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Vitali Sets

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Abstract. In this paper we introduce the idea of a Vitali set with the help of a specified system of covers of a set and obtain some of its basic properties.

1. Introduction

The idea of cover of a set has a profound influence on the basic developments in many areas. Heine-Borel theorem in Real analysis is the first fundamental result to focus prominently on the importance of the open covers. The concept of compactness is an abstraction arising out of the Heine-Borel theorem. In measure theory, there is another kind of cover of a set, known as a Vitali cover {[9], p.81} and Vitali theorem is the fundamental one to investigate properties of density of sets, approximate continuity, approximate differentiability, absolute continuity of functions etc. However , it appears that this idea of Vitali cover has not been so far fully explored to investigate various set theoretic properties.

In this paper we define a Vitali set with the help of Vitali covers suitably adapted for our purpose in a metric outer measure space and study some of its basic properties. In particular we show that a Vitali set has many interesting analytical and topological properties. The results of the present paper have no connection with those of [4].

2. Definitions and examples

Let X be a metric space and ψ be an outer measure defined on $P(X)$, the power set of X with $0 < \psi(X) < \infty$. Let T denote the class of all closed spheres in X (closed intervals, closed rectangles etc. in the case of Euclidean spaces). If $A \subset X$ then clearly there exists a sequence $\{E_n\}$ of sets from T such that $A \subset \bigcup_{n=1}^{\infty} E_n$.

Let $A \subset X$ be an arbitrary non-void set. Let ν be a family of non-void sets from T each of positive outer measure. Then ν is said to be a *Vitali cover* of A if for each $x \in A$ and $\varepsilon > 0$, there exist an $I \in \nu$ such that $x \in \mathrm{int}(I)$ and $\psi(I) < \varepsilon$.

Definition 1. *A set* $A \subset X$ *is said to be a Vitali set if every Vitali cover* $\mathcal{V}$ *of A contains a countable pairwise disjoint subcollection* $\{I_k\}$ *such that* $\psi[A - U_k I_k] = 0$.

In an *n*-dimensional Euclidean space {[10], p.109} or in a pseudo metric space [11] or even in a topological group [5], under certain regularity condition on the class of Vitali covers, it can be shown that any subset of these spaces is a Vitali set (with appropriate changes in the context). However, in this paper, we do not assume any such regularity condition on the Vitali covers.

The space *X* is said to be *smooth* with respect to ψ if the collection of closed spheres of positive outer measure is a Vitali cover of *X*.

We can easily see that the idea of smoothness is not purely hypothetical. For example, in the Method I of construction of outer measure ψ\{[8], p.90\} if the set function T is taken to be the diameter of the respective spheres, then for any closed sphere $S \subset X$, $\Psi(S) \leq \text{diam}(S)$ and so the smoothness condition is satisfied. Moreover, in the study of density of sets in a measure space, similar assumptions exist [7].

If $X = R$, the real line with the usual metric and ψ is the Lebesgue outer measure then X is smooth with respect to ψ. There are spaces which are not smooth. For example, let $\xi \in X$ be a fixed element and define $\Psi : P(X) \to [0, \infty)$ by $\psi(A) = 1$ if $\xi \in A$ and $\psi(A) = 0$ if $\xi \notin A$. Then ψ is an outer measure and *X* is not smooth with respect to ψ. We assume throughout that *X* is a smooth space with respect to ψ. Clearly in such a space any subset of X has a Vitali cover. Also if X is smooth then each singleton and so any countable set has outer measure zero. Sets *A*, *B* etc. are always assumed to be subsets of *X* and unless otherwise stated, sets are always non-void.

Example 1. If $\psi(A) = 0$ then A is a Vitali set. In particular any countable set is a Vitali set.

Example 2. If $\{A_i\}$ is a sequence of Vitali sets then $\bigcup_i A_i$ is a Vitali set.

Example 3. Let $X = [a, b] \subset R$, the real number space with the usual metric and ψ be the Lebesgue outer measure in $[a, b]$. By Vitali theorem {[9],p.81} one can show that every subset of $[a, b]$ is a Vitali set.

Example 4. Following the method of construction as given {[3],p.689; see also [6],p.112} one can exhibit the existence of bounded set in the 2-dimensional Euclidean space which is not a Vitali set.

3. ψ-derived set and Vitali set

In this section we define ψ-open and ψ-closed sets with the help of ψ-derived set and generate a topology. We prove several connections of Vitali sets with these ideas.

