0,\ \text{or} \\ t_{\min} = \min\limits_{j}\left[\dfrac{\pi}{Im\sqrt{\lambda_j}}\right] \text{for } \alpha\beta \in \boldsymbol{C} \end{cases}$$

$\boldsymbol{L} \in \boldsymbol{C}^{\boldsymbol{n}}$, $\Omega = 0$ for the complex affine potential and $\Omega \neq 0$ for the complex quadratic potential, λ_i is the eigenvalue of Ω. .

Accordingly $DF(\mathbf{z_0})(\mathbf{z}) = \sigma$, and as a result

$$\begin{aligned} g(\mathbf{z_0})(\mathbf{z}, \boldsymbol{w}) &= F^*(<,>)\,(\mathbf{z}, \boldsymbol{w}) \\ &= \left\langle T_{\mathbf{z_0}}F(\mathbf{z}),\, T_{\mathbf{z_0}}F(\mathbf{w})\right\rangle = \langle \sigma, \lambda\rangle \\ &= \frac{a^2}{t}\int_0^t \left\{\dot{\sigma}(s)\cdot\dot{\bar{\gamma}}(s) + \sigma(s)^T\left(\frac{\Omega}{m}\right)\bar{\lambda}(s)\right\} ds, \end{aligned}$$

and explicitly,

$$g(\boldsymbol{z_0})(\boldsymbol{z},\boldsymbol{z}) = \begin{cases} \frac{1}{2}\left|\left[\boldsymbol{q_0} - \left(\boldsymbol{q} - \frac{t^2\boldsymbol{L}}{2m}\right)\right]\right|^2 + \frac{2}{m}\left|[(\boldsymbol{q_0} - \boldsymbol{q}) \cdot \boldsymbol{L}]\right|, \quad \Omega = 0\,; \\ \left[\boldsymbol{Y} - \cos^{-1}\left[\sqrt{\Omega}t\right]\boldsymbol{X}\right]^T \sum^{-1}\left[\boldsymbol{Y} - \cos^{-1}\left[\sqrt{\Omega}t\right]\boldsymbol{X}\right] \\ \quad + \left((\boldsymbol{q_0} - \boldsymbol{q})^T \sqrt{\Omega}\,(\boldsymbol{q_0} - \boldsymbol{q})\right) + 2\left|[(\boldsymbol{q_0} - \boldsymbol{q}) \cdot \boldsymbol{L}]\right|, \ \Omega \neq 0, \\ \sum(t) = \frac{2\alpha}{t}(\Omega)^{-1}(\tan[\Omega t]), \ t \in (0, t_{\min}), \text{ where } t_{\min} = \min_j \frac{\pi}{\sqrt{\lambda_j}} \\ \text{for } \alpha\beta < 0, \text{ or } t > 0 \text{ for } \alpha\beta > 0, \text{ or } t_{\min} = \min_j \frac{\pi}{Im\left(\sqrt{\lambda_j}\right)} \text{ for } \alpha\beta \in \boldsymbol{C} \end{cases}$$

Theorem 5. *('Feynmannian' integral)*

Let $\varphi : \boldsymbol{C^n} \to \boldsymbol{C}$ be Lebesgue integrable and a Fourier transform of a bounded complex Borel measure on $\boldsymbol{C^n}$; then the solution of the complex diffusion equation (1.1) with a quadratic complex potential $V(\boldsymbol{q}) = \left(\frac{1}{2}\right)\boldsymbol{q^T}\,\Omega\,\overline{\boldsymbol{q}} + \boldsymbol{L}\cdot\boldsymbol{q} + P$, where Ω a nonsingular Hermitian matrix (if $\Omega \neq 0$), $\boldsymbol{L}$ vector, P a complex constant; can be reduced to a 'Feynmannian' path integral

$$R(\boldsymbol{q},t) = \int_{\Gamma^{kla}} \exp\left(\beta\int_0^t L(\sigma(s),\ \dot{\sigma}(s))\, ds\right) dF(\sigma), \tag{1.5a}$$

where $\quad L(\sigma(s),\ \dot{\sigma}(s)) = \left(\frac{1}{2}\right)m\left|\dot{\sigma}(s)\right|^2 - W(\sigma(s)), \quad W(\sigma) = V(\gamma - \boldsymbol{q}).$

