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# New Proofs of the Uniqueness of Extremal Noneven Digraphs

CHJAN C. LIM AND GREGORY K. VAN PATTEN
Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180
e-mail: limc@rpi.edu

**Abstract.** The authors give a new graph-theoretic proof of the uniqueness of a class of extremal noneven digraphs, a result originally obtained by Gibson. The method of proof is based on mathematical induction on the number $N$ of vertices *in* the digraphs, and the fact that there are a limited number of ways to add a new vertex and a fixed number of arcs to a given maximal noneven digraph to obtain a larger maximal noneven digraph. A by product of this approach is a new short proof of the extremal result: noneven digraphs on $N$ vertices can have at most $M(N) = \frac{1}{2}(N^2 + N - 2)$ arcs. The same approach is then adapted to prove the uniqueness of a class of near extremal maximal noneven digraphs on $N \geq 5$ vertices. To start the induction process, the authors show directly, without using a computer search, that there is exactly one class of near extremal maximal noneven digraphs on 5 vertices.

## 1. Introduction

In this paper we give new and direct graph-theoretic proofs of two uniqueness results for noneven digraphs. The first result, originally proved by Gibson [4], is the uniqueness of an equivalence class of noneven digraphs on $N$ vertices with $M(N) = \frac{1}{2}(N^2 + n - 2)$ non self-loop arcs. We call this class $T_c(N)$. It consists of the "Caterpillar" digraphs defined by Thomassen [23], and is related to a class of full binary trees by Lim [8]. As a corollary to our new proof of Gibson's result, we show that $M(N)$ is actually the largest possible number of non self-loop arcs in a noneven digraph on $N$ vertices. Gibson's result was given in the context of permanents and the conversion problem of Polya. The second result is the uniqueness of an equivalence class of the maximal noneven digraphs on $N > 4$ vertices with $M(N) - 1$ non self-loop arcs. We call this class $T_1(N)$, to remind the reader that it is also related to a class of full binary trees by the same algorithm as above [8].

Our method of proof is based on mathematical induction on the number $N$ of vertices in the digraph. The main idea of our approach to these results is the fact that there are very few ways to add a vertex, and a fixed number of arcs to a given maximal noneven digraph on $k$ vertices, and end up with another maximal noneven digraph on $k+1$

vertices. This fact is confirmed separately in Lemma 1 in the proof of the first uniqueness result, and Lemmas 2, 3, and 4 in the proof of Theorem 2. The proofs of these Lemmas are based largely on some counting arguments and the so-called pigeon-hole principle. Our proof of the second result, Theorem 2, is direct in the sense that we do not use Thomassen's result on the uniqueness of the noneven digraph with high total degree [23]. Indeed a by-product of our method yields a direct proof of Theorem 2.7 in [23](cf. also [10]). The same graph-theoretic method proves that there are exactly two classes of maximal noneven digraphs on $N$ vertices with $M(N)-2$ non self-loop arcs.

This paper is organized as follows. Section 2 gives the basic definitions that are used later; it also contains a brief summary of the historical development of this and related subjects. Section 3 contains an outline of the proofs of Theorems 1 and 2, as well as several basic results that we need in the proofs of the main theorems. In section 4, we give the proof of the first uniqueness result and in section 5 we give the proof of the second uniqueness result. In the appendix we give the outline to a proof of the result that there is exactly one class of maximal noneven digraphs on 5 vertices with $M(5)-1$ arcs.

## 2. Background

The following formulation of this problem in terms of sign-nonsingularity is standard [3], [1]. A *zero-pattern* is a matrix $A=[a_{i,j}]$ of 0's and 1's. A *signing* of the entries of $A$ is a matrix $A'=[a'_{i,j}]$ of 0's, 1's, and $-1's$, such that $\left|a'_{i,j}\right|=a_{i,j}$ for each entry $a_{i,j}$. When a zero-pattern $A$ is a $k\times k$ matrix, it is said to support *sign-nonsingularity* if there exists a signing of its entries such that the resulting matrix $A'$ is invertible, and every nonzero term in its standard determinant expansion has the same sign. That is, the product $sign(\sigma)a'_{1,\sigma(1)}a'_{2,\sigma(2)}a'_{k,\sigma(k)}$ has the same sign (or is zero) for every permutation $\sigma$ of $\{1,2,\cdots,k\}$, where $sign(\sigma)$ denotes the sign of $\sigma$. If $A$ supports sign-nonsingularity, then transposing $A$, resigning rows and columns of $A$, or permuting rows and columns of $A$ all result in a new zero-pattern which also supports sign-nonsingularity. Thus, if we say that two matrices are *equivalent*, we mean that one can be obtained from the other by such operations. Because the factors of any term in a matrix's standard determinant expansion are the diagonal entries of some equivalent matrix, every zero-pattern that supports sign-nonsingularity is equivalent to a matrix whose diagonal entries are all nonzero. A zero-pattern which supports sign-nonsingularity is *maximal* if either there are no zero entries, or changing any entry which is zero to a nonzero entry would result in a zero-pattern which does not support sign-nonsingularity. Note that if a matrix has a maximal zero-pattern then any equivalent matrix also has a maximal zero-pattern.

Given a $k\times k$ zero-pattern $A$, we may construct the following bipartite graph $B$ with bipartition $\{P,Q\}$, where $P$ contains the vertices $p_1,p_2,\cdots,p_k$ and $Q$ contains the vertices $q_1,q_2,\cdots,q_k$. An edge joins vertices $p_i$ and $q_j$ if and only if the entry $a_{ij}$ of $A$ is not zero. Note that all of the bipartite graphs which we thus associate with a particular equivalence class of zero-patterns are isomorphic. An *orientation* of $B$ is achieved by changing each edge to a directed arc. *B* is said to have a *pfaffian orientation* if there exists an orientation of *B* such that, as one traverses any simple closed path in *B*, the number of arcs on that path, which are directed the same way the path is traversed, is odd. It turns out that a zero-pattern *A* supports sign-nonsingularity if and only if the corresponding bipartite graph *B* has a pfaffian orientation. *B* is maximal when *A* has a maximal zero-pattern. The bigraph formulation is useful in light of Kasteleyn's theorem [6], which tells us that every planar bipartite graph has a pfaffian orientation, and a generalization of Kasteleyn's theorem by Little [13], which relates pfaffian orientation to the existence of a subdivision of $K_{3,3}$, the complete bipartite graph on six vertices.