Definition 2. *Let $A \subset X$. A point $\xi \in X$ is called a ψ-accumulation point of A if for every sequence $\{F_n\}$ from T with $\psi(F_n) > 0\ (n = 1,2,\cdots)$, $\psi(F_n) \to 0$ as $n \to \infty$ and $\xi \in \text{int}(F_n)\ (n = 1,2,\cdots)$, there exists a subsequence $\{F_{n_k}\}$ such that $\psi(A \cap F_{n_k}) > 0$ for $k = 1,2,\cdots$.*

The collection of all ψ-accumulation points of A is called the ψ-derived set of A and is denoted by $D_\psi(A)$.

Theorem 1. *$A - D_\psi(A)$ is a Vitali set if and only if $\psi[A - D_\psi(A)] = 0$.*

Proof. Let $A - D_\psi(A)$ be a Vitali set and $\xi \in A - D_\psi(A)$. There exists a sequence $\{F_i(\xi)\}$ from T such that $\psi(F_i(\xi)) > 0$, $\psi(F_i(\xi)) \to 0$ as $i \to \infty$, $\xi \in \text{int}(F_i(\xi))$, $(i = 1,2,\cdots)$ and $\psi[A \cap F_i(\xi)] = 0$ for $i = 1,2,\cdots$. The collection

$$\mathsf{V} = \left(\{F_i(\xi)\},\, i = 1,2,\cdots : \xi \in A - D_\psi(A)\right)$$

is a Vitali cover of $A - D_\psi(A)$ and so there exists a pairwise disjoint sequence $\{F_n\}$ from V such that

$$\psi\left[(A - D_\psi(A)) - \textstyle\bigcup_n F_n\right] = 0.$$

Now

$$A - D_\psi(A) \subset \left[(A - D_\psi(A)) - \textstyle\bigcup_n F_n\right] \cup [(\textstyle\bigcup_n F_n) \cap A]$$

and so $\psi[A - D_\psi(A)] = 0$. The converse part follows from *Example 1.* This proves the theorem.

Corollary 1. *If $\psi(A) > 0$ and $A - D_\psi(A)$ is a Vitali set, then almost all points of A are ψ-accumulation points of A. Also, a set of zero outer measure is a Vitali set.*

Lemma 1. *Let* $\{A_\alpha : \alpha \in \Lambda\}$ *be a collection of subset of X, where Λ is an index set. Then*

(i) $D_\psi(\bigcap_\alpha A_\alpha) \subset \bigcap_\alpha D_\psi(A_\alpha)$,

(ii) $D_\psi(\bigcup_{n=1}^N A_n) = \bigcup_{n=1}^N D_\psi(A_n)$,

where $A_n \, (n = 1,2,\cdots, N)$ *belong to the collection.*

The proof is omitted.

Definition 3. *A subset A is called ψ-closed if and only if* $D_\psi(A) \subset A$.

Definition 4. *A subset A is called ψ-open if and only if* $X - A$ *is ψ-closed.*

Clearly X and the empty set φ are ψ-closed and ψ-open.

Theorem 2. *The collection of all ψ-open sets forms a topology.*

The proof follows from Lemma 1.

We now characterize ψ-open sets in the following proposition.

Proposition 1. *If ψ is finitely additive then a subset A is ψ-open if and only if for every* $\xi \in A$ *there exists a sequence* $\{F_k\}$ *from* T *with* $\psi(F_k) > 0, \psi(F_k) \to 0$ *as* $k \to \infty$, $\xi \in \operatorname{int}(F_k)$ $(k = 1,2,\cdots)$ *such that* $\psi(A \cap F_k) = \psi(F_k)$ *for* $k = 1,2,\cdots$.

Proof. Suppose that A is ψ-open and $\xi \in A$. Then ξ is not a ψ-accumulation point of $X - A$. So there exists a sequence $\{F_k\}$ from T with $\psi(F_k) > 0, \psi(F_k) \to 0$ as $k \to \infty$, $\xi \in \operatorname{int}(F_k)(k = 1,2,\cdots)$ such that $\psi[F_k \cap (X - A)] = 0$ for $k = 1,2,\cdots$.

Now for $k = 1,2,\cdots$ we obtain

$$\begin{aligned} \psi(F_k) &= \psi[F_k \cap \{(X - A) \cup A\}] \\ &\leq \psi[F_k \cap (X - A)] + \psi(F_k \cap A) \\ &= \psi(F_k \cap A) \end{aligned}$$

and so $\psi(A \cap F_k) = \psi(F_k)$ for $k = 1,2,3,\cdots$.