The entity L represents the classical Lagrangian of a particle of mass

$$m = \left(\frac{-1}{2\alpha\beta}\right),$$

$\Gamma^{kla} = \{\sigma \in L^{2,2}([0,t];\ \boldsymbol{C^n}),\ m\ddot{\sigma} = -\nabla W(\sigma),\ \sigma(t) = \boldsymbol{0}$ *almost everywhere, and for any such functions σ, β the following inner product is well-defined as* $\langle\sigma|\rho\rangle = \frac{a^2}{t}\int_0^t\left(\dot{\sigma}(s)\cdot\dot{\overline{\rho}}(s) + \sigma(s)^T\left(\frac{\Omega}{m}\right)\overline{\rho}(s)\right)ds\}$.

Γ^{kla} *is the particle's space of classical paths, and the Feynmannian measure F is given by*

$$dF = [T_{\rho_0}(G)]^*(d\mu_{\varphi,q,s})\,; \quad \rho_0 \in \Gamma^{kla} \tag{1.5b}$$

G is defined as in Lemma 4 part (3), where

$$d\mu_{\varphi,q,s} = (S^{-1})^*(d\mu_{\varphi,q})$$

and $$S : \boldsymbol{C^n} \to \boldsymbol{C^n},\ \boldsymbol{z} \mapsto \boldsymbol{q_0} = \begin{cases} \left[\dfrac{1}{2\alpha\, t}\right]^{1/2} \cdot \boldsymbol{z}\,, \text{ if } \Omega = 0 \\ \left[2\alpha \left|\sqrt{\Omega}\, \tan\left(\sqrt{\Omega}\, t\right)\right|\right]^{-1/2} \cdot \boldsymbol{z}\,, \text{ if } \Omega \neq 0 \end{cases}$$

is the bounded complex Borel measure and the scalar mapping on $\boldsymbol{C^n}$:

$$d\mu_{\varphi,q}(\boldsymbol{q_0}) = \exp(k\boldsymbol{q} \cdot \boldsymbol{q_0})\, d\mu_\varphi(\boldsymbol{q_0})\,,$$

or equivalently

$$\Im(A) = \int_{(\boldsymbol{F}\circ\boldsymbol{T})^{-1}(A)} \exp(k\boldsymbol{q} \cdot \boldsymbol{q_0})\, d\mu_\varphi(\boldsymbol{q_0})\,.$$

$F : \boldsymbol{C^n} \to \Gamma^{kla},\ \boldsymbol{z} \mapsto \sigma$, *where σ is the classical path as in Lemma 4 part (3b) and*

$$T \equiv S^{-1} : \boldsymbol{C}^n \to \boldsymbol{C}^n,\ T(z) = \begin{cases} \left(2t\,|\,\alpha\,|\right)^{1/2} \cdot \boldsymbol{z} & , \text{ if } \Omega = 0 \\ \left(2\alpha\,|\sqrt{\Omega}\, \tan[\sqrt{\Omega}\, t]\,|\right)^{1/2} \cdot \boldsymbol{z} & , \text{ if } \Omega \neq 0 \end{cases}$$

Proof. The proof is given separately for the case $\Omega = 0$ (complex affine potential) and $\Omega \neq 0$ (complex quadratic potential).

Case $\Omega = 0$. From Shaharir and Zainal [20], the solution of the complex diffusion equation (1.1) is given by

$$R(\boldsymbol{q},t) = \int_{R^n} \left(\frac{1}{4\pi\alpha t}\right)^{n/2} \exp(-g(\mathbf{z},\mathbf{z})) \exp\Big(2\beta\int_0^t V(\sigma(s)\,ds\Big)\,\varphi(\boldsymbol{q}+\mathbf{z})\,d\mathbf{z}$$

$$= \int_{C^n} \left(\frac{1}{4\pi\alpha t}\right)^{n/2} \exp\left(-\frac{\mathbf{z}\cdot\bar{\mathbf{z}}}{4\alpha t} + \left(\frac{1}{2}\right)\rho t(\boldsymbol{q}+\mathbf{z})\cdot\boldsymbol{L} + \frac{L^2\beta^2\alpha t^3}{12} + \rho P t\right)$$

$$\left[\int_{C^n} \exp\,(k(\boldsymbol{q} + \mathbf{z})\cdot \boldsymbol{q_0}\, d\mu_\varphi(\boldsymbol{q_0})\right] d\mathbf{z},$$

