BULLETIN *of the* MALAYSIAN MATHEMATICAL SCIENCES SOCIETY

Bull. Malaysian Math. Sc. Soc. (Second Series) **26** (2003) 241–251

The Beginning of the Spectral Theory of Nevanlinna's Mapping from Topological Space to Endomorphisms Algebra of Banach Space and its Applications

MISIR B. RAGIMOV

Harran University, Art and Science Faculty, Department of Mathematics, 63200 Sanliurfa – Turkey
e-mail: rahimovmisir@yahoo.com

Abstract. The role of spectral theory of linear operators in qualitative theory of ordinary differential equations in Banach space is well known [1]. There wasn't the similar progress in theory of multidimensional differential equations bacause of abscence of satisfactory spectral theory, the linear mappings in the following form [8], [9].

$$A : E \to L(F) \tag{0.1}$$

where E is real, F is complex Banach space and $L(F)$ is Banach space of linear continuous operators, acting in F, and the values of operators Ah, $\forall h \in E$, commutate mutually. The interest for the study of spectral theory of such mappings is also explained by the demands of physics (spectral theory of commutative sets of self-adjoint operators (see momography of Berezansky [2])).

On the investigation of mapping in form (0.1) and especially in the case when Ah, $h \in E$, are unbounded operators, naturally arises the necessity of creation of spectral theory of some more general form of mappings, which include mappings (0.1) as a particular (private) case. While writing the spectral theory of such mappings we used Taylor's spectral theory [3, 4] of mutually commutative set of operators.

Main attention was paid to the application of such spectal theory to the solution of some questions of theory of multidimensional equations. Let's mark here, that in the case of finite dimensional spaces E and F (particularly F) there is a big analogy with ordinary differential equations what cannot be said about infinite dimensional space F.

1. Introduction

Let K be topological space and X be complex Banach space. Let's denote the set of strongly continuous mappings $\varphi : K \to L(X)$ of topological space K to Banach algebra $L(X)$ of continuous linear operators of Banach space X as $C(K; L(X))$. Let's denote the set of commuting and continuous mappings $\varphi : K \to L(X)$ of topological space K to

Banach algebra of $L(X)$ of continuous linear operators of Banach space X as $C_c(K;L(X))$. It is supposed that

$$\varphi(t)\varphi(s) = \varphi(s)\varphi(t), \quad \forall t, s \in K, \quad \forall \varphi \in C_c(K;L(X)).$$

By symbol $L(X;Y)$ we usually denote Banach space X of linear continuous operators from Banach space X to Banach space Y, supposing $L(X) = L(X;X)$ for $X = Y$.

Let's consider particularly two cases when K is real Banach space or cone from some Banach space. There are such interesting mappings $\varphi \in C_c(K;L(X))$ that

$$\varphi(t+s) = \varphi(t)\varphi(s) \quad (\forall t, s \in K)$$

in the case of cone and

$$\varphi(\alpha t + \beta s) = \alpha\varphi(t) + \beta\varphi(s), \ \forall \alpha, \beta \in C, t, s \in K,$$

in the case of Banach space.

Symbols R,C,N,Z usually denote the field of real numbers, the field of complex numbers, the set of non-negative numbers and the set of integers.

Let's mark the case when Λ is a set of mutually commutative linear operators from $L(X)$. If there are no any algebraic and topological conditions for Λ, then this set Λ will be considered as a subject from $C_c(\Lambda;L(X))$, where Λ is allocated with discrete topology.

Let's describe another interesting case when Λ is a set of unbounded closed operators acting in X and being producing operators of strongly continuous semigroups of mutually commutative operators from $L(X)$. In this case instead of Λ it is useful to consider the cone K, built by Banach space l_Λ^∞, consisting of bounded real functions on Λ. By definition

$$K = \left\{ \sum_{k=1}^{n} t_k \, \varphi_k(\lambda_k), \, t_k \geq 0, \, \lambda_k \in \Lambda, \varphi \in l_\Lambda^\infty \right\}$$

where $\varphi_k(\lambda) = 1$ for $\lambda = \lambda_k$ and $\varphi_k(\lambda) = 0$ for $\forall \lambda \neq \lambda_k$.

Then the set Λ can be studied by the help of subset of mappings $\varphi \in C_c(K;L(X))$ from $C_c(K;L(X))$ in the form:

$$\varphi\left(\sum_{k=1}^{n} t_k \, \lambda_k\right) = \sum_{k=1}^{n} T_k(t_k),$$

where $T_k(t)$ are subgroups built by operator $\lambda_k \in \Lambda$.

