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Methods of Destroying the Symmetries of a Graph

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Abstract. A note v of a graph G is called fixed if every automorphism of G sends v onto itself. A graph or digraph or other graphical structure is then called fixed if every node is fixed, i.e., its automorphism group is the identity. We present several methods for fixing a graph (destroying its automorphisms). These may not work for all graphs. The methods include orienting some of the edges, coloring some of the nodes with one or more colors and the same for the edges, labeling nodes or edges, and adding or deleting nodes or edges. These considerations lead to a multitude of new invariants and open questions.

1. Introduction

We follow in general the graph theoretic notation and terminology of the book [4] and that by Chartrand and Lesniak, [1]. Thus the graph $G = (V, E)$ has *node* set V and *edge* set E, *order* $n = |V|$ and *size* $m = |E|$.

If a graph already has the identity group, then it is *fixed*. If not, then *fixing a graph* G means altering it in some way to obtain a fixed graphical structure. The first published method of fixing a graph is apparently due to Tutte [13]. He applied his procedure to planar graphs only. He selected an arbitrary edge of the graph, oriented it in one of the two possible ways, and then drew at the center of the selected edge a small arrow perpendicular to the edge, oriented in one of the two possible directions. This served to fix the planar graph. The purpose of the small arrow was to specify the exterior region of the graph. Tutte's objective was to count triangulations of the plane without having to take their symmetries into consideration.

Our purpose is to present an exposition of several methods of fixing a graph:

1. Some graphs can be fixed by orienting a subset of its edges. This is not true of all graphs, for example, it cannot be done for the stars $K_{1,h}$ when $h \geq 4$. The result is a fixed mixed graph.
2. It is convenient to consider the nodes of a graph as colorless and draw them as small open circles. A graph G can sometimes be fixed by coloring a subset of its nodes with one color. However, every graph can be fixed when the number of colors assigned to the nodes is not restricted.

3. The same considerations can be applied to coloring edges of a graph. Obviously a graph cannot be fixed in this way if it has more than one isolated node. In particular, connected graphs can always be fixed by coloring edges.
4. A *labeled graph* has its nodes marked $1,2,3,\cdots,n$. A *partially labeled graph* has a subset of its nodes labeled, say from 1 to h. The same holds for labeling the edges of G. A graph can also be fixed by labeling some nodes and some edges.
5. Every graph G can be fixed by removing a sufficient number of nodes. This is trivially true because the removal of $n-1$ nodes leaves the trivial which is fixed.
6. However, not all graphs can be fixed by removing edges. For example, no path can be fixed in this way.
7. Similarly, some but not all can be fixed by adding new edges which join pairs of nodes that are not already adjacent.
8. One can admit the addition of edges to a given graph G that result in a multigraph. These additional edges do not necessarily join adjacent nodes of G.

By a *partial invariant* of G we mean an invariant that is definable for some but not all graphs. Each of the methods of fixing a graph listed above results in either invariants or partial invariants of G.

2. Orienting edges

To illustrate, any path can be fixed by orienting any one edge arbitrarily. Figure 1 shows two possible ways to fix the path P_4 by orienting one edge.

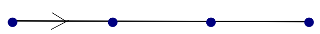

Figure 1. Fixing a path

Figure 2 shows two stars. The smaller of these can be fixed as shown. Obviously the larger one cannot be fixed no matter how any subset of its four edges are oriented.

Figure 2. Two stars; just one can be fixed

In a *mixed graph*, these are four possibilities for any ordered pair *u*, *v* of nodes: either *u* and *v* are not adjacent or they are joined either by an arc oriented from *u* to *v* or conversely, or they are joined by an undirected edge. Mixed graphs with given numbers of nodes, arcs and edges were counted by Harary and Palmer [8]. They also reported on this result in their book. *Graphical Enumeration,* [9]. Using the classic enumeration methods of Pólya [12], the number of graphs of given order and size as well as number of graphs were determined in [2]. Later, oriented graphs were counted in [3]. The formula for mixed graphs contains all of these results as special cases. A mixed graph with no arcs is a graph. Every mixed graph can be regarded as a digraph when each edge is replaced by a symmetric pair of arcs. An *oriented graph* is a mixed graph with no edges.