$$= \int_{C^n}\left(\int_{C^n}\left(\frac{1}{4\pi\alpha t}\right)^{n/2} \exp\left(-\frac{\omega\cdot\bar{\omega}}{4\alpha t}\right)d\omega\right) \exp\Big[\Big(-(\boldsymbol{q_0}-\boldsymbol{q_0})\,\alpha t + k\boldsymbol{q}\cdot\bar{\boldsymbol{q}}_{\mathbf{0}} - k\alpha\beta t^2\boldsymbol{q_0}\cdot\boldsymbol{L}$$

$$+ \left(\frac{L^2\beta^2\alpha t^3}{3}\right) + \rho P t\Big]\, d\mu_\varphi(\boldsymbol{q_0})\,; \quad \omega = z - 2\alpha t\,(k\boldsymbol{q_0} + \left(\frac{\beta t}{2}\right)\boldsymbol{L}\Big)$$

$$= \int_{C^n} \exp\big(-g(\boldsymbol{q_0},\boldsymbol{q_0})\big)\,\exp\left(\beta\int_0^t V(\sigma(s))\,ds\right)\exp\,(k\boldsymbol{q}\cdot\bar{\boldsymbol{q}}_{\mathbf{0}})\;d\mu_\varphi(\boldsymbol{q_0}),$$

upon consideration of Fubini's theorem and characteristics of a 'multinormal complex distribution' (see Shaharir [17] and Andersen *et al.* [5]). As a result we obtain

$$R(\boldsymbol{q},t) = \int_{C^n} \exp\,(-g(\boldsymbol{q_0}-\boldsymbol{q_0}))\,\exp\left(\beta\int_0^t V(\sigma(s))\,ds\right) d\mu_{\varphi,q}(\boldsymbol{q_0})\,;$$

and

$$d\mu_{\varphi,q}(\boldsymbol{q_0}) = \exp(k\boldsymbol{q}\cdot\boldsymbol{q_0})\,d\mu_\varphi(\boldsymbol{q_0}),$$

$$= \int_{C^n} \exp\left(-\frac{e^{kp}}{2}g(\mathbf{z},\mathbf{z})\right)\exp\left(\beta\int_0^t V(\sigma(s))\,ds\right) d\mu_{\varphi,q,s}(\mathbf{z}),$$

(where p represents the phase of $1/\alpha$) or in terms of the differential form, $d\mu_{\varphi,q,s} = (S^{-1})^*(d\mu_{\varphi,q})$, the pull-back of $d\mu_{\varphi,q}$ by S^{-1}.

Meanwhile we can write

$$R(\boldsymbol{q},t) = \int_{T_b C^n} \exp\big(-ag\,(\Im_{\rho_0}(\sigma)\cdot\Im_{\rho_0}(\sigma))\big)\;\exp\left(\beta\int_0^t V(\Im_{\rho_0}(\sigma(s))\,ds\right) d\mu_{\varphi,q,s}(\Im_{\rho_0}(\sigma)),$$

$$a = \left(\frac{1}{2}\right)e^{-kp},\; b = \alpha\rho_0(s_0),\; \rho_0 \in \Gamma^{kla}.$$

Together with the choice

$$\Im_{\rho_0}(\sigma) = T_{\rho_0} G(\sigma),$$

and G as in Lemma 4 part (3), and via Lemma 4 and the general theory of the transformation of a variable of an integral with respect to a differential form, we obtain

$$R(\boldsymbol{q},t) = \int_{T_b\Gamma^{kla}} \exp\left(-a < \sigma, \sigma >\right) \exp\left(\beta \int_0^t V(\sigma(s))\, ds\right) dF(\sigma) \tag{1.6a}$$

or

$$R(\boldsymbol{q},t) = \int_{T_b\Gamma^{kla}} \exp\left(-a < \gamma - \boldsymbol{q},\ \gamma - \boldsymbol{q} >\right) \exp\left(\beta \int_0^t V(\gamma(s) - \boldsymbol{q})\, ds\right) dF(\gamma - \boldsymbol{q}) \tag{1.6b}$$

The first part of the theorem is obtained when we identify $T_{\rho_0}\Gamma^{kla}$ *as* Γ^{kla} and upon considering the formulation of Albeverio and Hoegh-Krohn (as in [2]) which states that the normalised integral (or Fresnel integral) on H is invariant under the group transformation of Euclidean type, and the result by Parthasarathy [13] which states that a measure in the function space (eg. Wiener measure) is invariant under such transformation.

