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Homogeneous $2-\pi$ Metrical Structures on T^2M Manifold

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Abstract. On the geometrical model determined by the second order prolongation of a Riemannian space, we introduce for the first time the homogeneous Sasaki lift notion. We define almost $2-\pi$ homogeneous structures on the fibred of second order acceleration and study the normal conditions of the mentioned structures. Finally we determine a class of distinguished connections compatible with $2-\pi$ metrical homogeneous structure $\left(\overset{(0)}{G}, \overset{(0)}{F}\right)$.

Introduction

The term homogeneity has recently been discussed in Radu Miron's papers (see [10] – [11]). He introduces new geometrical models on Riemannian spaces and Finsler ones, respectively.

This paper discusses the second order prolongation of a Riemannian space. The basic concepts were introduced by Radu Miron in his monography [8]. On the mentioned geometrical models, the former author of this paper (see [15]) introduced and studied the notion $(\alpha\beta\gamma)$ – lift Sasaki of a Riemannian (M,γ) space to T^2M and then determined $(\alpha\beta\gamma)$ – the corresponding metrical linear connection; for the canonical metrical connection he determined the local components of the tensor fields of curvature and torsion. He has also introduced and studied the notion of μ – almost $2-\pi$ structure on T^2M and dealt with the linear connection compatible with such a structure, as well as with the conditions necessary normality. More, it has been considered also the d – gauge linear connections on T^2M, preparing the basis for the determination of the second order generalized EYM equations, and the gravitational field equations as well.

1. Homogeneous Sasaki lift of a (M,γ) Riemannian space to T^2M manifold

We will consider $R^n = (M,\gamma)$ a Riemannian space generated by a real, differentiable, *n*-dimensional manifold *M* and by a Riemannian metric γ on *M*, given by the local components $(\gamma_{ij}(x)),\ x \in U \subset M$. We will extend γ to $\pi^{-1}(U) \subset E = T^2M$, defining:

$$\left(\gamma_{ij} \circ \pi\right)(u) = \gamma_{ij}(x),\ u \in \pi^{-1}(U),\ \pi(u) = x \tag{1}$$

In this case $\gamma_{ij} \circ \pi$ are the local components of a tensor field on *E*. Usually, we will write these local components with γ_{ij} as well, and with $\gamma^i_{jk}(x)$ we will note the Christoffel symbols of the second species of γ metric. As we well know (see [8]), on *E* we can introduce a nonlinear connection determined only by γ metric. More, the coefficients of connection $\underset{(1)_j}{\overset{(0)^i}{N}}, \underset{(2)_j}{\overset{(0)^i}{N}}$ are determined by the following relations (see also [14]):

$$\begin{cases} \underset{(1)_j}{\overset{(0)^i}{N}}(x, y^{(1)}) = \gamma^i_{j0} \\ \underset{(2)_j}{\overset{(0)^i}{N}}(x, y^{(1)}, y^{(2)}) = \dfrac{1}{2}\left(\dfrac{\partial \gamma^i_{j0}}{\partial x^p} y^{(1)p} + \gamma^i_{0m} \cdot \gamma^m_{j0}\right) + \gamma^i_{j\bar{0}} \end{cases} \tag{2}$$

where "0" means the contraction by $(y^{(1)})$ and "$\bar{0}$" means the contraction by $(y^{(2)})$.

In the following section, we will partially avoid this particular nonlinear connection and we use one, more general, determined in:

Theorem 1.1. *If* $\underset{(1)_j}{\overset{(0)^i}{N}}, \underset{(2)_j}{\overset{(0)^i}{N}}$ *are the local components of the nonlinear connection determined only by Riemannian metric* γ*, and* X^i_j, Y^i_j *are the local components of any d- tensor field of* $(1,1)$ *type on E, then the functions:*

$$\begin{cases} \underset{(1)j}{N^i} = \underset{(1)j}{\overset{(0)^i}{N}} + X^i_j \\ \underset{(2)j}{N^i} = \underset{(2)j}{\overset{(0)^i}{N}} + \left(Y^i_m - \gamma^i_{m0}\right) \cdot X^m_j \end{cases} \tag{3}$$

are the local components of a nonlinear connection N on E.