Conversely if $\eta \in A$, there exists a sequence $\{F_k\}$ from T such that $\psi(F_k) > 0, \psi(F_k) \to 0$ as $k \to \infty, \eta \in \operatorname{int}(F_k)\,(k = 1,2,\cdots)$ and $\psi(A \cap F_k) = \psi(F_k)$ for $k = 1,2,\cdots$. Since ψ is finitely additive, we see that

$$\psi[(X - A) \cap F_k] = \psi[F_k - A \cap F_k] = \psi(F_k) - \psi(A \cap F_k) = 0$$

for $k = 1,2,\cdots$. Therefore η is not a ψ-accumulation point of $X - A$. Since $\eta \in A$ is arbitrary, $D_\psi(X - A) \subset X - A$ and so A is ψ-open. This proves the proposition.

Corollary 2. *If A is a non-empty ψ-open set then $\psi(A) > 0$.*

We need also the following proposition.

Proposition 2. *Let ψ be finitely additive, $A \subset X$ and $\xi \in X$. If there exists a ψ-open set U containing ξ such that $\psi(U \cap A) = 0$ then ξ is not a ψ-accumulation point of A.*

Proof. By Proposition 1 we can find a sequence $\{F_n\}$ of sets from T such that $\psi(F_n) > 0$, $\psi(F_n) \to 0$ as $n \to \infty$, $\xi \in \operatorname{int}(F_n)\,(n = 1,2,\cdots)$ and $\psi(F_n) = \psi(U \cap F_n)\,(n = 1,2,\cdots)$. Since $\psi[F_n - U \cap F_n] = \psi(F_n) - \psi[U \cap F_n] = 0$ and $F_n \cap A \subset \{(F_n - U \cap F_n) \cap A\} \cup (A \cap U)$,
we obtain

$$\psi(F_n \cap A) \le \psi[F_n - U \cap F_n] + \psi[A \cap U] = 0$$

for $n = 1,2,\cdots$. So ξ is not a ψ-accumulation point of A.

Theorem 3. *Let ψ be finitely additive. If $A - D_\psi(A)$ is a Vitali set, $A \cap D_\psi(A) \subset B \subset A$ and $B \cap D_\psi(X - A) = \varphi$ then B is ψ-open. Conversely if $B \subset A$ and B is ψ-open then $B \cap D_\psi(X - A) = \varphi$.*

Proof. The converse part follows from Proposition 2. Let $\xi \in B$. Then there exists a sequence $\{F_k\}$ in T such that $\xi \in \operatorname{int}(F_k)$, $\psi(F_k) > 0$, $\psi(F_k) \to 0$ as $k \to \infty$ and $\psi[F_k \cap (X - A)] = 0$ for $k = 1, 2, \cdots$. Since

$$\begin{aligned} F_k &= (F_k \cap A) \cup (F_k \cap (X - A)) \\ &= (F_k \cap B) \cup (F_k \cap (A - B)) \cup (F_k \cap (X - A)) \\ &\subset (F_k \cap B) \cup (F_k \cap (A - D_\psi(A)) \cup (F_k \cap (X - A)), \end{aligned}$$

we obtain by Theorem 1

$$\psi(F_k) \le \psi(F_k \cap B) + \psi(A - D_\psi(A)) + \psi[F_k \cap (X - A)] = \psi(F_k \cap B)$$

and so by Proposition 1, B is ψ-open.

Corollary 3. *If* $A - D_\psi(A)$ *is a Vitali set and* $A \cap D_\psi(A) \cap D_\psi(X - A) = \varphi$ *then A can be expressed as a disjoint union of a ψ-closed and a ψ-open set , provided ψ is finitely additive.*

Corollary 3 follows from Theorems 1and 3.

Theorem 4. *Let A be a Vitali set. If for some B,* $A \cap D_\psi(B) = \varphi$, *then* $\psi(A \cap B) = 0$. *In particular, if no point of A is a ψ-accumulation point of A, then* $\psi(A) = 0$.

Proof. For each $\xi \in A$ there exists a sequence $\{F_k(\xi)\}$ from T such that $\psi(F_k(\xi)) > 0$, $\psi(F_k(\xi)) \to 0$ as $k \to \infty$, $\xi \in \text{int}(F_k(\xi))$ and $\psi(F_k(\xi) \cap B) = 0$ for $k = 1,2,\cdots$.

