Certain Properties of Parabolic Starlike and Convex Functions of Order ρ

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Abstract. We investigate starlike and convex functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ with the property that $\dfrac{zf'(z)}{f(z)}$ and $1 + \dfrac{zf'(z)}{f'(z)}$ lie inside a certain parabola. We give some particular examples of functions having the required properties and it is shown that these functions are invariant under particular integral operators. We also determine the radii of uniformly convexity and starlikeness for certain functions. Such type of work was carried out by [1] and we are motivated by this work.

1. Introduction

Let A denote the class of all functions $f(z)$ which are analytic in the open unit disc $U = \{z : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$ and let S denote the subclass of A consisting of functions which are also univalent in U. A great deal of attention has been given in recent years to the uniformly starlike and convex functions introduced by Goodman [4]. He introduced the class UCV of uniformly convex functions which have the additional property that for every circular arc γ contained in U with center also in U the image arc $f(\gamma)$ is convex.

Ma and Minda [7] and Ronning [9] independently proved that $f \in UCV$ if and only if

$$\text{Re}\,\frac{zf''(z)}{f'(z)} + 1 > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in U). \tag{1.1}$$

Furthermore Ronning [9] defined the class S_p of functions $f \in A$ for which

$$\text{Re}\,\frac{zf'(z)}{f(z)} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in U). \tag{1.2}$$

In [1] the class S_p was generalized by introducing a parameter ρ. For $0 \leq \rho < 1$, let Ω_p be the region

$$\Omega_\rho = \left\{ u + iv : v^2 \leq 4(1 - \rho)(u - \rho) \right\} = \left\{ w : \left| w - 1 \right| \leq 1 - 2\rho + \operatorname{Re} w \right\} \quad (1.3)$$

and suppose $S_p(\rho)$ be the subclass of A consisting of functions f such that

$$\frac{zf'(z)}{f(z)} \in \Omega_\rho \qquad (z \in U)$$

and also let $K_p(\rho)$ be the subclass of A consisting of functions f such that $zf' \in S_p(\rho)$. It is easily seen that $S_p(1/2) = S_p$ and $S_p(\rho)$ is a subset of starlike functions.

A function f belonging to $S_p(\rho)$ and $K_p(\rho)$ is called parabolic starlike and convex of order ρ, respectively.

In [1] Ali and Singh, obtained sharp upper bounds for n-th coefficient of functions in $S_p(\rho)$ and for the inverse function $f^{-1}(w) = w + d_2 w^2 + \cdots$ when $n = 2, 3,$ and 4. Also they obtained a general Littlewood type of bound on $\left| a_n \right|$.

Motivated by the work of Ali and Singh [1] we study the radius problem for certain functions and we introduce some examples for the classes $S_p(\rho)$ and $K_p(\rho)$.

It is also shown that these classes are invariant under particular integral operators. Further results obtained in [10] will be special cases of our results.

2. Integral operators

The function which maps U onto the parabolic region Ω_ρ is given by

$$q_\rho(z) = 1 + \frac{4(1 - \rho)}{\pi^2} \left[\log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right]^2.$$

(The branch of $\sqrt{z}$ is chosen such that $\operatorname{Im} \sqrt{z} \geq 0$.)

It is clear then that $f \in S_p(\rho)$ and $K_p(\rho)$, respectively, if and only if, $\frac{zf'(z)}{f(z)}$ and $1 + \frac{zf''}{f'(z)}$ are subordinate to $q_\rho(z)$, (see [1]), which we denote by

$$\frac{zf'(z)}{f(z)} \prec q_\rho(z), \quad 1 + \frac{zf''(z)}{f'(z)} \prec q_\rho(z).$$

The convolution of two power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is defined as the power series

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

In our next investigation we need the following Lemma of Ruscheweyh [12].

Lemma 1 ([12]). *Let φ be convex and g be starlike. Then for each function F, analytic in U, the image of U under $\frac{\varphi * Fg}{\varphi * g}$ is a subset of the convex hull of $F(U)$.*

Theorem 1. *If $f \in S_p(\rho) \,(or\, K_p(\rho))$ then so is $f * \varphi$ for any function $\varphi(z) = z + \cdots$ analytic and convex in U.*

Proof. We know $f \in S_p(\rho)$ if and only if, for $z \in U$,

$$\frac{zf'(z)}{f(z)} \prec q_\rho(z).$$

But $q_\rho(z)$ is convex and f is starlike of order ρ. So an application of Lemma 1 yields

$$\frac{\varphi * (\frac{zf'}{f})f}{\varphi * f} = \frac{z(\varphi * f)'}{\varphi * f} \prec q_\rho(z).$$

Hence it follows that $\varphi * f \in S_p(\rho)$. The result for $K_p(\rho)$ now follows from the relationship $f \in K_p(\rho)$ if and only if $zf' \in S_p(\rho)$.