The nonlinear connection N assures the existence of basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ adapted to the tangent space $T_u E$. The vector fields of the adapted basis are defined with the help of the following relations:

$$\delta_k = \frac{\partial}{\partial x^k} - \underset{(1)k}{N^i} \cdot \frac{\partial}{\partial y^{(1)i}} - \underset{(2)k}{N^i} \cdot \frac{\partial}{\partial y^{(2)i}} \tag{4}$$

$$\overset{(1)}{\delta}_k = \frac{\partial}{\partial y^{(1)k}} - \underset{(1)k}{N^i} \cdot \frac{\partial}{\partial y^{(2)i}}, \quad \overset{(2)}{\delta}_k = \frac{\partial}{\partial y^{(2)k}} \tag{5}$$

For further developments, we need the following result:

Theorem 1.2. *Lie brackets of the vector fields of the adapted basis* $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$ *are given by*:

$$\left[\delta_j, \delta_k\right] = \underset{(01)jk}{R^i} \cdot \overset{(1)}{\delta}_i + \underset{(01)jk}{R^i} \cdot \overset{(2)}{\delta}_{(i)} \tag{6}$$

$$\left[\delta_j, \overset{(1)}{\delta}_k\right] = \underset{(11)jk}{B^i} \cdot \overset{(1)}{\delta}_i + \underset{(12)jk}{B^i} \cdot \overset{(2)}{\delta}_{(i)} \tag{7}$$

$$\left[\delta_j, \overset{(2)}{\delta}_k\right] = \underset{(21)jk}{B^i} \cdot \overset{(1)}{\delta}_i + \underset{(22)jk}{B^i} \cdot \overset{(2)}{\delta}_{(i)} \tag{8}$$

$$\left[\overset{(1)}{\delta}_j, \overset{(1)}{\delta}_k\right] = \underset{(12)jk}{R^i} \cdot \overset{(2)}{\delta}_i, \quad \left[\overset{(1)}{\delta}_j, \overset{(2)}{\delta}_k\right] = \underset{(21)jk}{B^i} \cdot \overset{(2)}{\delta}_{(i)} \tag{9}$$

where:

$$\underset{(01)jk}{R^i} = \underset{(01)jk}{\overset{(0)}{R}{}^i} + X^i_{jk}, \quad \underset{(02)jk}{R^i} = \underset{(02)jk}{\overset{(0)}{R}{}^i} + (XY)^i_{jk}, \quad \underset{(12)jk}{R^i} = \underset{(12)jk}{\overset{(0)}{R}{}^i} + \overset{(1)}{X}{}^i_{[jk]} \tag{10}$$

$$\underset{(11)}{B}{}^{i}_{jk} = \overset{(0)}{\underset{(11)}{B}}{}^{i}_{jk} + \overset{(1)}{X}{}^{i}_{jk} \,, \quad \underset{(12)}{B}{}^{i}_{jk} = \overset{(0)}{\underset{(12)}{B}}{}^{i}_{jk} + \underset{(12)}{(XY)}{}^{i}_{jk} \tag{11}$$

$$\underset{(21)}{B}{}^{i}_{jk} = \overset{(0)}{\underset{(21)}{B}}{}^{i}_{jk} + \overset{(2)}{X}{}^{i}_{jk} \,, \quad \underset{(22)}{B}{}^{i}_{jk} = \overset{(0)}{\underset{(22)}{B}}{}^{i}_{jk} + \underset{(22)}{(XY)}{}^{i}_{jk} \tag{12}$$

with the following notations:

$$x^{i}_{jk} = \frac{\delta X^{i}_{j}}{\delta x^{k}} - \frac{\delta X^{i}_{k}}{\delta x^{j}} \,, \quad \overset{(1)}{X}{}^{i}_{jk} = \frac{\delta X^{i}_{j}}{\delta y^{(1)k}} \,, \quad \overset{(2)}{X}{}^{i}_{jk} = \frac{\delta X^{i}_{j}}{\delta y^{(2)k}} \tag{13}$$