Given a $k \times k$ zero-pattern *A*, we may also construct the following digraph *G*, with vertices $v_1, v_2, \cdots, v_k$. An arc is directed from $v_i$ *i* to $v_j$, when $i \neq j$, if and only if the entry $a_{ij}$ of *A* is not zero. In general, the family of digraphs that we thus associate with a particular equivalence class of zero-patterns are not all isomorphic. However, we say that two digraphs are equivalent when they correspond to equivalent matrices. *A* digraph is called noneven if there is a signing of its arcs with $\{1, -1\}$ such that the product of the signs along each simple directed cycle is negative. Otherwise, a digraph is called even. It turns out that, when every diagonal entry $a_{i,i}$ of the $k \times k$ zero-pattern *A* is nonzero, *A* supports sign-nonsingular if and only if the associated digraph *G* is noneven. *G* is maximal if and only if *A* has a maximal zero-pattern. The digraph formulation is useful in light of various theorems which equate noneveness with the existence of certain structures in a digraph. Seymour and Thomassen [20], show that a digraph is even if and only if it contains a subdivision of $C_{2n+1}$ (i.e., contains a "weak odd double-cycle"). This is equivalent to Little's result mentioned in the previous paragraph involving subdivisions of $K_{3,3}$.

The bipartite graph *B* which we associate with a given zero-pattern can be transformed into the associated digraph *G* as follows. Replace each edge joining $p_i$ with $q_j$, where $i \neq j$, with an arc directed from $p_i$ to $q_j$. Delete any edge joining $p_i$ with $q_j$ when $i = j$. Then replace each pair of vertices $\{p_i, q_i\}$ with the single vertex $\{v_i\}$, so that all arcs that were directed from $p_i$ are now directed from $v_i$, and all arcs that were directed toward $q_i$ are now directed toward $v_i$. In this way the digraph *G* is obtained. Conversely, when every diagonal entry $a_{i,i}$ of *A* is nonzero, the bipartite graph *B* can be obtained by first splitting each vertex of *G*, and then replacing all directed arcs by undirected edges.

Previous work in this area appears to be along two distinct but related branches, namely (a) sign-solvability of linear systems and their applications in mathematical biology and economics,and (b) enumeration of perfect matchings or 1-factors of bipartite graphs and their applications in theoretical physics. Samuelson's work [19] in economics provided the important impetus for the branch of results on sign-solvable problems [1], [7], [21], [8], [20], [15]. Kastelyn's methods on the Ising Model [6] started the line of work on the enumeration of perfect matchings by computing pfaffians, permanents, and determinants [12], [4], [2], [24], [9], [14], [22], [18]. This field continues to be a very active one, and recently McCuaig, Robertson, Seymour and Thomas [16] proved an important structure theorem for noneven digraphs.

## 3. Preliminary results

In carrying out our plan, we find it most convenient to use the digraph formulation. Our first result states that there is exactly one equivalence class of noneven digraphs on $N$ vertices with $M(N)$ arcs, which is in fact an equivalence class of maximal noneven digraphs. This result allows us to show, as a corollary, that $M(N)$ is the most arcs that a noneven digraph on $N$ vertices may have. Using the same techniques, we then prove our second result which states that, for $N \geq 5$ there is exactly one equivalence class of maximal noneven digraphs on $N$ vertices with $M(N)-1$ arcs. We know that for $N = 4$ there are exactly two such equivalence classes and for $N = 5$ there is exactly one such class from the computer search of [11]. These preliminary results for $N = 4$ and 5 are confirmed rigorously in the proof of Lemma 5 in the appendix. There are no such equivalence classes for $N \leq 3$.

We now formally define the terminology and notation which appear in our proofs. A *digraph* is a set of vertices $\{v_1, v_2, \cdots\}$, along with a set of *arcs*. An arc is an ordered pair of vertices, $\langle v_1, v_2 \rangle$. It is customary to write such an arc as $v_1 \to v_2$, and this is how we denote arcs in our proofs. The arc $v_1 \to v_2$ is said to be *directed* from $v_1$ to $v_2$. $v_1$ is called the *tail* of the arc $v_1 \to v_2$, and $v_2$ is called its *head*. In our proofs, the head and tail of an arc are always distinct vertices. $V(v_i)$ indicates the degree of vertex $v_i$ in a digraph, which is the total number of arcs directed toward and from $v_i$. If we delete from digraph $G$ a set of vertices $S$ along with any arcs directed toward or from some member of $S$, the resulting subdigraph $G\backslash S$ is called an *induced subdigraph* of $G$. A *directed cycle* is a set of arcs of the form $\{v_1 \to v_2, v_2 \to v_3, \cdots, v_k \to v_1\}$. We indicate a directed cycle by the notation $(v_1, v_2, \cdots, v_k)$. In particular, $(v_1, v_2)$ is the 2-cycle, $\{v_1 \to v_2, v \to v_1\}$. $C_n^*$ is the digraph consisting of the 2-cycles $\{v_1, v_2), (v_2, v_3), \cdots, (v_n, v_1)\}$.