Certainly, in the case if operators from Λ are producing operators of group of operators, then K can be considered as abel group. At last, if, besides that, Λ is finite set then K is real finite dimensional subspace. In this case the problem can be reduced to the construction of spectral theory of representations of finite dimensional space. The problem is somehow important because in this case many questions of theory of many variables are mentioned. For example, for the case of group of shifts these are classic questions of theory of functions.

Now let's pass to the different definitions of spectrum of mappings from $C_c(K; L(X))$. The variety of the definitions of the spectrum is usually explained by different types of tasks, and depending on it that or this definition of spectrum can be useful (see [3, 4]).

Let $T = (T_1, \cdots, T_n)$ be a finite set of linear operators from algebra $L(X)$. Let $\Lambda^p(\sigma, X)$ be a space of external forms with finite basis $\sigma = (s_1, \cdots, s_n)$ and with coefficients in X.

By definition $\Lambda^0(\sigma, X) = \Lambda^n(\sigma, X) = X$. Let's consider the sequence

$$0 \to \Lambda^0(\sigma, X) \xrightarrow{\alpha_1 - z} \Lambda^1(\sigma, X) \xrightarrow{\alpha_2 - z} \cdots \xrightarrow{\alpha_n - z} \Lambda^n(\sigma, X) \xrightarrow{\alpha_n - z} 0 \qquad (1.1)$$

where operators $\alpha_i - z : \Lambda^i(\sigma, X) \to \Lambda^{i+1}(\sigma, X)$ are defined by the formula

$$(\alpha - z)\varphi = \sum_{i=1}^{n} (T_i - z_i)s_i \wedge \varphi,$$

where $\varphi \in \Lambda^i(\sigma, X)$ and $z \in C^n$.

Definition 1.1. *Let's correspond vector $z \in C^n$ to resolvent set $\rho(T)$ of set T if sequence (1.1) is exact, i.e. cohomology groups $H^p(\alpha - z, X) = \ker(\alpha_p - z) / im(\alpha_{p-1} - z)$ are zero. Set $\sigma(T) = C^n \setminus \rho(T)$ is called Taylor's spectrum of commutative set*

$$T = (T_1, T_2, \cdots, T_n).$$

Let's list the main properties of Taylor's spectrum without proof. These properties will be most often used.

2. *If $T' = (T_1, \cdots, T_n, T_{n+1}) = T \cup T_{n+1}$, then $\sigma(T) = \pi(\sigma(T'))$, where $\pi : C^{n+1} \to C^n$ is projection on the first n coordinates. If $F(\sigma(T))$ is an algebra of analytic functions in the vicinity of $\sigma(T)$, then there exists the homomorphism: $F(\sigma(T)) \to L(X)$, such that $1 \to I$ and $z_i \to T_i$ for any $i = 1, 2, \cdots, n$ and, more*

over, $\sigma(f(T)) = f(\sigma(T))$. *If* $\sigma(T)$ *is unification of two disjoint spectral parts* σ_1 *and* σ_2, *then there exist projectors* P_i *commuting with* $T_i, i = 1, 2, \cdots, n,$ *such that*

$$\sigma(TP_i) = \sigma_i, \; i = 1, \; 2, \cdots, n,$$

where

$$TP_i = (T_1P_i, \cdots, T_nP_i).$$

Let A be an arbitrary element of $C_c(K; L(X))$.

Definition 1.2. *Let's call the set of functionals* $\lambda : K \to C$ *such that for any finite set* $(t_1, \cdots, t_n) \subset K$ *the point*

$$(\lambda(t_1), \cdots, \lambda(t_n)) \in C^n = C \times \cdots \times C$$

belongs to Taylor's spectrum of the set

$$A(t) = (A(t_1), \cdots, A(t_n))$$

as a spectrum $\sigma(A)$ *of mapping A.*

It is clear that the set $\sigma(A)$ coincide with the compliment of the set $\rho(A)$ which is called resolvent set of the mapping *A*, consisting of the functionals $\lambda : K \to C$ such that there exists a set $(t_1, \cdots, t_n) \subset K$ for which vector $z_0 = (\lambda(t_1), \cdots, \lambda(t_n)) \in C^n$ belongs to the set $(A(t_1), \cdots, A(t_n))$.