Since $[\{F_k(\xi)\},\ k = 1,2,\cdots : \xi \in A]$ is a Vitali cover of *A*, there exists a countable pairwise disjoint subcollection $\{F_n\}$ such that $\psi[A - \bigcup_{n=1}^{\infty} F_n] = 0$. Since

$$\begin{aligned} A \cap B &\subset \left\{(A \cap B) - \left(\bigcup_{n=1}^{\infty} F_n\right) \cap B\right\} \cup \left\{\left(\bigcup_{n=1}^{\infty} F_n\right) \cap B\right\} \\ &= \left\{\left(A - \bigcup_{n=1}^{\infty} F_n\right) \cap B\right\} \cup \left\{\left(\bigcup_{n=1}^{\infty} F_n\right) \cap B\right\} \\ &\subset \left(A - \bigcup_{n=1}^{\infty} F_n\right) \cup \left\{\bigcup_{n=1}^{\infty} (F_n \cap B)\right\}, \end{aligned}$$

it follows that $\psi(A \cap B) = 0$.

Theorem 5. *Let ψ be finitely additive and A and B be ψ-open sets. If* $A \cap B = \varphi$ *then* $A \cap D_\psi(B) = \varphi$. *Conversely if A is a Vitali set and* $A \cap D_\psi(B) = \varphi$ *then* $A \cap B = \varphi$.

Proof. If $A \cap B = \varphi$, by Proposition 2, $A \cap D_\psi(B) = \varphi$. Conversely if *A* is a Vitali set and $A \cap D_\psi(B) = \varphi$, by Theorem 4, $\psi(A \cap B) = 0$. Since $A \cap B$ is ψ-open, by Corollary 2, $A \cap B = \varphi$.

Corollary 4. *If* $X = A \cup B$ *where A and B are ψ-open, A is a Vitali set and* $A \cap D_\psi(B) = \varphi$ *then X is not connected, provided ψ is finitely additive.*

Proof. By Theorem 5, $A \cap B = \varphi$ and so *X* is not connected.

Corollary 5. *If* $D_\psi(A \cap B) \neq \varphi$ *and A is a Vitali set then* $A \cap D_\psi(B) \neq \varphi$.

Proof. If $A \cap D_\psi(B) = \varphi$ then by Theorem 4, $\psi(A \cap B) = 0$ and so $D_\psi(A \cap B) = \varphi$, a contradiction.

4. A criterion for a Vitali set

In this section we obtain a criterion for a Vitali set in terms of countable intersection property in outer measure. For this, we first introduce the following definition.

Definition 5. *Let* U *be a nonvoid collection of subsets of X, each being the complement of some closed sphere, having the following property: If* $U \in \mathcal{U}$ *then for every* $x \in X - U$ *there exists a sequence* $\{U_n\}$ *from* U *such that* $x \in \text{int}(X - U_n), (n = 1,2,\cdots)$ *and* $\psi[X - U_n] \to 0$ *as* $n \to \infty$. *We call such a collection* U a *complementary Vitali class in X.*

Definition 6. *Let* U *be any complementary Vitali class in X and* $\psi\left(\bigcap_{n=1}^{\infty} U_n\right) > 0$ *for every sequence* $\{U_n\}, U_n \in \mathcal{U}$, $n = 1,2,\cdots; (X - U_i) \cap (X - U_j) = \phi$, $i \neq j$. *Then X is said to possess the countable intersection property in outer measure if* $\psi(\bigcap_{U \in \mathcal{U}} U) > 0$.

Theorem 6. *X has the countable intersection property in outer measure if and only if* $X - A$ *is a Vitali set for every A with* $\psi(A) = 0$.

Proof. Suppose that $X - A$ is a Vitali set whenever $\psi(A) = 0$. Let U be a complementary Vitali class such that the intersection of each countable subfamily $\{U_n\}$ of U has positive outer measure where $(X - U_i) \cap (X - U_j) = \varphi$, $i \neq j$. If possible, let $\psi(\bigcap_{U \in \mathcal{U}} U) = 0$.

Let $V = X - U$ where $U \in \mathcal{U}$. Then $\psi[\cap(X - V)] = 0$, i.e. $\psi[X - \bigcup V] = 0$.

Let $V_0 = X - \bigcup V$. Then $\psi(V_0) = 0$. We consider the collection

$$\nu = \{V : X - V \in \mathcal{U}\}.$$

This collection covers $X - V_0$. If $x \in V$ for some V then $x \in X - U$ for some $U \in \mathcal{U}$. Thus for each $\varepsilon(> 0)$ there exists $U_\varepsilon \in \mathcal{U}$ such that $x \in \text{int}(X - U_\varepsilon) = \text{int}(V_\varepsilon)$, say and $\psi(V_\varepsilon) = \psi(X - U_\varepsilon) < \varepsilon$. Therefore ν is a Vitali cover of $X - V_0$. By hypothesis, $X - V_0$ is a Vitali set. So there exists a pairwise disjoint sequence $\{V_n\}$ from ν such that

$$\psi\left[X - V_0 - \bigcup_{n=1}^{\infty} V_n\right] = 0$$

and so

$$\psi\left[X - \bigcup_{n=1}^{\infty} V_n\right] \leq \psi\left[X - V_0 - \bigcup_{n=1}^{\infty} V_n\right] + \psi(V_0) = 0.$$