Each of the following two operations on a noneven digraph produces a new noneven digraph which has one more vertex and one more arc. An *arc subdivision* of a digraph $G$ is achieved by marking a new vertex $vn$ on an arc $v_i \to v_j$ of $G$, which is replaced by the two new arcs $v_i \to v_n$ and $v_n \to v_j$. A *vertex splitting* of a digraph $G$ is achieved by replacing a vertex $v_o$ of $G$ by two new vertices $v_i$ and $v_j$, replacing each arc $v_o \to v_x$ in $G$ by $v_j \to v_x$, replacing each arc $v_x \to v_o$ in $G$ by $v_x \to v_i$, and adding arc $v_i \to v_j$. We indicate arc subdivisions of $C_n^*$ by the cycle notation

$\{(v_1, v_2, v_3), (v_4, \cdots\}$, while we use figures to illustrate vertex splittings, which can be more difficult to find. Suppose that digraph $G'$ is the result of repeated application of arc subdivisions and vertex splittings to some original digraph *G*. Then a digraph $G''$ which contains digraph $G'$ is said to *weakly contain G*.

We make use of the following Theorems and Lemmas in our proofs:

**Counting Lemma.** *For any digraph on N vertices,* $\frac{1}{2}\sum_{i=1}^{N} V(v_i) = A$, *where A is the total number of arcs in the digraph.*

**Noneven subdigraph Lemma.** *A subdigraph of a noneven digraph is noneven.*

**Significance of** $\boldsymbol{C^*_{2n+1}}$**:** A digraph is even if and only if it weakly contains $C^*_{n+1}$ for some positive integer *n* [20].

**Maximal subdigraph Lemma.** *Let D be a noneven digraph and M a maximal proper subdigraph. If vertex v is not in M, then at most one vertex of M is in a 2-cycle with v.*

## 4. First uniqueness result

We would like to demonstrate that there is exactly one equivalence class of noneven digraphs on *N* vertices with *M*(*N*) arcs. In fact, we show that this class is exactly $T_c(N)$, the so-called "Caterpillar" equivalence class of digraphs on *N* vertices [8], [23]. Members of the $T_c(N)$ equivalence class are maximal noneven digraphs with *M*(*N*) arcs.

We frequently refer to the "bouquet" form of $T_c(N)$ digraphs, which is illustrated in Figure 1. We always refer to the central vertex of such a digraph as *C*. Vertices other than *C* are called "petal vertices". The arcs which form the 2-cycles radiating out from *C* are called "radial arcs". The arcs joining the petal vertices are called "connecting arcs". If a connecting arc is directed from petal vertex *X* to petal vertex *Y*, we say that *X* is *higher* than *Y*, or that *Y* is *lower* than *X*. We number the petal vertices of the bouquet of $T_c(N)$ from lowest to highest, as $1, 2, \cdots, N-1$.

The petal vertices of the bouquet form of $T_c(N)$ form a totally ordered set, with respect to arc direction. Let us interpret "higher than" as the order relation >. Given any two petal vertices, $v_1$ and $v_2$, exactly one of the statements $v_1 > v_2$ and $v_2 > v_1$ is true. Given three petal vertices, $v_1, v_2$ and $v_3$, the statements $v_1 > v_2$ and $v_2 > v_3$ imply $v_1 > v_3$. If we add a new vertex *n* to a Caterpillar digraph in bouquet form, and join that vertex by arcs to petal vertices of the Caterpillar, then whenever an arc is directed from petal vertex *i* to *n*, and from *n* to petal vertex *j*, there cannot be an arc directed from *j* to *i* without forming a $C^*_3$. In this sense, the new vertex cannot violate the transitivity of the total order on the petal vertices. We make use of this observation in our proofs.

**Lemma 1.** *If G is a noneven digraph obtained by adding a new vertex and* $k+1$ *new arcs to a* $T_c(k)$ *digraph, then G is a* $T_c(k+1)$ *digraph.*

*Proof.* Let *n* denote the new vertex in *G*. Since $T_c(k)$ digraphs is maximal, one end of each arc added must be at *n*. By the pigeon-hole principle, *n* must be in a 2-cycle with at least one vertex of the $T_c(k)$ digraph, and by the maximal subdigraph lemma, at most one vertex of the $T_c^*$ digraph may be in a 2-cycle with *n*. We let *D* denote the vertex which is in the 2-cycle with *n*. A single arc joins *n* with every other vertex of the $T_c^*$ digraph.

Suppose *D* is a petal vertex. Then any arc which joins *n* with a petal vertex *j* lower than *D* must be directed toward *j* or else $\{(n, D, j), (j, C), (C, D)\}$ is an arc subdivision of $C_3^*$, and similarly any arc which joins *n* with a petal vertex higher than *D* must be directed toward *n*. This forces *D* to be either the highest or lowest petal vertex, or else $\{(n, p, C, q), (n, D), (C, D)\}$ is an arc subdivision of $C_3^*$, where *p* is lower than *D* and *q* is higher than *D*. If *D* is the highest petal vertex, then an arc joining *n* and *C* must be directed toward *C* or else $\{(n, p, C), (n, D), (D, C)\}$ is an arc subdivision of $C_3^*$, and similarly if *D* is the lowest petal vertex then an arc joining *n* and *C* must be directed toward *n*. In either case, the resulting digraph is a $T_c(k+1)$ digraph, in bouquet form with one branch.

The other possibility is that *n* is in the 2-cycle with *C* rather than with a petal vertex. Let petal vertex *i* be higher than petal vertex *j*, with an arc directed from *i* to *j*. Suppose that an arc is directed from *n* to *i*, and that another arc is directed from *j* to *n*. Then $\{(n, i, j), (i, C), (j, C)\}$ is a $C_3^*$. So, letting *h* be the highest petal vertex such that an arc is directed from *n* to *h*, an arc must be directed from *n* to all petal vertices lower than *h*. An arc is directed toward *n* from each of the remaining petal vertices. Thus we see that *n* simply forms a petal vertex of a $T_c^*(k+1)$ digraph.

We can now prove the first theorem by mathematical induction.