Let's give three more useful in this situation definitions of spectrum of the mappings from $C(K; L(X))$. The first of them is most close to Definition 1.4.

For arbitrary complex functional $\lambda : K \to C$ let's denote as $D(\lambda)$ the linear manifold

$$D(\lambda) = \{f \in L(K; X) : (\lambda h - Ah) fk = (\lambda k - Ak) fh\}$$

for any $h, k \in K$. Here K is finite dimensional linear space.

For operator $A \in C_c(K; L(X))$ and functional $\lambda : K \to C$ let's define operator $S_\lambda : X \to D(\lambda)$ by the formulae

$$(S_\lambda y)h = (\lambda h - Ah)y, \; \forall h \in K, \; y \in X.$$

It is clear that $S_\lambda X \subset D(\lambda)$.

Definition 1.3. *Let K be a finite dimensional space. Let's call the functional* $\lambda : K \to C$ *regular for* $A \in C_c(K; L(X))$, *if* S_λ *is linear homeomorphism X on* $D(\lambda)$. *Let's denote the set of all regular functionals as* $\rho(A)$ *(regular set). The compliment to the set of regular functionals is called spectrum of mapping A and is denoted* $\sigma_1(A)$.

Let's extend Definition 1.3 on the case, when *K* is arbitrary topological space.

Definition 1.4. *Let's call as spectrum* $\sigma_1(A)$ *of the mapping* $A \in C_c(K; L(X))$ *the set of functionals* $\lambda : K \to C$ *such that for any finite set* $(t_1, \cdots, t_n) \subset K$ *the spectrum* $\sigma_1(\overline{A})$ *of the operator* $\overline{A} \in C(R^n; L(X))$ *defined by the formulae*

$$\overline{A}(h) = \sum_{i=1}^{n} h_i A(t_i),\ h = (h_1, \cdots, h_n) \in R^n$$

contains the functional

$$\overline{\lambda}(h) = \sum_{i=1}^{n} h_i \lambda(t_i).$$

Definition 1.5. *Let* $A \in C_c(K; L(X))$, *where X is separable topological space. Let's denote as* $\sigma_2(A)$ *the set of functionals* $\lambda : K \to C$, *for which there exists a normed sequence* $(x_n) \subset X$, *i.e.* $\| x_n \| = 1, \forall n \geq 1$, *such that*

$$\lim(A(t)x_n - \lambda(t)x_n) = 0$$

for any fixed $t \in K$.

It is clear that this definition serves as a generalization of the concept of boundary spectrum of bounded operator. It is clear that the definition of separability is, in general, extra, but in the case of non-separable space *X* it will be necessary to use generalized sequences of elements from *X* instead of sequences in Definition 1.5.

Let $A \in C_c(K; L(X))$ and *U* be the minimal (closed) subalgebra from algebra $L(X)$, containing all the operators $A(t), t \in K$. As $M\{U\}$. Let's denote the space of maximal ideals of this algebra.

Definition 1.6. *As* $\sigma_3(A)$ *let's denote the set of functionals* $\lambda : K \to C$ *in the form*

$$\lambda(t) = \lambda_M(t) = A(t)(M) : K \to C,$$

where M runs the space $M\{U\}$.

2. Comparison of various definitions of spectrum of mappings of topological space into algebra of endomorphisms of Banach space

As it was marked above, this or that definition of spectrum is usually used depending on the type of the considered task. However, it is clear that all the above formulated definitions of spectrum, being formally far from each other, are connected in definite way. We can follow this connection in this paragraph.

Theorem 2.1. *For any mapping* $A \in C_c(K; L(X))$ *there have place the following inclusions*

$$\sigma_2(A) \subset \sigma_1(A) \subset \sigma(A) \subset \sigma_3(A).$$

Besides that, the set $\sigma_1(A)$ *coincide with the compliment to the set of those functionals* $\lambda : K \to C,$ *for which there exists such finite set* $(t_1, \cdots, t_n) \subset K$ *that*

$$H^0(\alpha - z, X) = H^1(\alpha - z, X) = \{0\},$$

where $z = (\lambda(t_1), \cdots, \lambda(t_n))$ *and all the designations are taken from the definition of spectrum* $\sigma_1(A)$.