We prove the last part of the theorem via the same arguments as proposed in the results of the second part of Lemma 4 and the theory relating to the Borel measure, together with the fact that the trasformation T and F are proper mappings. The measure F is obtained via theorem of Radon-Nikodym (see Halmos [9]).

Case Ω ≠ 0. Referring to results obtained in Shaharir and Zainal [20], the solution of the complex diffusion equation (1.1) is given by

$$R(\boldsymbol{q},t) = \int_{R^n} \left(\frac{1}{2\pi\left|\sum(t)\right|}\right)^{n/2} \exp(-g(\boldsymbol{z},\boldsymbol{z})) \exp\left(2\beta \int_0^t V(\sigma(s))\, ds\right) \varphi\left(\boldsymbol{z} + \cos^{-1}\left[\sqrt{\Omega}\, t\right] \boldsymbol{X}\right) d\boldsymbol{z},$$

$$\boldsymbol{X} = \boldsymbol{q} + \left(\sqrt{\Omega}\right)^{-1} \boldsymbol{L}, \quad \boldsymbol{Y} = \boldsymbol{q_0} + \left(\sqrt{\Omega}\right)^{-1} \boldsymbol{L}.$$

or

$$= \int_{R^n} \left(\frac{\left|\sqrt{\Omega}\right|}{4\pi\alpha \left|\tan\left[\sqrt{\Omega}\, t\right]\right|}\right)^{n/2} \left(\cos\left[\sqrt{\Omega} t\right]\right)^{-n/2} \exp\left[\frac{-1}{4\pi\alpha} \boldsymbol{z}^{\boldsymbol{T}} \left(\sqrt{\Omega}\, \tan^{-1}\left[\sqrt{\Omega} t\right]\right) \bar{\boldsymbol{z}}\right.$$

$$+ \frac{1}{4\alpha} \boldsymbol{X}^{\boldsymbol{T}}\left(\sqrt{\Omega}\tan\left[\sqrt{\Omega}\,t\right]\right)\overline{\boldsymbol{X}}\Big] \exp\left(n\beta\left(P - \frac{1}{2}\boldsymbol{L}^{\boldsymbol{T}}\left(\sqrt{\Omega}\right)^{-1}\overline{\boldsymbol{L}}\,t\right)\right)\varphi\left(\boldsymbol{z} + \cos^{-1}\left[\sqrt{\Omega}\,t\right]\boldsymbol{X}\right)d\boldsymbol{z}$$

$$= \int_{C^n}\left(\int_{C^n}\left(\frac{\left|\sqrt{\Omega}\right|}{4\pi\alpha\left|\tan\left[\sqrt{\Omega}\,t\right]\right|}\right)^{n/2}\exp\left[\frac{-1}{4\pi\alpha}\omega^{\boldsymbol{T}}\left(\sqrt{\Omega}\ \tan^{-1}\left[\sqrt{\Omega}\,t\right]\right)\overline{\omega}\right]d\omega\right)\left(\cos\left[\sqrt{\Omega}\,t\right]\right)^{-n/2}$$

$$\exp\left(-\alpha\boldsymbol{Y}^{\boldsymbol{T}}\left(\left(\sqrt{\Omega}\right)^{-1}\tan\left[\sqrt{\Omega}\,t\right]\right)\overline{\boldsymbol{Y}} + k\,\boldsymbol{X}^{\boldsymbol{T}}\cos^{-1}\left[\sqrt{\Omega}\,t\right]\overline{\boldsymbol{X}}\right.$$

$$\left. + \frac{1}{4\alpha}\boldsymbol{X}^{\boldsymbol{T}}\left(\sqrt{\Omega}\tan\left[\sqrt{\Omega}t\right]\right)\overline{\boldsymbol{X}}\right)\exp\left(n\beta\left(P - \frac{1}{2}\boldsymbol{L}^{\boldsymbol{T}}\left(\sqrt{\Omega}\right)^{-1}\overline{\boldsymbol{L}}\right)t\right)du_{\varphi}(\boldsymbol{Y}),$$

$t \in (0, t_{\min})$, $t_{\min} = \min_j \frac{\pi}{\sqrt{\lambda_j}}$ for $\alpha\beta < 0$; or $t > 0$ for $\alpha\beta > 0$; or