**Theorem 1.** *For* $N \geq 3$, $T_c(N)$ *is the unique equivalence class of noneven digraphs on N vertices with M(N) arcs.*

*Proof.* Our proof is by induction on the number of vertices. It is easily verified that $T_c(3)$ is the only equivalence class of noneven digraphs on three vertices with *M*(3) arcs. We now assume that $T_c(N)$ is the only equivalence class of noneven digraphs on *N* vertices with *M*(*N*) arcs for $N = 3, \cdots, k$ and show that it is the only such equivalence class for $N = k+1$ as well. Consider a noneven digraph *G* on $k+1$ vertices with

$M(k+1)$ arcs. Suppose that for every vertex *i* in *G*, $V(i) > k+1$. Then by our counting lemma, *G* has at least $\frac{1}{2}(k+1)(k+2)$ arcs, which is more than the number of arcs, $M(k+1)$, that we are assuming *G* has. So *G* contains at least one vertex $i_1$ such that $V(i_1) \leq k+1$. Next, suppose there is some vertex $i_2$ in *G* with $V(i_2) \leq k+1$. Consider the induced subdigraph $G \setminus \{i_2\}$ on *k* vertices, obtained by deleting vertex $i_2$. The noneven subdigraph lemma tells us that $G \setminus \{i_2\}$ is noneven. The number of arcs in $G \setminus \{i_2\}$ is given by subtracting $V(i_2)$ from the number of arcs in *G*, *M*(*k*+1). Thus $G \setminus \{i_2\}$ has at least $M(k+1) - k$ arcs, which is more than $M(k)$. Thus we may delete arcs from $G \setminus \{i_2\}$ until we are left with a noneven digraph on *k* vertices with $M(k)$ arcs. Our induction hypothesis tells us that we must then have a $T_c(k)$ digraph. However, a $T_c(k)$ digraph is maximal, so that it could not be a subdigraph of a noneven digraph on the same number of vertices with more arcs. Thus, every vertex *i* of *G* satisfies $V(i) \geq k+1$. So the vertex $i_1$, which has total degree $V(i_1)$ less than or equal to $k+1$, actually satisfies $V(i_1) = k+1$. Consider the induced noneven subdigraph $G \setminus \{i_1\}$ obtained by deleting vertex $i_1$ from *G*. It has $M(k)$ arcs and thus by our induction hypothesis is a member of $Tc(k)$. Then, however, by Lemma 1, we know that *G* is actually a member of $T_c(k+1)$.

We now use Theorem 1 to show,

**Corollary.** *No noneven digraph on N vertices can have more than M(N) arcs.*

*Proof.* Suppose we have such a digraph, *G* on *N* vertices. Then, we can delete arcs from *G* until we have a noneven subdigraph $G'$ on *N* vertices with *M*(*N*) arcs, which by Theorem 1 is a $T_c(N)$ digraph. But a $T_c(N)$ digraph cannot be a proper subdigraph of a noneven digraph on *N* vertices, because $T_c(N)$ digraphs are maximal. So *G* could not have more than $M(N)$ arcs to begin with.

## 5. Second uniqueness result

Now we would like to demonstrate, using a method similar to the one used in Theorem 1 that, for $N \geq 5$, there is exactly one equivalence class of maximal noneven digraphs on *N* vertices with $M(N) - 1$ arcs, which we denote as $T_1(N)$ for $N \geq 5$. $T_1(N)$ digraphs is first constructed.

Here is a description of the standard form of $T_1(N)$ digraphs that we refer to for the rest of this paper. Figure 2 shows a $T_1(N)$ digraph in this form. Start with a $T_c(N-1)$ digraph, for some $N \geq 4$. The vertices and arcs of this $T_c(N-1)$ digraph are collectively called the "underlying Caterpillar" within the $T_1(N)$ digraph which we are about to construct. Introduce a new vertex called *n*. Now add arcs directed from *n* to all vertices of the underlying Caterpillar, except for the highest petal vertex. The second highest petal vertex of the underlying Caterpillar is called *D*. To complete the construction, add the arc $D \to n$. When we mention *C* or a petal vertex in a $T_1(N)$ digraph, we are referring to the corresponding vertices in the underlying Caterpillar.

We need the following lemmas for the induction proof of Theorem 2. Lemma 2 tells us that when we start with a $T_c(k)$ digraph, for $k \geq 4$, if we add one new vertex and *k* arcs to form a maximal noneven digraph *G* on $k+1$ vertices, then $G \in T_1(k+1)$. Lemma 3 tells us that when we start with a $T_c(k)$ digraph which is missing one arc, where $k \geq 4$, if we add one new vertex and *k*+1 new arcs such that each new arc has one end at the new vertex, to form a maximal noneven digraph *G* on $k+1$ vertices, then $G \in T_1(k+1)$. Lemma 4 tells us that when we start with a $T_1(k)$ digraph, where $k \geq 5$, if we add one new vertex and $k+1$ arcs to form a maximal noneven digraph *G* on $k+1$ vertices, $G \in T_1(k+1)$. Lemma 5 tells us that $T_1(5)$ is the only equivlence class of digraphs on 5 vertices with $M(5)-1$ arcs that is noneven and maximal.

**Lemma 2.** For $k \geq 4$, when a new vertex, *n*, and *k* new arcs are added to a $T_c(k)$ digraph to form a maximal noneven digraph *G*, then $G \in T_1(k+1)$.

*Proof.* Without loss of generality we assume the $T_c(k)$ digraph is in bouquet form. Suppose a single arc joins each vertex of the $T_c(k)$ digraph with *n*. As in the last paragraph of the proof of Lemma 1, we know that there is a highest petal vertex *h* such that an arc is directed from *n* to *h*, that an arc is directed from *n* toward any petal vertices lower than *h*, and that an arc is directed toward *n* from any remaining petal vertices. In this case, the resulting digraph is not maximal, because by adding a second arc joining *n* and *C*, a $T_c^*(k+1)$ digraph is formed.