Proof. At first let's establish the validity of the last statement. Let

$$H^0(\alpha - Z, X) = H^1(\alpha - Z, X) = \{0\},\ z = (\lambda(t_1), \cdots, \lambda(t_n))$$

for some finite set $t_1, \cdots, t_n \subset K.$

Let's consider the functional $\bar{\lambda} : R^n \to C,$ defined by the formulae

$$\bar{\lambda}(h) = \sum_{i=1}^{h} h_i \lambda(t_i),\ h = (h_1, \cdots, h_n).$$

Then

$$D(\bar{\lambda}) = \{ f \in L(R^n; X) : (\bar{\lambda}(h) - \bar{A}(h)) fk = (\bar{\lambda}(k) - \bar{A}(k) fh\},\quad \forall h,\ k \in R^n,$$

where

$$\bar{A}(h) = \sum_{i=1}^{n} h_i A(t_i),\quad \forall h, \in R^n,\quad h = (h_1, \cdots, h_n).$$

Evidently, the condition of equality of the group $H^0(\alpha - z, X) = \{0\}$ to zero means injectiveness of the mapping $\overline{A} - \overline{\lambda} : R^n \to D(\overline{\lambda}),\ \overline{A}(h) = \overline{A}_h,\ \forall h \in R^n$, and the condition $H^1(\alpha - z, X) = \{0\}$ means surjectiveness of this mapping. The opposite is also true. Therefore, the last statement is proved.

Directly from the proved and Definition 1.2 and Item 1 it follows that $\sigma_1(A) \subset \sigma(A)$.

Let's prove the inclusion $\sigma_2(A) \subset \sigma_1(A)$.

Directly from the definition of set $\sigma_2(A)$ it follows that for $\forall \lambda \in \sigma_2(A)$ there exists a normal sequence $(x_n) \subset X$ such that

$$A(t)x_n - \lambda(t)x_n \to 0\ \forall t \in K.$$

Then it is clear that in the case $H^1(\alpha - z, X) \neq \{0\}$, $z = (\lambda(t_1), \cdots, \lambda(t_n))$, $\alpha = \sum_{i=1}^{n} A(t_i)\, S_i$ for any finite set $(t_1, \cdots, t_n) \subset K$. In the opposite case operator $\overline{A} - \overline{\lambda}$ (see the proof of the last statement) was surjective. But this can't be by the virtue of the condition.

$$A(t)x_n - \lambda(t)x_n \to 0,\ \forall t \in K.$$

Let's prove inclusion $\sigma(A) \subset \sigma_3(A)$. If functional $\lambda : K \to C$ doesn't belong to $\sigma_3(A)$ then λ doesn't belong to joint spectrum $\sigma(A(t_1), \cdots, A(t_n))$ of the operators $A(t_1), \cdots, A(t_n)$ for some finite set $(t_1, \cdots, t_n) \subset K$. This means that there exists such a set of operators $(B_1, \cdots, B_n) \subset U$ (algebra U was entered before Definition 1.6) that

$$A(t_1)B_1 + \cdots + A(t_n)B_n = 1.$$

So from Taylor's results [3,4] it follows that in this case vector $z = (\lambda(t_1), \cdots, \lambda(t_n))$ doesn't belong to spectrum $\sigma(A(t))$ of Taylor's set $A(t) = (A(t_1), \cdots, A(t_n))$. The latter means that $\lambda \overline{\in} \sigma(A)$. The theorem is proved.

Let's notice at once that inclusions in Theorem 1.1 can be strict.

Example 2.1. Let $A \in L(X)$, i.e. in fact *K*, consist of one element and spectrum of operator *A* coincide with $\sigma(A)$. Then $\sigma_2(A) \neq \sigma(A)$. As an example of such operator we can take operator A of unilateral shift in space l_2 of infinitely summable with square sequences $Ax = (0, x_1, x_2, \cdots)$, if

$$x = (x_1, x_2, \cdots, x_n, \cdots); \ \|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}\,.$$

Example 2.2. Let K be two-pointed set (t_1, t_2), $A(t_i) = A_i : X \to X$, where X; is Banach space of analytical functions inside the cylinder $\max\ \{|\ z_1\ |, |\ z_2\ |\} \le 1$ of bounded functions with norm $\|f\| = \max_{|z_1|,|z_2| \le 1} |f(z_1, z_2)|$, and operators $A_i, i = 1, 2$ are defined by formulas

$$A_1 f(z_1, z_2) = z_1 f(z_1, z_2), \quad f \in X,$$

$$A_2 g(z_1, z_2) = z_2 g(z_1, z_2), \quad g \in X.$$

Then, evidently, $H^0(\alpha, X) = \{0\}$. If

$$A_1 f = A_2 g$$

then

$$f = A_2 h, \ g = A_1 h,$$

where

$$h(z_1, z_2) = \frac{1}{z_2} f(z_1, z_2) = \frac{1}{z_1} g(z_1, z_2) \in X, \text{ i.e. } H^1(\alpha, X) = \{0\}.$$