It is very well known that a graph and its arcs have precisely the same automorphism group, regarded as a permutation group acting on the node set. Figure 3 shows two different ways to orient K_5 which result in a fixed mixed graph. The first orients four edges and the second only three edges. This figure shows the arcs of these mixed graphs. The first is a spanning directed path and the second is a directed linear forest. This leads to the first partial invariant which is defined only for those graphs *G* that can be fixed by orienting some of its edges. For such a graph, let $f(G, or)$ be the smallest number of edges which can be oriented in such a way that the resulting mixed graph is fixed. Then clearly $f(K_5, or) = 3$.

Figure 3. The arcs of two fixings of K_5

Jacobson and Harary [5] determine the smallest number of edges of a complete graph that must be oriented in order to obtain a fixed mixed graph using the enumeration of fixed oriented trees by Robinson and Harary [11]. Their result utilized the methods of Harary and Prins [10]. In view of the considerations of Figure 3, we have the following observation.

Theorem A. *The number* $f(K_n, or)$ *is equal to the smallest number of arcs in a fixed orientation of a linear forest of order n.*

For each partial invariant of a graph, there is an associated open question. Characterize those graphs for which the partial invariant is well defined. Consider $f(G, or)$ for trees. We already saw that this number is defined only for those stars having at most four nodes. This observation generalizes to arbitrary trees in terms of the number of ways to fix a given branch B and the number of branches isomorphic to B. The criterion can be stated correctly but is not yet expressed elegantly.

However, one can consider other types of arrowheads. These can be drawn when there are only two such types as in Figure 4 and the extension to any number of types is immediate.

Figure 4. Two types of arrowheads: single and double

With these considerations, we now have edge orientation fixing invariants:

1. The total number of edges which need to be oriented with type of arrowhead.
2. The number of types of arrowheads.
3. Call the cost of a single arrowhead 1; of a double arrowhead 2, etc.

Then we can define the *cost of fixing a graph by edge orientations* as the smallest possible sum of the costs of the orienting arrows.

3. Coloring the nodes

Considering the nodes of a given graph G as colorless, a partial invariant for fixing G is the number of nodes which need to be colored using only one color. For a path, any one node can be colored except for the paths of odd order, in which case coloring the central node does not fix the path. For a complete graph of order $n > 3$, one color will not serve. To fix K_n, $n-1$ colors are needed and the last node is left colorless. The associated open questions are:

1. Characterize those graphs that can be fixed by assigning only one color to some of the nodes. In particular, settle this question for trees. When only one node needs to be colored, the result is equivalent to the rooted graph with the root at that node. When G can be defined by coloring nodes with one color, what is the smallest possible number of such nodes? The associated invariant is this smallest number of nodes.

2. Since the complete graph cannot be fixed with just one color, we ask for the maximum possible size of a graph with given order n that can be so fixed.

3. When more than one color is required, what is the smallest number of colors and the smallest number of nodes that need to be colored to fix G?

4. When then colors are presented by the numbers $1,2,3,\cdots$ the color cost of fixing a graph is defined as expected. This is entirely analogous to the color cost of an edge orientation mentioned above.

The number of node colors needed to fix C_n, which can be written $f(C_n, 0, \text{colors})$ is 2 when $n = 3$ or 4 and is 1 when $n > 6$. Figure 5 shows that the hexagon C_6 can be fixed by coloring only three nodes with one color, and the same holds for all bigger cycles.