Corollary 1. *If $f \in S_p(\rho), K_p(\rho)$, then so is*

$$\frac{1+r}{z^r} \int_0^z t^{r-1} f(t) dt, \quad \mathrm{Re}(r) > 0.$$

Proof.   Since

$$\frac{1+r}{z^r}\int_0^z t^{r-1} f(t)\,dt = f * \sum_{n=1}^{\infty} \frac{1+r}{n+r}\, z^n.$$

The result follows from Theorem 1 and noting that $\sum_{n=1}^{\infty} \frac{1+r}{n+r}\, z^n$ is convex in U. See [11].

Corollary 2.   *If* $f \in S_p(\rho), K_p(\rho),$ *then so is*

$$\int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi}\, d\xi,\ |x| \le 1,\ x \ne 1.$$

Proof.   We may write

$$\int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi}\, d\xi = f * h$$

where

$$h(z) = \sum_{n=1}^{\infty} \frac{1 - x^n}{(1 - x)n}\, z^n = \frac{1}{1 - x}\log\left[\frac{1 - xz}{1 - z}\right],\ |x| \le 1,\ x \ne 1.$$

Since h is convex the result follows from Theorem 1.

Corollary 3.   *Let* $\mu \ge 0$ *and*

$$|A| \le \begin{cases} \sqrt{\dfrac{1 - 2\rho}{2(1 - \rho)}} & if \quad 0 \le \rho < 1/2 \\[2ex] \rho_0 & if \quad 1/2 \le \rho \le 5/6 \end{cases}$$

Then the function

$$f_\mu(z) = \sum_{n=1}^{\infty} \frac{A^{n-1}}{(n+1)^\mu}\, z^n$$

belongs to $K_p(\rho)$, $0 \le \rho \le \frac{5}{6}$, where ρ_0 is the smallest positive root of the equation

$$-(1 + \rho)x^4 - 2(1 + 2\rho)x^3 + 2(1 - 2\rho)x + (1 - \rho) = 0.$$

Proof. Since the function $f(z) = \frac{z}{1-Az}$ belongs to $K_p(\rho)$ with condition mentioned above for $|A|$ and function $\theta_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{(1+n)^\mu}$ is convex (see [6]). Hence the result follows from Theorem 1.

Theorem 2. *Let* $f_i \in K_p(\rho)$ *and let* α_i *be real numbers such that* $\alpha_i \geq 0$ *and* $\sum_{i=1}^{n} \alpha_i \leq 1$. *Let* $\beta \geq 1$ *then the function* $g(z)$ *defined by*

$$g(z) = \int_0^z \left(\prod_{i=1}^{n} f_i'(\xi)^{\alpha_i} \right)^{1/\beta} d\xi$$

also belongs to $K_p(\rho)$.

Proof. Since $f_i \in K_p(\rho)$ we have

$$\mathrm{Re} \left(\beta \frac{zg''(z)}{g'(z)} \right) + \beta(2 - 2\rho) \geq \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i''(z)}{f_i'(z)} \right| \geq \beta \left| \frac{zg''(z)}{g'(z)} \right|,$$

which implies that $g \in K_p(\rho)$.

Remark 1. The special case of Theorem 2 for $\rho = 1/2$, $\beta = 1$ was proved by Shanmugam and Ravichandran [10].

Theorem 3. *Suppose* $f \in A$ *is such that* $\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{1-\rho}{t}$, *where* $t > 0$ *and* $0 \leq \rho < 1$. *Then*

$$g(z) = \int_0^z \left(\frac{f(\xi)}{\xi} \right)^t d\xi$$

belongs to $K_p(\rho)$.

Proof. Since

$$\frac{zg''(z)}{g'(z)} = t \left(\frac{zf'(z)}{f(z)} - 1 \right)$$

we can write

$$\mathrm{Re} \frac{zg''(z)}{g'(z)} + (2 - 2\rho) \geq 1 - \rho \geq \left| \frac{zg''(z)}{g'(z)} \right|.$$

Hence $g \in K_p(\rho)$.