Thus we know that *n* is in at least one 2-cycle with a vertex of the $T_c(k)$ digraph. From the maximal subdigraph lemma we know that *n* may be in a 2-cycle with at most one vertex of the $T_c(k)$ digraph. So *n* is in exactly one 2-cycle with a vertex of the $T_c(k)$ digraph. If that 2-cycle is with *C*, then the resulting digraph is not maximal. This is because the arcs joining *n* with the petal vertices do not violate the transitivity of the order on the petal vertices, so adding one more arc joining *n* with the remaining petal vertex forms a $T_c(k+1)$ digraph. Thus, *n* is in exactly one 2-cycle with some petal vertex which we denote as *D*.

As in the proof of Lemma 1, any arc which joins *n* with a petal vertex lower than *D* must be directed toward that petal vertex, and any arc which joins *n* with a petal vertex higher than *D* must be directed toward *n*, so that there is a $C_3^*$ by arc splitting if there are both at least two petal vertices above *D* and at least two petal vertices below *D*. If *D* is the highest or the lowest pedal vertex, then the resulting digraph is not maximal because by adding one more arc we can form a $T_c(k+1)$ digraph. Thus the only two possible ways to place all *k* arcs to get a maximal noneven digraph are obtained by either:

(1) letting $D$ be the second highest petal vertex, not joining $n$ with the highest petal vertex, and joining $n$ with all other vertices of the $T_c(k)$ digraph by arcs directed toward each of those vertices, or (2) letting $D$ be the second lowest petal vertex, not joining $n$ with the lowest petal vertex, and joining $n$ with all other vertices of the $T_c(k)$ digraph by arcs directed toward $n$. The last step is to show that these two digraph forms, described by (1) and (2) respectively, are actually in the same equivalence class. Number the vertices, from 1 to $N$, of both digraph forms. Next, reverse the directions of all arcs in the form described by (1); this corresponds to transposing the entries of its adjacency matrix. Now the two forms are identical as digraphs, with different labelings. Relabeling the vertices of a digraph corresponds to a row and collumn permutation of its adjacency matrix. So we have demonstrated that both forms of digraph are in the $T_1(k+1)$ equivalence class.

**Lemma 3.** *When we start with a $T_c(k)$ digraph which is missing one arc, where $k \geq 4$, if we add one new vertex, n, and $k+1$ new arcs so that one end of each new arc is at the new vertex, to form a maximal noneven digraph G, then $G \in T_1(k+1)$.*

*Proof.* Without loss of generality we assume the $T_c(k)$ digraph missing one arc is actually a $T_c(k)$ digraph in bouquet form missing one arc. For brevity, we refer to the $T_c(k)$ digraph missing one arc as "the incomplete Caterpillar". We consider separately the cases when the missing arc is either a connecting arc or a radial arc. For radial arcs, without loss of generality, we assume they are directed toward $C$, for otherwise it can be treated similarly due to symmetry.

First we establish that $n$ can be in a 2-cycle with at most one vertex of the incomplete Caterpillar. Suppose the missing arc is a connecting arc, and that $n$ is in 2-cycles with two vertices of the incomplete Caterpillar. $n$ cannot be in 2-cycles with both $C$ and a petal vertex $j$ or else $\{(n,j),(j,c),(C,n)\}$ is a $C_3^*$. Suppose $n$ is in 2-cycles with two petal vertices, $i$ and $j$. Since $k$ is at least 4, $n$ is joined by an arc with a third vertex of the incomplete Caterpillar. If this third vertex is $C$ then either $\{n,C,i),(n,j),(j,C)\}$ or $\{(C,n,i),(n,j),(j,C)\}$ is an arc subdivision of $C_3^*$, depending on whether the arc joining $n$ and $C$ is directed toward $n$ or $C$; if this third vertex is another petal vertex $h$ then either $\{(n,h,C,j),(n,i),(i,C)\}$ or $\{(j,C,h,n),(n,i),(i,C)\}$ is an arc subdivision of $C_3^*$, depending on whether the arc joining $n$ and $h$ is directed toward $n$ or $h$. So if the missing arc is a connecting arc, $n$ is in exactly one 2-cycle with the incomplete Caterpillar.

Now suppose the missing arc would be a radial arc directed toward $C$ if present. We denote the petal vertex which is not in a 2-cycle with $C$ by $p$. $n$ cannot be in 2-cycles with $C$ and a petal vertex $j$ different from $p$ or else $\{(i,n,j),(j,C),(C,i)\}$ is a $C_3^*$, and $n$ cannot be in 2-cycles with two pedal vertices, $i$ and $j$, which are both different from $p$ or else $\{(i,n,j),(j,C),(C,i)\}$ is an arc subdivision of $C_3^*$. In all remaining instances when $n$ is in more than one 2-cycle, $n$ is in a 2-cycle with $p$. We consider first those cases when $p$ is not the lowest petal vertex. $n$ cannot be in 2-cycles with both $p$ and a petal vertex $i$

lower than $p$ or else $\{(n,p),(C,p,i),(n,i)\}$ is an arc subdivision of $C_3^*$. If petal vertex $i$ is lower than $p$, then $n$ cannot be in 2-cycles with both $p$ and a petal vertex $j$ higher than $p$ or else $\{j,p,i,C),(n,j),(n,p)\}$ is an arc subdivision of $C_3^*$. If petal vertex $i$ is lower than $p$ then $n$ cannot be in 2-cycles with $p$ and $C$ or else $\{(p,i,C),(n,p),(n,C)\}$ is an arc subdivision of $C_3^*$. Now we consider those cases when $p$ is the lowest petal vertex. If $p$ is the lowest petal vertex and $n$ is in 2-cycles with $p$ and a petal vertex which is not the second lowest petal vertex, then the digraph weakly contains $C_3^*$ by a combination of arc subdivision and vertex splitting (see Figure 3). If $p$ is the lowest petal vertex and $n$ is in 2-cycles with $p$ and the second lowest petal vertex $s$, then any arc joining $n$ with another petal vertex $j$ is directed toward $n$ or else $\{(s,n,j),(C,s),(C,j)\}$ is an arc subdivision of $C_3^*$, and if $C$ is joined with $n$ by an arc, that arc must be directed toward $n$ or else $\{(s,n),(s,C),(n,c,p)\}$ is an arc subdivision of $C_3^*$. Thus, when $p$ is the lowest petal vertex and $n$ is in 2-cycles with $p$ and the second lowest petal vertex, the resulting digraph is not maximal because it may be extended to a $T_c(k+1)$ digraph; so this is not allowed. Finally, if $p$ is the lowest petal vertex and $n$ is in 2-cycles with $C$ and $p$, then any arc which joins $n$ with another petal vertex $j$ must be directed toward $n$ or else $\{(n,j,p),(j,C),(n,C)\}$ is an arc subdivision of $C_3^*$. Thus, if $p$ is the lowest petal vertex and $n$ is in 2-cycles with $C$ and $p$, the resulting digraph is not maximal because it may be extended to a $T_c(k+1)$ digraph, and so this case is not allowed.