On the other hand, $H^2(\alpha, X) \neq X$ for function $\varphi(z_1, z_2) \equiv 1$ doesn't belong to the set of values of both operators, so

$$\sigma_2(A) \subset \sigma(A).$$

Let's notice that the example of strict inclusion $\sigma(A) \subset \sigma_3(A)$ can be easily given if you use the corresponding Taylor's example for the case K, consisting of finite number of elements (see [3,4]).

In conclusion, of this paragraph let's give one criteria of coincidences of all the spectrums for one special subset of mappings from $C(K; L(X))$. But at first let's formulate lemma in which $\sigma_3(A)$ is identified with the space $M\{U\}$ by the formulae from Definition 1.6 and, therefore, $\sigma_3(A)$ is allocated with topology of space $M\{U\}$ (this identification will hold).

Lemma 2.1. *Let* $A \in C_c(K; L(X))$ *and spectrum* $\sigma_3(A) = \sigma_1 \cup \sigma_2$, *where* $\sigma_1 \cap \sigma_2 = \emptyset$. *Then there exist* $P_1, P_2 \in L(X)$ *such that*

(1) $P_1 + P_2 = I$, $P_1, P_2 \in U$

(2) $\sigma_3(A \setminus X_i) = \sigma_i$, $i = 1, 2$, *where* $X_i = P_i X$ *and* $A \setminus X_i$ *denote the mapping* $A_i \in C_c(K; L(X_i))$ *defined by the formulae* $A_i(t)x = A(t)x$ *for* $x \in X_i$, $i = 1, 2$.

Proof. By Shilov's theorem [6] about idempotents under the theorems conditions there exist projectors $P_1, P_2 \in U$ such that $P_1 + P_2 = I$ and $P_i(M) = 1$, $\forall M \in \sigma_i \subset M\{U\}$ (spectrum $\sigma_3(A)$ is identified with the space of maximal ideals, i.e. $M\{U\} = \sigma_3(A) = \sigma_1 \cup \sigma_2$).

Statement (2) directly follows from the definition of spectrum $\sigma_3(A)$ and the fact that algebras U, built by operators A_i, coincide with algalgebras $P_i U$ with joint to them units. It is necessary to notice, that $M\{P_i U\} = \sigma_i$ by the same Shilov's theorem. Lemma is proved.

For the case if K is finite dimensional real space (or K is finite set) then there takes place the following deep result.

Lemma 2.2. *If* $\sigma(A) = \sigma_1 \cup \sigma_2$, *where* $\sigma_1 \cap \sigma_2 = \emptyset$, $\sigma_i (i = 1, 2)$ *are closed sets, then there exist such closed spaces* X_1 *and* X_2 *that*

(1) $X = X_1 \oplus X_2$

(2) X_1 *and* X_2 *are invariant corresponding to each operator commutating with all the operators* $A(t)$, $t \in K$;

(3) $\sigma(A \mid X_1) = \sigma_1$ *and* $\sigma(A \mid X_2) = \sigma_2$.

On the other hand this statement contains (on less suggestions than Lemma 2.1) less information about spaces X_1 and X_2 than Lemma 2.1.

Theorem 2.2. *Let the mapping* $A \in C_c(K; L(X))$ *has the property: every* $A(t), t \in K$, *can be represented in the form* $\alpha(t) I_X + B(t)$, *where* $\alpha \in K'$, $B(t)$ *is completely continuous operator,* $\forall t \in K$. *Then*

$$\sigma(A) = \sigma_i(A), \ i = 1, 2, 3.$$

Proof. Let U be the least subalgebra from $L(x)$, containing all the operators $A(t), t \in K$ and let $M\{U\}$ be the space of maximal ideals of this algebra. Without restriction of a generality we can think that U contains a unit I (otherwise it is always possible to attach it to U). From Theorem 2.1 it follows that $\sigma(A)$, $\sigma_1(A)$ and $\sigma_2(A)$ contain in $\sigma_3(A)$, which can be identified with the subset from $M\{U\}$ (in fact $\sigma_3(A) = M\{U\}$). Let's prove that $M\{U\}$ is quite non-connected compact topological space.