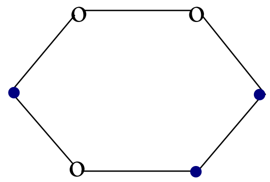

Figure 5. Fixing a hexagon by coloring nodes

4. Coloring the edges

As in the case of node coloring, it is natural to ask which graphs can be fixed by coloring selected edges with only one color, and what is the smallest number of edges that work. Continuing by analogy, one can consider the number of colors needed for the edges when one color will not do, and the associated edge color cost for a given graph. A difference between fixing a graph by coloring nodes with as many colors as needed and doing so by coloring the edges is that every graph *G* can be fixed by node colorings, but only those graphs with at most one isolated node can be fixed by edge coloring. Thus this includes every connected graph. To illustrate, the cube Q_3 can be fixed by coloring three edges with one color, but cannot be fixed by coloring only two edges.

5. Labeling nodes or edges

A partial labeling of a graph using $1,2,3,\cdots$ is equivalent to coloring nodes so that each color can only be used for one node. of course every labeled graph is fixed and so is every graph with all but one node labeled. Not every graph needs to be so labeled in order to fix it, in fact, the only such graph is the complete graph. Again, as for edge colorings considered above, a graph can be fixed by edge labeling if an only if it has at most one isolated node.

Since a label on a node *u* served to fix *u*, the number of its nodes that must be labeled to fix *G* can be determined by inspecting the permutations in the automorphism group of *G*. The *metric dimension* of a connected nontrivial graph *G* was studied by both Harary and Melter [7]. It is the smallest number *h* of nodes in a "distance basis", v_1 to v_h, such that for each node u, the distance vector $(d,(u,v_1),\cdots,d(u,v_h))$ gives coordinates for u that are different from those of any other node. It follows at once that the number of nodes that must be labeled to fix G is at most the metric dimension of G.

6. Adding or removing nodes or edges

As mentioned above, every graph G can be fixed by removing all but one node. Hence an invariant for all graphs is the smallest number of nodes which need to be removed to obtain a fixed induced subgraph.

For edges there are three possibilities, all of which result in partial invariants:

1. Edges can be removed.
2. They can be added.
3. They can be altered, meaning some edges are removed and new edges added. As for all partial invariants, the main question is to determine the set of all graphs for which it is meaningful.

We illustrate with a smallest nontrivial fixed graph, that of Figure 6a of order 6. The theta-graph of Figure 6b can be fixed by adding one edge, and the fan, $K_1 + P_5$, of Figure 6c needs to have one edge removed.

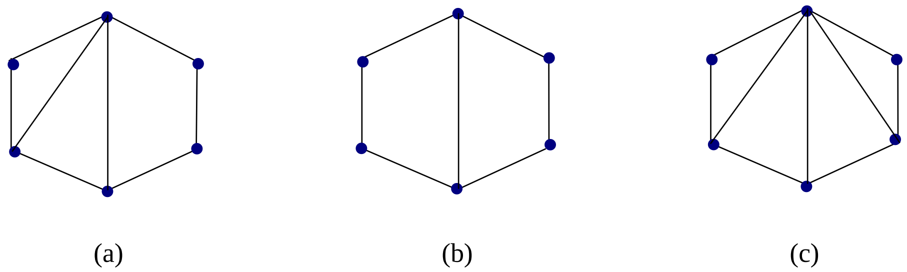

Figure 6. Fixing a graph by adding or removing edges

Figure 6a also shows that the number of additional edges needed to fix a hexagon is 2, and this holds for all larger cycles as well. For smaller cycles, this partial invariant is meaningless.

7. Constructing a supermultigraph

It has been proved that for any connected graph G with $n \geq 4$ nodes, it is possible to add new multiple edges to the existing edges in such a way that the resulting supermultigraph M is irregular, that is, the nodes of M have distinct degrees. This was investigated by

Harary, Jacobson, Kubicka, Kubicki and Oellernann, [6]. It is well known that no graph can have all its degrees different. Obviously, any multigraph with distinct degrees is fixed. Therefore it is always possible to construct a fixed supermultigraph of G whose underlying graph is G itself. This is nothing new. It is equivalent to weighting the edges of G, regarding the number of multiple edges joining a pair of adjacent nodes of G as the weight of that edge. This has already been encountered in Section 3.