We have established that $n$ is in exactly one 2-cycle with a vertex of the incomplete Caterpillar. We denote this vertex by $D$. One end of the missing arc must be a petal vertex. Suppose that a petal vertex $q$ is different from $D$, and that the missing arc would have one end at $q$ if present. Then $V(q)=k$. If we delete $q$, the resulting digraph has $M(k)$ arcs and thus is a $T_c(k)$, by the noneven subdigraph lemma and also Theorem 1. But we know from Lemma 2 that when a new vertex and $k$ new arcs are added to a $T_c(k)$ digraph, the only maximal digraphs which may result are all equivalent to $T_1(k+1)$. The only cases left to consider, then, are when there is no such petal vertex $q$. That is, only one end of the missing arc is a petal vertex, and that petal vertex is $D$. So for the rest of the proof of Lemma 3 we only consider cases when $D$ is a petal vertex and the missing arc is a radial arc with one end at $D$.

We refer to the first petal vertex lower than $D$ as $L$ (if there is a petal lower than $D$), and we refer to the first petal vertex higher than $D$ as $H$ (if there is a petal vertex higher than $D$). $n$ joins any petal vertex petal $i$ lower than $L$ by an arc directed toward that petal vertex or else $\{i,n,D,L),(I,C),(i,C)\}$ is an arc subdivision of $C_3^*$, and similarly $n$ joins any petal vertex higher than $H$ by an arc directed toward $n$. Thus if $k>4$ there cannot be both a petal vertex $i$ lower than $L$ and a petal vertex $j$ higher than $H$ or else $\{(n,D),(n,i,C,j),(D,L,C)\}$ is an arc subdivision of $C_3^*$. Thus for $k\geq 4$ at least one of the following four statements is true:

(1) $D$ is the highest petal vertex.
(2) $D$ is the lowest petal vertex.
(3) $H$ is the highest petal vertex.
(4) $L$ is the lowest petal vertex.

We treat the four cases separately. We analyze case (3) only for $k \geq 5$. Cases (3) and (4) are the same when $k = 4$, and the proof when $k = 4$ is included in the analysis of case (4).

(1) For $k \geq 4$, if $D$ is the highest petal vertex then the arc joining $n$ with $C$ must be directed toward $C$ or else $\{(n,i,C), (n,D), (D,L,C)\}$ is an arc subdivision of $C_3^*$, where $i$ is a petal vertex lower than $L$. Then the only arc whose direction is still undetermined is that joining $n$ with $L$. This arc cannot be directed from $L$ to $n$ or else $\{(D,i,C), (C,L,n), (n,D)\}$ is an arc subdivision of $C_3^*$, where $i$ is a petal vertex lower than $L$, and it cannot be directed from $n$ to $L$ or else the resulting digraph is not maximal because it can be extended to a $T_c(k+1)$ digraph in bouquet form with one branch by adding the missing arc. So we have shown that case (1) is not possible for $k \geq 4$.

(2) For $k \geq 4$, when $D$ is the lowest petal vertex, we must consider four subcases.

(i) The arc joining $n$ and $C$ is directed toward $C$, and the arc joining $n$ and $H$ is directed toward $H$. This subcase is ruled out because $(n,C,j), (n,H,D), (H,C)\}$ is an arc subdivision of $C_3^*$, where $j$ is a petal vertex higher than $H$.

(ii) The arc joining $n$ and $C$ is directed toward $n$, and the arc joining $n$ and $H$ is directed toward $H$. This subcase is ruled out because the resulting digraph extends to a $T_c(k+1)$ digraph by adding an arc directed from $H$ to $n$ and thus is not maximal.

(iii) The arc joining $n$ and $C$ is directed toward $C$, and the arc joining $n$ and $H$ is directed toward $n$. This subcase is ruled out because the resulting digraph extends to a $T_c(k+1)$ digraph by adding and arc directed from $C$ to $n$ and thus is not maximal.

(iv) The arc joining $n$ and $C$ is directed toward $n$, and the arc joining $n$ and $H$ is directed toward $n$. This subcase is ruled out because the resulting digraph extends to a $T_c(k+1)$ digraph by adding an arc directed from $n$ to $H$ and thus is not maximal. So we have shown that case (2) is not possible for $k \geq 4$.

(3) For $k \geq 5$, when $H$ is the highest petal vertex, there are two subcases to consider.

(i) The arc joining *n* with *H* is directed toward *H*. This subcase is ruled out because $\{(D,n,H),(H,C),(D,L,C)\}$ is an arc subdivision of $C_3^*$.

(ii) The arc joining *n* with *H* is directed toward *n*. This subcase is ruled out because there is a vertex splitting of $C_3^*$ (see Figure 4). So we have shown that case (3) is not possible for $k \geq 5$.