$t_{\min} = \min_j \frac{\pi}{Im\left(\sqrt{\lambda_j}\right)}$ for $\alpha\beta \in \boldsymbol{C}$, where $\omega = \boldsymbol{z} - 2k\alpha\left(\left(\sqrt{\Omega}\right)^{-1}\tan\left[\sqrt{\Omega}t\right]\right)\boldsymbol{Y}$,

together with certain manipulation with respect to the second factor of the integrand and the condition that $(\alpha(\sqrt{\Omega})^{-1}\tan(\sqrt{\Omega}t))$ is a positive semidefinite complex (hermitian) matrice with non negative real eigenvalues (see Gantmacher [7], Andersen *et al*. [5]);

$$R(\boldsymbol{q}, t) = \int_{C^n} \exp\left(-g(\boldsymbol{Y}, \boldsymbol{Y})\ \exp\left(\beta\int_0^t V(\sigma(s))\,ds\right)\ d\mu_{\varphi,x}(\boldsymbol{Y})\right);$$

where

$$d\mu_{\phi,x}(\boldsymbol{Y}) = \exp\left(k\boldsymbol{Y}^{\boldsymbol{T}}\cos^{-1}\left[\sqrt{\Omega}\,t\right]\boldsymbol{Y}\right)\ d\mu_{\phi}(\boldsymbol{Y}),$$

and taking into account Fubini's theorem and characteristics of the 'complex multinormal distribution'. We can use the same arguments as the case $\Omega = 0$ to obtain the result (1.5).

Remarks 6

(a) $T_{\rho_0}\Gamma^{kla}$ represents the tangent vector set at $\beta_0 \in \Gamma^{kla}$. The space $T_{\rho_0}\Gamma^{kla}$ is identified naturally with Γ^{kla} by associating a tangent vector with an element of Γ^{kla}. Furthermore, $T_{\rho_0}\Gamma^{kla}$ being a vector space, it can be naturally identified as isomorphic with respect to Γ^{kla}. Nevertheless this natural isomorphism is no longer assumed when we generalise to a manifold. In this case it is necessary to treat the tangent spaces at various points as different (see for example, Crampin and Pirani [6]).

(b) Referring to the formulation of Albeverio and Hoegh-Krohn [2], the potential function V as in Theorem 5 (affine complex and quadratic complex potentials) is derivable from the space $F(\boldsymbol{C^n})$, Fresnel integrable function space on $\boldsymbol{C^n}$. Briefly V is a Fourier transformation of μ, which is a bounded complex Borel measure on $\boldsymbol{C^n}$, i.e.

$$V(\boldsymbol{q}) = \int_{C^n} \exp(k\,\boldsymbol{q}\cdot\bar{\boldsymbol{r}}\,)\,d\mu(\boldsymbol{r}). \tag{1.7}$$

With reference to Parthasarathy [13],

(i) $V(\boldsymbol{q})$ is uniformly continuous in a normed topology,

(ii) if $V_1(\boldsymbol{q}) = V_2(\boldsymbol{q})$ for all $\boldsymbol{q} \in \boldsymbol{C^n}$, then $\mu_1 = \mu_2$,

(iii) convolution $(V_1 * V_2)(\boldsymbol{q}) = V_1(\boldsymbol{q})V_2(\boldsymbol{q})$ for all $\boldsymbol{q} \in \boldsymbol{C^n}$ and $\mu_1,\ \mu_2$ are bounded Borel measures on $\boldsymbol{C^n}$, and

(iv) $\bar{V}(\boldsymbol{q}) = \overline{V(\boldsymbol{q})}$.

In addition, $V : \boldsymbol{C^n} \to \boldsymbol{C}$ is a function S^∞ whereby the first differential is bounded and all the higher order derivatives are most likely experiencing linear growth, i.e. (see Albeverio and Brzezniak [4])

$$\left|D^{v}V(\boldsymbol{q})\right| \le \begin{cases} M & , \quad |v| = 1 \\ m(1+|\boldsymbol{q}|) & , \quad |v| > 1 \end{cases} \tag{1.8}$$