However, when we admit the addition to G of new edges or multiple edges on pairs of nodes that are not adjacent in G as well as the preceding addition of multiple edges to edges already in G, we obtain a new method of constructing an irregular supermultigraph of G , and hence, a new way to fix G. It is easy to see that three new multiple edges must be added to any cycle in order to fix it.

8. Illustrations

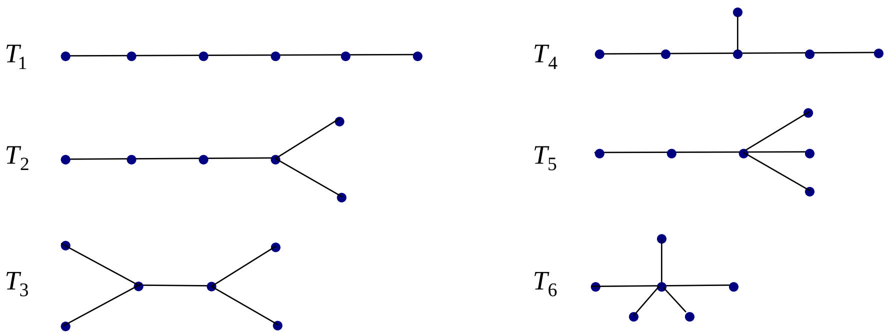

Figure 7. The six trees of order 6

Figure 7 shows all the trees of order 6. We present values of various fixing invariants for these trees. The following four fixing invariants (a, b, c, d) are equal for each of these trees. These numbers are listed in the first column of Table 1:

(a) The number of nodes that must be labeled.
(b) The number of edges that must be oriented. Only for the star are two types of arrowhead needed.
(c) The number of edge colors needed.
(d) The number of edges to be colored. Thus in each tree, no edge color appears more than once.
(e) The coloring of nodes tells a different story. In the second column, (e), an ordered pair is shown for each tree: (number of colors needed, numbers of nodes to color). Only for T_3 do we get a nonreflexive ordered pair, namely (1.3). Even there (2,2) works but is not listed since fixing with just one node is prepared.

The last two columns of Table 1 consider adding new multiple edges (f) to those of the tree, and (g) new edges and multi-edges anywhere. The numbers provide an interesting contrast.

Table 1. Fixing invariants for the trees of order 6

tree	**(a,b,c,d)**	**(e)**	**(f)**	**(g)**
1	1	1,1	1	1
2	1	1,1	1	1
3	1	1,3	3	3
4	1	1,1	1	1
5	2	2,2	3	2
6	4	4,4	10	5

For the partial fixing invariant given by the number of edges not in *G* that need to be added to *G*, it is convenient to refer to [4, pp. 220-221]. There, the four fixed graphs (Figure 8) of order 6 are listed. A comparison of Figures 7 and 8 shows that this number is 2 for the first five trees and does not exist for the star.

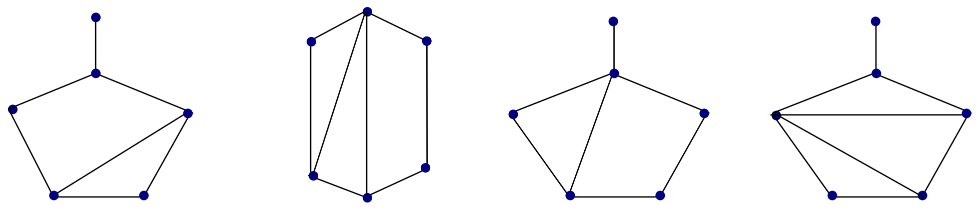

Figure 8. The fixed graphs of order 6

9. Conclusion

Many open questions are proposed. Further, several new invariants and partial invariants are introduced. for each of these, the usual problems associated with the investigation of a new graphical concept suggest themselves. These include the determination of computational complexity and the canonical extremal problem of finding the maximum and minimum values among all graphs with given order and size, as well as constructing the associated extremal graphs.

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