From this let's suggest that there exists connected subset α from $M\{U\}$, which contains more than one point. But then there exists an element $B \in U$, such that function $\Phi(M) = B(M) : M(U) \to C$ is nonconstant on the set α. But then as function Φ is continuous, the set $\Phi(\alpha)$ is connected subset from C, containing in $\sigma(B)$. As algebra U consists of operators, multiple to identical operator plus completely continuous one, then operator B is represented in the form $\beta I + B_0$, where $\beta \in C$ and B_0 is completely continuous operator. It is clear that we came to contradiction because the spectrum of operator B having not more than one limit point (i.e. β is a candidate for this limit point) doesn't contain connected parts. So $M\{U\}$ is completely disconnected space.

By virtue of the Theorem 2.1 it is clear that for the proof of this theorem it is enough to prove the equality $\sigma_2(A) = \sigma_3(A)$. Let's assume the opposite, i.e. $\sigma_2(A) \subset \sigma_3(A)$ and $\sigma_2(A) \neq \sigma_3(A)$. Directly from the definition of set $\sigma_2(A)$ it follows that $\sigma_2(A)$ is closed subset from $M\{U\}$. And from the results of Lubich [5] it follows that $\sigma_2(A) = \emptyset$ if and only $A \equiv 0$ (in the case of nonseparable space X this is Domar-Lindal's result [7] in this case generalized orientations are used in Definition 1.5). By the virtue of disconnectedness of $M\{U\}$ there exists open-closed subset Δ from $\sigma_3(A)$, dis-adjoint with $\sigma_2(A)$.

By Shilov's theorem about idempotents there exists projector $P \in U$ such that $P(M) = 1, \forall M \subset \Delta$ and $P(M) = 0, \forall M \overline{\in} \sigma_3(A) \setminus \Delta$. Let's consider the mapping $A_p(t) = A(t)P : K \to L(X)$. It is clear that $A_p \in C_c(K; L(X))$ (see proof of Lemma 2.1), because

$$A_p(t)(M) = A(t)(M)P(M) = \begin{Bmatrix} A(t)(M) & , & \text{if } M \subset \Delta, \\ 0 & , & \text{if } M \overline{\subset} \Delta, \end{Bmatrix}$$

i.e. $\sigma_3(A_p) = \Delta$ and so $A_p \neq 0$. On the other hand directly from Definition 1.5 it follows that $\sigma_2(A_P) \subset \Delta$, because spectrum $\sigma_2(A \mid Y)$ of restriction of the mapping A on any subspace $Y \subset X$ contains in $Y = PX$. But A_P can be considered as A/Y where $Y = PX$. From Theorem 2.1 it follows that $\sigma_2(A_P) \subset \Delta$ and so $\sigma_2(A_P) = \emptyset$, i.e. $A_P \equiv 0$. We came to contradiction. The theorem is proved.

The author thanks Professor Nasir Ganikhodjaev for useful discussions and referee for his or her useful suggestions.

References

1. S.G. Krein, *Linear Differential Equations in Banach Space*, Moscow, Nauka, 1967.
2. Yu.M. Berezansky, *Selfajoint Operators in the Spaces of Functions of Infinite Numbers of Variables*, Naukova Dumka, Kiev, 1978.
3. J.L. Taylor, A joint spectrum for several commuting operators, *J Funct. An* **6** (1970), 172–191.
4. J.L. Taylor, The analytic-functional calculus for several commuting operators, *Acta Math* **125** (1970), 1–2.
5. Yu.I. Lubich, On the spectrum of representation of topological abel group, *DAN SSSR* **200 (4)** (1971), 777–779.
6. G.Ye. Shilov, On decomposition of commutative normed ring into direct sum of ideals, *Math art.* **32:2** (1953), 353–364.
7. Y. Domar and L.A. Lindahl, Three Spectral Notions for Representations of Commutative Banach Algebr., *Uppsala University, Department of Mathematics* (1974).
8. M.B. Ragimov, Multidimensional spectral analysis of linear operators, *Doctor Dissertation* (1997).
9. F. und R. Nevanlinna, *Absolute Analysis*, Berlin, 1959.