(4) For $k \geq 4$, when *L* is the lowest petal vertex, the arc joining *n* with *H* must be directed toward *n* or else $\{(n,H,D),(H,C),(D,L,C)\}$ is an arc subdivision of $C_3^*$. This forces the arc joining *n* with *L* to be directed toward *n* or else there is a vertex splitting of $C_3^*$ (see Figure 5). This in turn forces the arc joining *n* with *C* to be directed toward *n* or else $\{(n,C,H),(n,D,L),(L,C)\}$ is an arc subdivision of $C_3^*$. Now that the directions of all arcs have been determined, we see that this case is not maximal because the resulting digraph extends to a $T_c(k+1)$ digraph by adding an arc directed from *L* to *D*. So we have shown that case (4) is not possible for $k \geq 4$. No step in the analysis of case (4) requires arcs entering or leaving petal vertices higher than *H*, so that this is a valid proof for the case $k = 4$ as well.

**Lemma 4.** *For $k \geq 5$, when we start with a $T_1(k)$ digraph and add a new vertex and $k+1$ new arcs to form a maximal noneven digraph G, then $G \in T_1(k+1)$.*

*Proof.* We denote the vertex which we add to our $T_1(k)$ digraph by *nn*. Since $T_1(k)$ is maximal, one end of each new arc must be at *nn*. The pigeon hole principle tells us that at least one vertex of the $T_1(k)$ digraph is in a 2-cycle with *nn*, and our maximal subdigraph lemma tells us that at most one vertex of the $T_1(k)$ digraph may be in a 2-cycle with *nn*. Thus, providing it is possible to add all $k+1$ new arcs without forming an even digraph, *nn* is in a 2-cycle with exactly one vertex of the $T_1(k)$ digraph, and we denote this vertex by *DD*.

If *DD* is a petal vertex of the underlying Caterpillar, then we know from our proof of Lemma 1 that it must be either the highest or the lowest petal vertex. Thus the four vertices which *DD* might be are:

(i) the highest petal vertex of the underlying Caterpillar,

(ii) the lowest petal vertex of the underlying Caterpillar,

(iii) *n*, or

(iv) *C*.

Suppose *DD* is the highest petal vertex of the underlying Caterpillar. If the arc joining *nn* with *n* is directed toward $n$ then $\{(n,i,C,DD,nn),(n,D),(D,C)\}$ is an arc subdivision of $C_3^*$, where *i* is a petal vertex of the underlying Caterpillar lower than *D*, and if the arc joining *nn* with *n* is directed toward *nn* then $\{(n,nn,DD,D),(D,C),(C,D,D)\}$ is an arc subdivision of $C_3^*$. So $DD$ is not the highest petal vertex of the underlying Caterpillar.

Suppose *DD* is the lowest petal vertex. Then we know from the proof of Lemma 1 that any arc joining *nn* with the other vertices of the underlying Caterpillar must be directed toward *nn*. The arc joining *nn* with *n* must also be directed toward *nn* or else $\{(nn,n,D,C),(nn,DD),(DD,C)\}$ is an arc subdivision of $C_3^*$. Since we have now established that each new arc which does not have an end at *DD* may be directed only one way, if the resulting digraph on $K=1$ vertices is noneven, then it is maximal. When *n* is deleted from that digraph, the induced subdigraph is just is just a $T_c(k)$ digraph in bouquet form with one branch. Therefore by Lemma 2 the digraph we constructed on $k+1$ vertices is a $T_1(k+1)$ digraph.

Suppose *DD* is *n*. Then the arc joining *nn* with the highest petal vertex must be directed toward *nn* or else $\{(nn,C,D,n),(D,C),(C,h)\}$, where *h* is the highest petal vertex, is an arc subdivision of $C_3^*$. Also, any arc joining *nn* with a petal vertex *i* lower than *D* must be directed toward that petal vertex or else $\{(i,nn,n,D),(i,C),(C,D)\}$ is an arc subdivision of $C_3^*$. Now consider the arc joining *nn* with *D*. If this arc is directed toward $D$ then $\{(nn,D,C,h),(nn,n),(n,D)\}$, where *h* is the highest petal vertex of the underlying Caterpillar, is an arc subdivision of $C_3^*$. If this arc is directed toward *nn* then $\{(D,nn,n),(nn,i,C,h),(D,C)\}$, where *i* is a petal vertex of the underlying Caterpillar lower than *D* and *h* is the highest petal vertex of the underlying Caterpillar, is an arc subdivision of $C_3^*$. So *DD* cannot be *n*.

Suppose *DD* is *C*. The arc joining *nn* with *n* must be directed toward *nn* or else $\{(nn,n,i,C),(n,D),(D,C)\}$, where *i* is a petal vertex lower than *D*, is an arc subdivision of $C_3^*$. This forces the arc joining *nn* with $D$ to be directed toward *nn* or else $\{(D,n,nn),(D,C),(C,nn)\}$ is arc subdivision of $C_3^*$, and this forces the arc joining *nn* with the highest petal vertex *h* of the underlying Caterpillar to be directed toward *nn* or else $\{(n,nn,h,D),(D,C),(C,nn)\}$ is an arc subdivision of $C_3^*$. The directions of the remaining arcs are now determined just as in the last paragraph of the proof of Lemma 1, in order to avoid arc subdivisions of $C_3^*$. Thus, *nn* is now just another petal vertex in a $T_c(k)$ digraph in bouquet form which has been extended to a $T_1(k+1)$ digraph.

**Lemma 5.** *$T_1(5)$ is the unique equivalence class of maximal noneven digraphs on 5 vertices with $M(5)-1$.*

In the appendix we outline a proof using Lemmas 1, 2, 3, and 4 as in the proof of Theorem 2, which does not require an exhaustive computer search of all possible cases. In the course of this proof, one can see that there are exactly two equivalence classes of maximal noneven digraphs on 4 vertices, with $M(4)-1$ arcs, namely the wheel $C_4^*$ and the digraph $T_1(4)$ which is equivalent to the balanced binary tree with four leaves (cf. Lim [8]).

Using the above lemmas, we now prove the second theorem.