$$(XY)^{i}_{jk} = \overset{(0)}{\underset{(1)}{N}}{}^{i}_{m} \cdot X^{m}_{jk} + X^{i}_{m} \cdot \overset{(0)}{\underset{(01)}{R}}{}^{m}_{jk} + X^{i}_{m} \cdot X^{m}_{jk}$$
$$+ \left(\frac{\delta(Y^{i}_{q} \cdot X^{q}_{j})}{\delta x^{k}} - \frac{\delta(Y^{i}_{q} \cdot X^{q}_{k})}{\delta x^{j}} \right) - \left(\frac{\delta(\gamma^{i}_{q0} \cdot X^{q}_{j})}{\delta x^{k}} - \frac{\delta(\gamma^{i}_{q0} \cdot X^{q}_{k})}{\delta x^{j}} \right) \tag{14}$$

$$\underset{(12)}{(XY)}{}^{i}_{jk} = \overset{(0)}{\underset{(1)}{N}}{}^{i}_{m} \cdot \overset{(1)}{X}{}^{m}_{jk} + X^{i}_{m} \cdot \overset{(0)}{\underset{(11)}{B}}{}^{m}_{jk} + X^{i}_{m} \cdot \overset{(1)}{X}{}^{m}_{jk} + \frac{\delta Y^{i}_{j}}{\delta y^{(1)k}} - \frac{\delta Y^{i}_{k}}{\delta y^{(1)j}} \tag{15}$$

$$\underset{(22)}{(XY)}{}^{i}_{jk} = \overset{(0)}{\underset{(1)}{N}}{}^{i}_{m} \cdot \overset{(2)}{X}{}^{m}_{jk} + X^{i}_{m} \cdot \overset{(0)}{\underset{(21)}{B}}{}^{m}_{jk} + X^{i}_{m} \cdot \overset{(2)}{X}{}^{m}_{jk} + \frac{\delta Y^{i}_{j}}{\delta y^{(2)k}} \tag{16}$$

$$\overset{(1)}{X}{}^{i}_{[jk]} = \overset{(1)}{X}{}^{i}_{jk} - \overset{(1)}{X}{}^{i}_{kj} \tag{17}$$

Theorem 1.3. ([8]) *The pair* $\mathrm{Prol}^{(2)} R^{(n)} = \left(\tilde{T}^{2}M, G\right)$ *where*:

$$G = \gamma_{ij}(x) \cdot dx^{i} \otimes dx^{j} + \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \tag{18}$$

is a Riemannian space of 3n dimension, with G metric structure depending only on $\gamma(x)$ *Riemannian structure, apriori on Riemannian space* $R^{(n)} = (M, \gamma)$.

We will say that *G* is Sasaki lift of γ Riemannian structure. We define the homotety $h_{t} : (x, y^{(1)}, y^{(2)}) \to (x, ty^{(1)}, t^{2} y^{(2)})$, $t \in R^{*}$ on the fibres of $T^{2}M$. We also mention that *G* is transformed in accordance with:

$$G \circ h_t(x, y^{(1)}, y^{(2)}) = \gamma_{ij}(x) \cdot dx^i \otimes dx^j +$$

$$t^2 \cdot \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + t^4 \cdot \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \quad (19)$$

The above remark makes us affirm that the Sasaki lift G is nonhomogeneous on the fibres of T^2M.

In the following part we concentrate upon a new lift of Sasaki type, called homogeneous Sasaki lift and noted as $\overset{(0)}{G}$:

$$\overset{(0)}{G} = \gamma_{ij}(x) \cdot dx^i \otimes dx^j +$$

$$\frac{1}{F^2} \cdot \gamma_{ij}(x) \cdot \delta y^{(1)i} \otimes \delta y^{(1)j} + \frac{1}{F^4} \cdot \gamma_{ij}(x) \cdot \delta y^{(2)i} \otimes \delta y^{(2)j} \quad (20)$$

where $F^2 = \lambda_{ij}(x) \cdot y^{(1)i} \cdot y^{(1)j}$.