Remark 2. The special case of Theorem 3 for $\rho = \frac{1}{2}$, $t = 2$ was proved by Shanmugan and Ravichandran [10].

We will need the following lemma in the next theorem.

Lemma 2 ([2]). *Let* $\beta, \gamma \in C$. *Let* $h(z) = c + h_1 z + \cdots$ *be a convex (univalent) function in U, with* $\mathrm{Re}\,(\beta h(z) + \gamma) > 0$, $z \in U$. *If* $p(z) = c + p_1 z + \cdots$ *is analytic in U, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \ \text{ implies } \ p(z) \prec h(z).$$

Theorem 4. *Let* $f \in A$ *and* $\alpha > 0$, $f \in S_p(\rho)$ *then*

$$F(z) = \left[\alpha \int_0^z \frac{f^\alpha(t)}{t}\, dt \right]^{1/\alpha} \tag{2.1}$$

also belongs to $S_p(\rho)$.

Proof. Differentiation of (2.1) w.r.t. to z, leads to

$$(\alpha - 1)\frac{zF'(z)}{F(z)} + \frac{zF''(z)}{F'(z)} = \alpha\,\frac{zf'(z)}{f(z)} - 1. \tag{2.2}$$

Let $p(z) = \frac{zF'(z)}{F(z)}$, then (2.2) is equivalent to

$$p(z) + \frac{1}{\alpha}\,\frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)}. \tag{2.3}$$

Since $f \in S_p(\rho)$, it follows by (2.3)

$$p(z) + \frac{1}{\alpha}\,\frac{zp'(z)}{p(z)} \prec q_p(\rho).$$

But it is easy to see that $\mathrm{Re}(\alpha q_p(\rho)) > 0$ and $q_p(\rho)$ is convex (univalent). By Lemma 2 it follows

$$p(z) \prec q_p(\rho).$$

Hence $F \in S_p(\rho)$.

Finally in this section we investigate sections of elements $K_p(\rho)$.

Theorem 5. *If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in K_p(\rho)$, *then* $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$
belongs to $K_p(\rho)$ *for* $|z| < \frac{1}{4}$.

Proof. Let $g_n(z) = z + \sum_{k=2}^{n} z^k = \frac{z - z^{n+1}}{1-z}$, it is well known g_n is convex for
$|z| < \frac{1}{4}$. Hence function $h_n(z) = 4g_n(z/4)$ is convex in U. By making use of Theorem
1 it follows that $h_n * f \in K_p(\rho)$ or $f_n \in K_p(\rho)$ for $|z| < \frac{1}{4}$. Hence the proof is
complete.

3. Radius problem

Let $M(\beta)$ denote the class of all analytic functions $p(z)$ defined on U, with $p(0) = 1$
satisfying $\left| \arg p(z) \right| < \frac{\pi\beta}{2}$, $(z \in U)$, where $\beta > 0$.

Definition 1. *A function $f(z)$ in the class A is said to be a member of the class*
$C^*(M_\beta, \alpha)$ *if and only if there is a function $g(z) \in S^*(\alpha)$ (starlike of order α) such*
that $\frac{f(z)}{g(z)} \in M(\beta)$.

Definition 2. *The $K_p(\rho)$ radius of S denoted $R_p(\rho)$ is the radius of the largest disc*
$|z| < R_p(\rho)$ *in which* $1 + \frac{zf''(z)}{f'(z)} \in \Omega_\rho$ *holds for all* $f \in S$.

The $S_p(\rho)$ radius of $C*(M_\beta, \alpha)$, denoted $R_p^*(\rho)$ is the radius of the largest disc
$|z| < R_p^*(\rho)$ in which $\frac{zf'(z)}{f(z)} \in \Omega_\rho$ holds for all $f \in C^*(M_\beta, \alpha)$.