3. Conclusions

Theorem 5 generates a solution for the complex diffusion equation (1.1) with a complex quadratic potential $V(\boldsymbol{q}) = \left(\frac{1}{2}\right)\boldsymbol{q}^{\boldsymbol{T}}\Omega\boldsymbol{q} + \boldsymbol{L}\cdot\boldsymbol{q} + P$ (Ω is nonsingular Hermitean matrix, if $\Omega \neq 0$). This solution is in the form of a Feynmannian path integral (1.5a) with the measure (1.5b) on the classical path space Γ^{kla}. This result enables the extension of our framework to formulate the Feynman integral in order to include complex quadratic potential in unitary space $\boldsymbol{C^n}$. As a matter of fact, this result shows qualitatively the connection between the Feynmannian integral with the original Feynman path integral and its real integral form. Clearly this is shown *via* the pull-back operation in Γ^{kla} with respect to the path integral solution of the complex diffusion equation with quadratic potential in $\boldsymbol{C^n}$. The extension to the quadratic potential in Riemannian manifold can be done similarly.

Acknowledgement. Professor Shaharir Mohamad Zain has furnished me with some intense discussion and words of encouragement. I dearly appreciated these. The author likes to thank the referee whose comments led to some improvement of the presentation of the results.

References

1. R. Abraham, J.E. Marsden and T. Ratiu, *Manifolds, Tensor Analysis and Applications,* Addison-Wesley, Reading, 1983.
2. S.A. Albeverio and R.J. Hoegh-Krohn, Mathematical theory of Feynman path integral. *Lect. Notes in Maths,* Springer-Verlag, Berlin, **523** (1976).
3. S.A. Albeverio and R.J. Hoegh-Krohn, Feynman path integrals and corresponding method of stationary phase, in S.A. Albeverio & *et al.* (eds.), Feynman path integrals*, Lect. Notes in Phys*., Springer-Verlag, Berlin **108** (1979), 3-57.
4. S.A. Albeverio and Z. Brzezniak, Finite dimensional approximation approach to oscillatory integrals and stationary phase in infinite dimensions, *J. Funct. Anal.* **113** (1993), 117-334.
5. H.H. Andersen, M. Hojbjerre, D. Sorensen and P.S. Eriksen, *Linear and Graphical Models for the Multivariate Complex Normal Distribution,* Springer-Verlag, New York, 1995.
6. M. Crampin and F.A.E. Pirani, *Applicable Differential Geometry,* Cambridge Univ. Press, Cambridge, 1986.
7. F.R. Gantmacher, *The Theory of Matrices,* Chelsea Pub. Co., New York, 1960.
8. D.H. Griffel, *Applied Functional Analysis,* Ellis Horwood, Chichester, 1981.
9. P. Halmos, *Measure Theory,* Springer-Verlag, Berlin, 1974.
10. A. Kufner, O. John and S. Fucik, *Function Spaces,* Noordhoff Pub., Leyden, 1977.
11. S. Lang, *Differential Manifolds,* Springer-Verlag, New York, 1985.
12. M.A. Naimark, *Normed Rings,* Noordhoff Pub., Groningen, 1959.
13. K.P. Parthasarathy, *Probability Measures on Metric Spaces,* Academic Press, New York, 1967.
14. E. Prugovecki, *Quantum Mechanics in Hilbert Space,* Academic Press, New York, 1981.
15. K. Rektorys, *Variational Methods in Mathematics, Science and Engineering,* D. Reidel Pub., Dordrecht, 1980.
16. M.Z. Shaharir, New framework for the Feynman integral, *Int. Jour. Theor. Phys.* **10** (1986), 1075-1094.
17. M.Z. Shaharir, On complex normal distribution, *Sains Malaysiana* **16** (1987), 397-408.
18. M.Z. Shaharir, A unifying method of obtaining the Feynman, Kac, Cameron and Albeverio & Hoegh-Krohn solution to the heat and Schroedinger equations, in H. Ning and W. Chong-shi (eds.), Particle and nuclear physics*, World Scientific, Singapore* (1987), 217-226.
19. M.Z. Shaharir, An improved version of the Feynman-Kac solution to a complex diffusion equation with a quadratic potential, in Subanar *et al* (eds.), *Proc. of the Mathematical Analysis and Statistics Conference,* Gadjahmada Univ.,Yogyakarta, Indonesia (1995), 268-276.
20. M.Z. Shaharir and A.A. Zainal, Penyelesaian kamiran nyata bagi persamaan resapan kompleks berpotensi kuadratik teritlak kompleks (*in Malay*), *Matematika* **13** (1997), 29-40.