This implies that $\psi\left[\bigcap_{n=1}^{\infty}(X - V_n)\right] = 0$, i.e. $\psi\left[\{\bigcap_{n=1}^{\infty} U_n\}\right] = 0$ where $U_n = X - V_n \in \mathcal{U}$. This contradiction shows that $\psi(\bigcap_{U \in \mathcal{U}} U) > 0$.

Conversely suppose that X has the countable intersection property in outer measure. Let ν be a Vitali cover of $X - A$ where $\psi(A) = 0$ and let $\mathrm{U} = \{U : X - U \in \nu\}$. Then clearly U is a complementary Vitali class in X.

Since $X - A \subset \bigcup_{V\in\nu} V$, we have $X - \bigcup_{V\in\nu} V \subset A$ i.e. $\bigcap_{U\in \mathrm{U}} U \subset A$, i.e $\psi(\bigcap_{U\in\mathrm{U}} U) = 0$. This implies by hypothesis that there exists a countable subfamily $\{U_n\}$ from U, $(X - U_i) \cap (X - U_j) = \phi$, $i \neq j$ such that $\psi\left[\bigcap_{n=1}^{\infty} U_n\right] = 0$. If $V_n = X - U_n$ then $V_n \in \nu, V_i \cap V_j = \phi$, $i \neq j$ and

$$\begin{aligned}
&\psi\left[X - A - \bigcup_{n=1}^{\infty} V_n\right] = \psi\left[\left(X - \bigcup_{n=1}^{\infty} V_n\right) - A\right] \\
&= \psi\left[\bigcap_{n=1}^{\infty} U_n - A\right] \\
&\leq \psi\left[\bigcap_{n=1}^{\infty} U_n\right] \\
&= 0
\end{aligned}$$

and so $X - A$ is a Vitali set. This proves the theorem.

5. Vitali type set in product spaces

Throughout the section we assume that T is a class of closed spheres which are ψ-measurable. If A and B are Vitali sets, we consider the product set $A \times B$ in the product space $X \times X$. Let M be the class of all ψ-measurable sets in X. By Theorem 2{[2], p.17}, M is a σ-ring and the restriction of ψ to M is a measure and so (X, M, ψ) is a measure space. It is known {[1];[2], p.129} that if $(X, S, \mu))$ and (Y, T, ν) are arbitrary measure spaces, there exists a unique measure Π on $S \times T$ such that

$$\Pi(P \times Q) = \mu(P)\,\nu(Q) \tag{1}$$

for every finite rectangle $P \times Q \in S \times T$. In this section we consider the unique measure Π on $M \times M$ (where $X = Y$, $S = T = M$ and $\mu = \nu = \psi$).

Definition 7. *If the requirement 'pairwise disjoint' in Definition* 1 *is withdrawn, then A is said to be a Vitali type set.*

Theorem 7. *Suppose that A, B* $\in M$. *If* $A \times B$ *is a Vitali type set in* $X \times X$ *with respect to the measure* Π *then at least one of A and B is a Vitali type set.*

Proof. Let ν_1 and ν_2 be Vitali covers of A and B respectively. If $U \in \nu_1, V \in \nu_2$ and if $\psi(U) < \varepsilon$, $\psi(V) < \varepsilon(\varepsilon > 0)$ then by (1) $\Pi(U \times V) < \varepsilon^2$.

Clearly $\nu_1 \times \nu_2$ is a Vitali cover of $A \times B$. So there exists a countable family $\{I_k \times J_1\}$, $I_k \in \nu_1$, $J_1 \in \nu_2$ such that

$$\Pi\left[A \times B - \bigcup_{k,l}(I_k \times J_1)\right] = 0.$$

Now $(A - \bigcup_k I_k) \times (B - \bigcup_1 J_1) \subset A \times B - \bigcup_{k,1}\{I_k \times J_1\}$ and so

$$\Pi\left[(A - \bigcup_k I_k) \times \left(B - \bigcup_1 J_1\right)\right] = 0.$$

Since by (1)

$$\Pi\left[(A - \bigcup_k I_k) \times (B - \bigcup_1 J_1)\right] = \psi(A - \bigcup_k I_k)\psi(B - \bigcup_1 J_1),$$

at least one of the factors in the right hand side is zero and this proves the theorem.

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