**Theorem 2.** *For $N \geq 5$, $T_1(N)$ is the unique equivalence class of maximal noneven digraphs on N vertices with $M(N)-1$ arcs.*

*Proof.* Our proof is by induction on the number of vertices. Lemma 5 starts our induction for the case $N = 5$. Now we assume that $T_1(N)$ is the only such class possible for $N = 5, \cdots, k$ and then show that it is the only such class for $N = k+1$ as well. Let $G$ be a maximal noneven digraph on $k+1$ vertices with $M(k+1)-1$ arcs. Suppose that for every vertex $i$ in $G$, $V(i) > k+1$. Then by our counting lemma, $G$ has at least $\frac{1}{2}(k+1)(k+2)$ arcs, which is more than the number of arcs, $M(k+1)$, that we are assuming $G$ has. So $G$ contains at least one vertex with degree less than or equal to $k+1$. By the corollary following Theorem 1, no vertex of $G$ can have total degree less than $k$, or else the subdigraph induced by deleting that vertex would have more arcs than is possible for a noneven digraph on $k$ vertices. Thus at least one of the following is true: $G$ contains a vertex of degree $k$, or $G$ contains a vertex of degree $k+1$. If $G$ contains a vertex $i1$ with $V(i_1) = k$ then by Theorem 1 the induced noneven subdigraph $G \setminus \{i_1\}$ is a $T_c(k)$ digraph, so that $G$ is in $T_1(k+1)$ by Lemma 2. If $G$ does not contain a vertex of degree $k$, then it must contain a vertex $i1$ of degree $k+1$. If $G \setminus \{i_1\}$ is not maximal, then it is a $T_c(k)$ digraph missing one arc, so that $G$ is in $T_1(k+1)$ by Lemma 3. If $G \setminus \{i_1\}$ is maximal then by our induction hypothesis it is in $T_1(k)$, so that $G$ is in $T_1(k+1)$ by Lemma 4. This concludes the proof.

## Appendix

### Outline of proof of Lemma 5

First note that $M(5)-1=13$. Now suppose we have a maximal digraph $G$ on 5 vertices with 13 arcs. If all the vertices have degree greater than 5, then our counting lemma tells us that $G$ has at least 15 arcs, which is too many. So $G$ has a vertex with degree less than or equal to 5. No vertex of $G$ can have degree less than 4, because the induced subdigraph obtained by deleting such a vertex from $G$ would have at least $13-3=0$ arcs, and we know by our corollary to Theorem 1 that no noneven digraph on 4 vertices can have more than $M(4)=9$ arcs. So $G$ contains either a vertex of degree 5 or a vertex of degree 4.

First suppose that $G$ contains a vertex of degree 4. The induced subdigraph obtained by deleting that vertex has $13-3=4=9=M(4)$ arcs, and so by Theorem 1 that induced subdigraph is a $T_c(4)$ digraph. We know by Lemma 2 that if a new vertex and 4 arcs are added to a $T_c(4)$ digraph, a $T_1(5)$ digraph results.

Next suppose that $G$ contains a vertex of degree 5, and that the induced subdigraph obtained by deleting that vertex is not maximal. Since that induced subdigraph has 8 arcs, it must be a $T_c(4)$ digraph missing one arc. We know by Lemma 3 that if a new vertex and 5 arcs are added to a $T_c(4)$ digraph missing one arc, a $T_1(5)$ digraph results.

The last possibility is that $G$ contains a vertex of degree 5, and that the induced subdigraph obtained by deleting that vertex is maximal. Now we must check the various possible digraphs on 4 vertices with 8 arcs, and see what happens when we add a new vertex and 5 new arcs to those digraphs which are maximal. The order in which we check these digraphs is by number of 2-cycles, starting with those digraphs with the most 2-cycles.

The first such digraphs are those on 4 vertices with four 2-cycles. One of these is the $C_4^*$ digraph. This is maximal if it is noneven because the addition of any new arc results in a $C_3^*$ by arc splitting. However, it is not possible to add a new vertex and 5 new arcs to it without forming a $C_3^*$ by arc splitting. The only other digraph on 4 vertices with four 2-cycles has a $C_3^*$.

Next are all digraphs on 4 vertices with three 2-cycles. Exactly two of these are noneven and maximal, and these two are in the same equivalence class, $T_1(4)$.

Finally, of the digraphs on 4 vertices with two 2-cycles. Only one of these is maximal, and it is equivalent to $C_4^*$.

Figure 1

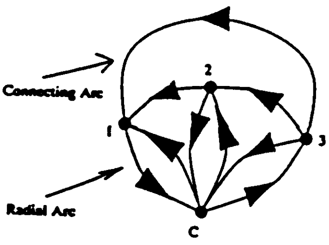



Figure 2

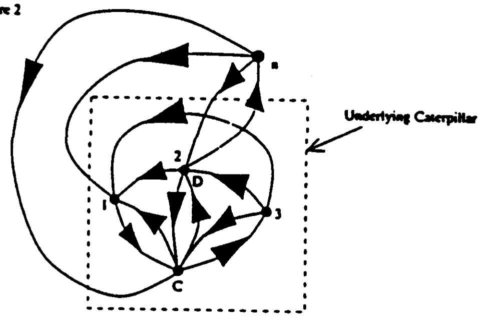



Figure 3

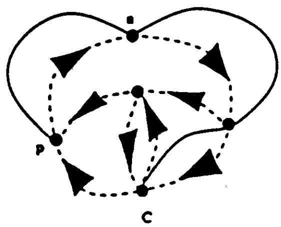



Figure 4

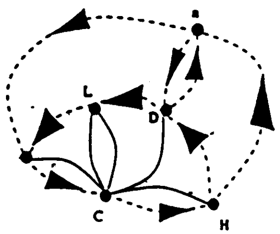
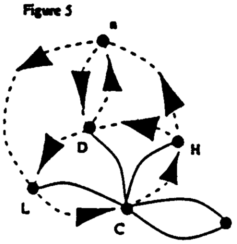



Figure 5

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