Let $0 \le \rho < 1$, $a > \rho$ and let $m_a(\rho)$ be the largest number in which disc
$D(a, m_a(\rho)) = \{w \in C; \left| w - a \right| < m_a(\rho)\}$ lies completely inside region Ω_ρ. A direct
calculation gives us

$$m_a(\rho) = \begin{cases} a - \rho & \text{if} \quad \rho < a < 2 - \rho \\ 2\sqrt{(1-\rho)(a-1)} & \text{if} \quad a \ge 2 - \rho \end{cases}.$$

If we restrict the value of a by $\frac{1+\rho}{2} < a < (2-\rho) + \sqrt{\rho^2 - \rho + \frac{5}{2}}$ then disc contains
the point 1. Hence it follows:

Lemma 3. *Let* $f \in A$. *If for any a with* $\frac{1+\rho}{2} < a < 5 - 4\rho$

$$\left| \frac{zf'(z)}{f(z)} - a \right| < m_a(\rho)$$

for all $z \in U$. *Then* $f \in S_p(\rho)$.

To prove the next theorems we need the following results.

Lemma 4. *Suppose* $p(z) = 1 + c_n z^n + \cdots$ *is analytic and belongs to* $M(\beta)$, *then*

$$\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2\beta n r^n}{1 - r^{2n}} \quad for \ |z| = r < 1.$$

Proof. Define $g(z) = p(z)^{1/\beta}$. Then $\arg g(z) = \frac{1}{\beta} \arg p(z)$ and $\operatorname{Re} g(z) > 0$. It is known [8] that

$$\left| \frac{zg'(z)}{g(z)} \right| \leq \frac{2nr^n}{1 - r^{2n}}, \quad for \ |z| = r < 1.$$

Hence

$$\left| \frac{zp'(z)}{p(z)} \right| = \beta \left| \frac{zg'(z)}{g(z)} \right| \leq \frac{2\beta n r^n}{1 - r^{2n}}, \quad for \ |z| = r < 1,$$

and the proof is complete.

Lemma 5 ([3]).   *If* $p(z) = 1 + c_1 z + c_2 z + \cdots$ *is analytic and* $\operatorname{Re} p(z) > \alpha$ *for* $z \in U$ *then*

$$\left| p(z) - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 - \alpha)r}{1 - r^2} \quad for \ |z| = r < 1.$$

Theorem 6. *The* $K_p(\rho)$ *radius of S is*

$$R_p(\rho) = \frac{2 - \sqrt{3 + \rho^2}}{1 + \rho}.$$

Equality occurs for $f(z) = \frac{z}{(1-z)^2}$.

Proof. For $f \in S$, it is known [5] that

$$\left| \left(1 + \frac{zf''(z)}{f'(z)} \right) - \frac{1 + r^2}{1 - r^2} \right| < \frac{4r}{1 - r^2}, \quad \text{for } |z| = r < 1.$$

This represents a circular disc intersecting the real axis in

$$x_1 = \frac{1 - 4r + r^2}{1 - r^2} \quad \text{and} \quad x_2 = \frac{1 + 4r + r^2}{1 - r^2}.$$

For $r = R_p(\rho)$ we have $\frac{1-4r+r^2}{1-r^2} = \rho$ and for r less than this value the disc lies completely inside the parabolic region Ω_ρ, by Lemma 3. Hence the proof is complete.

Theorem 7. *The $S_p(\rho)$ radius $R_p^*(\rho)$ for the class $C^*(M_\beta, \alpha)$ is*

$$R_p^*(\rho) = \frac{1 - \rho}{(1 + \beta - \alpha) + \sqrt{1 + \beta - \alpha)^2 - (1 - \rho)(1 + \rho - 2\alpha)}}$$

where $\beta \geq \frac{3}{4}(1 - \rho)$ and $0 \leq \alpha < 1$, $0 \leq \rho < 1$.

Proof. Let g be a starlike function of order α such that $h(z) = \frac{f(z)}{g(z)} \in M(\beta)$. Since $\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} + \frac{zh'(z)}{h(z)}$, by Lemmas 4 and 5, we get

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\alpha)r^2}{1 - r^2} \right| \leq \frac{2(1 + \beta - \alpha)r}{1 - r^2} \tag{3.1}$$

This circular disc touches the parabola Ω_ρ if $\frac{1-2(1+\beta-\alpha)r+(1-2\alpha)r^2}{1-r^2} = \rho$.

This gives the value of $R_p^*(\rho)$. For r less than this value the circular disc is inside the parabola Ω_ρ. The result is sharp for function $f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \left(\frac{1+z}{1-z} \right)^\beta$, which satisfies the hypothesis with $g(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$.

Remark 3.   The special case of Theorem 7 for $\rho = \frac{1}{2}$ and $\beta = 1$ was proved by Shanmugam and Ravichandran [10].

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