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Some Properties of a Function Connected to a Double Series

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Abstract. In this paper, we investigate some fundamental properties of the function associated with double series in a metric space.

1. Introduction

Let X denote the set of all permutations of the set of positive integers endowed with Frechet metric

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}$$

where $x = \{x_k\}_{k=1}^{\infty}$ and $y = \{y_k\}_{k=1}^{\infty}$ are points of X.

Agnew [1] proved that the set of all sequences of positive integers endowed with the Frechet metric is a metric space of second category. But it is not complete.

H.M. Sengupta [4] studied some properties of a function defined on some subset of the above mentioned space X relating with a conditionally convergent series of real terms. His result leads us to define a function which maps the space X into an interval related to a double series.

Before the definition of the function, we need the following preliminaries. Let $\sum_{m,n} a_{mn}$ be a double series of real terms. We determine the position of a term a_{mn} in the series $\sum_{m,n} a_{mn}$ according to the following plan. Let the double series $\sum_{m,n} a_{mn}$ be written in full length like a single series as

$$a_{11} + (a_{21} + a_{22} + a_{12}) + (a_{31} + a_{32} + a_{33} + a_{23} + a_{13}) + \cdots + (a_{n1} + a_{n2} + \cdots + a_{nn} + a_{(n-1)n} + \cdots + a_{2n} + a_{1n}) + \cdots \tag{1}$$

If a_{mn} occupies the *r*th position in this single series then the position of the term in the double series also be counted as *r*th position.

Now to each $x = \{x_k\}_{k=1}^{\infty} \in X$ there corresponds a subseries $\sum_{m,n} a_{mn}(x)$ which is obtained as follows:

We take those terms of the double series $\sum_{m,n} a_{mn}$ if the integers corresponding to the position of the terms of the double series are present in the sequence of integers $x = \{x_n\}_{n=1}^{\infty}$; otherwise we discard the term. Thus to each point $x = \{x_n\}_{n=1}^{\infty} \in X$ there corresponds a series $\sum_{m,n} \varepsilon_{mn}(x)\, a_{mn}$ where $\varepsilon_{mn}(x)$ takes the value 0 or 1 according to the integers corresponding to the position of terms are absent or present in the sequence of positive integers $\{x_n\}$. Also for each subseries of $\sum_{m,n} a_{mn}$ there exists a point of X which corresponds to it.

Let $\sum_{m,n} a_{mn}$ be a non-absolutely convergent double series with $a_{mn} \to 0$ as $m,\ n \to \infty$ in Pringshein sense. We define a function $f \equiv f\left(\sum_{m,n} a_{mn}\right)$ on X depending on the double series $\sum_{m,n} a_{mn}$ below:

$$f(x) = \begin{cases} \dfrac{\sum \varepsilon_{mn}(x)\, a_{mn}}{1 + \left|\sum_{m,n} \varepsilon_{mn}(x)\, a_{mn}\right|} & , \quad \text{if } \sum_{m,n} \varepsilon_{mn}(x)\, a_{mn} \quad \text{converges,} \\ 0 & , \quad \text{otherwise} \end{cases} .$$

We now investigate some interesting properties of the function $f : X \to (-1, 1)$.

Theorem 1. *If α is any real number in $(-1,1)$ then the set $\{x \mid x \in X, f(x) = \alpha\}$ is dense in X.*

Proof. Let α be any real number in $(-1,1)$. Then there exists a series which converges to $\dfrac{\alpha}{1-\alpha}$ or $\dfrac{\alpha}{1+\alpha}$ according as $\alpha > 0$ or $\alpha \le 0$, of the series $\sum_{m,n} a_{mn}$ and hence there exists an $x = \{x_n\}_{n=1}^{\infty} \in X$ which corresponds to this series. Therefore to each $\alpha \in (-1,1)$ there exists an $x = \{x_n\}_{n=1}^{\infty} \in X$ such that $f(x) = \alpha$. Thus we obtain a subseries $\sum_{m,n} \varepsilon_{mn}(x)\, a_{mn}$ corresponding to this point $x \in X$ where $\varepsilon_{mn}(x) = 0$ or 1.

Denote by $A(x)$, the set of all $y = \{y_n\}_{n=1}^{\infty} \in X$ for which there exists a positive integer $\mu = \mu(y)$ such that for $m, n \geq \mu$, we have $\varepsilon_{mn}(x) = \varepsilon_{mn}(y)$. We will now prove that $A(x)$ is dense in X. Let $z = \{z_i\} \in A(x)$ and $\varepsilon > 0$. It is sufficient to show that $A(x) \cap S(z,\varepsilon) \neq \phi$ where $S(z,\varepsilon)$ is the ε-sphere centered at z. Let l be the smallest positive integer for which $\sum_{i=l+1}^{\infty} 2^{-i} < \varepsilon$. We define the sequence $y = \{y_i\}_{i=1}^{\infty}$ in the set $A(x)$ as follows:

$y_t = z_t$, $t = 1,2,\cdots,l$. If $z_l \leq x_{l+1}$ then $y_t = x_t$; $t = l+1, l+2,\cdots$ and if $z_l > x_{l+1}$ then $y_t = x_l$ for $t = l+1, l+2,\cdots, m-1$ where m is the smallest positive integer with the property $z_m \leq x_{m+1}$ and $y_t = x_t$ for $t = m, m+1, m+2,\cdots$.
Obviously, $\rho(y,z) < \varepsilon$ holds for the sequence $y = \{y_i\}_{i=1}^{\infty}$ and there is a positive integer t_0 such that $y_t = x_t$, $t = t_0, t_0+1, t_0+2,\cdots$. Therefore $\sum_{m,n} \varepsilon_{mn}(y)\, a_{mn} = \sum_{m,n} \varepsilon_{mn}(x)\, a_{mn}$ for $m, n > t_0$. Hence $y = \{y_i\}_{i=1}^{\infty} \in A(x)$ and $y \in S(z,\varepsilon)$ i.e., $A(x) \cap S(z,\varepsilon) \neq \phi$. Hence $A(x)$ is dense in X. Also the subseries $\sum_{m,n} \varepsilon_{mn}(y)\, a_{mn}$ converges if and only if the subseries $\sum_{m,n} \varepsilon_{mn}(x)\, a_{mn}$ converges. Hence $f(y) = f(x) = \alpha$ for each $y \in A(x)$. Hence the theorem follows.

Theorem 2. *The function f is discontinuous everywhere.*

Proof. Let $x_0 \in X$ and $f(x_0) = \alpha_0$, by the consequence of the above theorem, it can be shown that for $\delta > 0$ the sphere $S(x_0,\delta)$ contains a point $x \in X$ for which $f(x) = \alpha_0 + 1$. Hence the result follows.

Corollary. *The function does not belong to the first Baire class.*

Proof. It follows from the fact that the set of points of discontinuity of a function belonging to the first Baire class is set of first category [3].

Theorem 3. *The function f belongs exactly to the third Borel class.*

Proof. Since f is discontinuous everywhere, it cannot belong to the first Baire class. Therefore it suffices to show that for each real number a, each of the sets $A^a = \{x \in X : f(x) < a\}$ and $A_a = \{x \in X : f(x) > a\}$ belong to the third additive Borel class.

We first investigate for the set A^a. If $a \geq 1$ or $a \leq -1$ then $A^a = X$ or ϕ and therefore the above assertion is obvious.

Let $0 < a < 1$. Then we have $A^a = \{x \in X_1 : f(x) < a\} \cup \left\{x \in X_2 : f(x) > a\right\}$. Where $X_1 = \{ x \in X : \sum_{m,n} \varepsilon_{mn}(x)\, a_{mn} < +\infty\}$ and $X_2 = X - X_1$. For each point $x \in X_1$,

we have
$$f(x) = \underset{m,n\to\infty}{Lt} S_{mn}(x),$$

where
$$S_{mn}(x) = \frac{\sum_{p=1}^{m}\sum_{q=1}^{n} \varepsilon_{pq}(x) a_{pq}}{1 + \left|\sum_{p=1}^{m}\sum_{q=1}^{n} \varepsilon_{pq}(x) a_{pq}\right|}.$$

Then
$$\left\{x \in X_1 : f(x) < a\right\} = \bigcup_{k=1}^{\infty}\bigcup_{r=1}^{\infty}\bigcap_{n=r}^{\infty}\bigcap_{m=r}^{\infty} M(k,m,n)$$

where
$$M(k,m,n) = \left\{x \in X_1 : S_{mn}(x) < a - \frac{1}{k}\right\}.$$

It is evident that for fixed $k, m, n, M(k,m,n)$ is closed in X and consequently the set $\{x \in X_1 : f(x) < a\}$ is an F_σ set in X_1. By Cauchy's criterion of convergence of a double series we have

$$X_1 = \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{q=1}^{\infty}\bigcap_{p=1}^{\infty}\left\{x \in X : \left|\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+q} \varepsilon_{ij}(x) a_{ij}\right| \le 1/k\right\}$$
$$= \bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}\bigcap_{q=1}^{\infty}\bigcap_{p=1}^{\infty} H(m,n,p,q,k)$$

where
$$H(m,n,p,q,k) = \left\{x \in X_1 : \left|\sum_{i=m+1}^{m+p}\sum_{j=n+1}^{n+q} \varepsilon_{ij}(x) a_{ij}\right| \le 1/k\right\}.$$

It is evident that for fixed m, n, p, q, k, the set $H(m,n,p,q,k)$ is closed in X. Hence X_1 is $F_{\sigma\delta}$ in X and also the set $\{x \in X_1 : f(x) < a\}$ is F_σ in X_1. Therefore X_2 is $G_{\delta\sigma}$ in X. Hence X_2 is $F_{\sigma\delta\sigma}$ in X [2]. Hence the set A^a is an $F_{\sigma\delta\sigma}$ set in X.

If $-1 < a \le 0$, then we have $A^a = \{x \in X : f(x) < a\}$ and A^a can be shown to be an $F_{\sigma\delta\sigma}$ set in X in a similar manner. Using analogous considerations, it can be shown that $A_a = \{x \in X : f(x) > a\}$ is $F_{\sigma\delta\sigma}$ in X. The theorem follows.

We now define Darboux's property of a function.

Definition. *Let f be a real valued function defined on a metric space X. Then f is said to have* ***Darboux's property*** *if for each* $\varepsilon > 0$, *any real number c and every* $u, v \in S(x,\varepsilon)$ *with* $f(u) < c < f(v)$ *there exists a* $w \in S(x,\varepsilon)$ *such that* $f(w) = c$.

Theorem 4. *The function f has the Darboux's property.*

Proof. By Theorem 1, for each $x \in X$ and any $\varepsilon > 0$ we have $f(S(x,\varepsilon)) = (-1, 1)$. Hence the result.

Note. The function f is open in the sense that it maps each open set into an open set.

References

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