Theorem 1.4. *The following properties holds*:

(a) *The pair* $\left(T\widetilde{T}^2M, \overset{(0)}{G}\right)$ *is a Riemannian space*;

(b) $\overset{(0)}{G}$ *depend only by the* $\gamma(x)$ *Riemannian metric*;

(c) *The distributions N,* V_1, V_2 *are ortogonal with respect to* $\overset{(0)}{G}$.

Definition 1.1. *A D linear connection on* $\widetilde{T}^2M$ *is call* (0) - *metrical connection with respect to* $\overset{(0)}{G}$ *if* $D\overset{(0)}{G} = 0$ *and D preserves by parallelism the N horizontal distribution*.

With respect to adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k\right)$, any D linear connection on E can be representated as follows:

$$D_{\delta_k}\delta_j = \overset{(H)}{L}{}^i_{jk} \cdot \delta_i + \overset{(1)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(2)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \quad (21)$$

$$D_{\delta_k}\overset{(1)}{\delta}_j = \overset{(3)}{L}{}^i_{jk} \cdot \delta_i + \overset{(V_1)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(4)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \quad (22)$$

$$D_{\delta_k}\overset{(2)}{\delta}_j = \overset{(5)}{L}{}^i_{jk} \cdot \delta_i + \overset{(6)}{L}{}^i_{jk} \cdot \overset{(1)}{\delta}_i + \overset{(v_2)}{L}{}^i_{jk} \cdot \overset{(2)}{\delta}_i \quad (23)$$

$$D_{\overset{(1)}{\delta}_k}\delta_j = \overset{(H)}{F}{}^{i}_{jk}\cdot\delta_i + \overset{(1)}{F}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(2)}{F}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{24}$$

$$D_{\overset{(1)}{\delta}_k}\overset{(1)}{\delta}_j = \overset{(3)}{F}{}^{i}_{jk}\cdot\delta_i + \overset{(V_1)}{F}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(4)}{F}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{25}$$

$$D_{\overset{(1)}{\delta}_k}\overset{(2)}{\delta}_j = \overset{(5)}{F}{}^{i}_{jk}\cdot\delta_i + \overset{(6)}{F}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(v_2)}{F}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{26}$$

$$D_{\overset{(2)}{\delta}_k}\delta_j = \overset{(H)}{C}{}^{i}_{jk}\cdot\delta_i + \overset{(1)}{C}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(2)}{C}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{27}$$

$$D_{\overset{(2)}{\delta}_k}\overset{(1)}{\delta}_j = \overset{(3)}{C}{}^{i}_{jk}\cdot\delta_i + \overset{(v_1)}{C}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(4)}{C}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{28}$$

$$D_{\overset{(2)}{\delta}_k}\overset{(2)}{\delta}_j = \overset{(5)}{C}{}^{i}_{jk}\cdot\delta_i + \overset{(6)}{C}{}^{i}_{jk}\cdot\overset{(1)}{\delta}_i + \overset{(v_2)}{C}{}^{i}_{jk}\cdot\overset{(2)}{\delta}_i \tag{29}$$

The set consisting of the functions $\overset{(H)}{L}{}^{i}_{jk}, \cdots, \overset{(v_2)}{C}{}^{i}_{jk}$ represents the set of the coefficients of D linear connection. In what regards the notion of (0) - metrical connection, there can be proved the following result:

Theorem 1.5. *There exist D (0) - metrical connections on $\tilde{T}^2M$, which depend only on the γ Riemannian tensor field. One of these has its coefficients given by*:

$$\overset{(1)}{L}{}^{i}_{jk} = \overset{(2)}{L}{}^{i}_{jk} = \overset{(3)}{L}{}^{i}_{jk} = \overset{(4)}{L}{}^{i}_{jk} = \overset{(5)}{L}{}^{i}_{jk} = \overset{(6)}{L}{}^{i}_{jk} = 0 \tag{30}$$

$$\overset{(1)}{F}{}^{i}_{jk} = \overset{(2)}{F}{}^{i}_{jk} = \overset{(3)}{F}{}^{i}_{jk} = \overset{(4)}{F}{}^{i}_{jk} = \overset{(5)}{F}{}^{i}_{jk} = \overset{(6)}{F}{}^{i}_{jk} = 0 \tag{31}$$

$$\overset{(1)}{C}{}^{i}_{jk} = \overset{(2)}{C}{}^{i}_{jk} = \overset{(3)}{C}{}^{i}_{jk} = \overset{(4)}{C}{}^{i}_{jk} = \overset{(5)}{C}{}^{i}_{jk} = \overset{(6)}{C}{}^{i}_{jk} = 0 \tag{32}$$

$$\overset{(H)}{L}{}^{i}_{jk} = \gamma^{i}_{jk}\ ,\ \overset{(v_1)}{L}{}^{i}_{jk} = \gamma^{i}_{jk} + \frac{1}{F^2}\cdot\theta^{i}_{jk}\ ,\ \overset{(v_2)}{L}{}^{i}_{jk} = \gamma^{i}_{jk} + \frac{2}{F^2}\cdot\theta^{i}_{jk} \tag{33}$$

$$\overset{(H)}{F}{}^{i}_{jk} = 0\ ,\ \overset{(v_1)}{F}{}^{i}_{jk} = -\frac{1}{F^2}\cdot\Lambda^{i}_{jk}\ ,\ \overset{(v_2)}{F}{}^{i}_{jk} = -\frac{2}{F^2}\cdot\Lambda^{i}_{jk} \tag{34}$$

$$\overset{(H)}{C}{}^{i}_{jk} = \overset{(v_1)}{C}{}^{i}_{jk} = \overset{(v_2)}{C}{}^{i}_{jk} = 0 \tag{35}$$

with the following notations:

$$\theta^i_{jk} = \left(X^t_k \cdot \delta^i_j + X^t_j \cdot \delta^i_k - X^t_s \cdot \gamma_{jk} \cdot \gamma^{is} \right) \cdot y^{(1)}_t \tag{36}$$

$$\Lambda^i_{jk} = \delta^i_{jk} \cdot y^{(1)}_k + \delta^i_k \cdot y^{(1)}_j - \gamma_{jk} \cdot y^{(1)i} \tag{37}$$

Theorem 1.6. *The set of all* (0)*-metrical connections is given by the coefficients* $\overset{(H,*)}{L}{}^{i}_{jk}, \cdots, \overset{(v_2,*)}{C}{}^{i}_{jk}$, *whose expression is show by the following relations*:

$$\overset{(H,*)}{L}{}^{i}_{jk} = \overset{(H)}{L}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(H)}{I}{}^{r}_{hk}, \quad \overset{(v_\alpha,*)}{L}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(v_\alpha)}{I}{}^{r}_{hk}, \quad \alpha = 1, 2 \tag{38}$$

$$\overset{(H,*)}{F}{}^{i}_{jk} = \overset{(H)}{F}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(H)}{J}{}^{r}_{hk}, \quad \overset{(v_\alpha,*)}{F}{}^{i}_{jk} = \overset{(v_\alpha)}{F}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(v_\alpha)}{J}{}^{r}_{hk}, \quad \alpha = 1, 2 \tag{39}$$

$$\overset{(H,*)}{C}{}^{i}_{jk} = \overset{(H)}{C}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(H)}{H}{}^{r}_{hk}, \quad \overset{(v_\alpha,*)}{C}{}^{i}_{jk} = \overset{(v_\alpha)}{C}{}^{i}_{jk} + \Omega^{ih}_{rj} \cdot \overset{(v_\alpha)}{H}{}^{r}_{hk}, \quad \alpha = 1, 2 \tag{40}$$

where:

$$\Omega^{ih}_{rj} = \frac{1}{2} \cdot \left(\delta^i_r \cdot \delta^h_j - \gamma_{rj} \cdot \gamma^{ih} \right) \tag{41}$$

and $\overset{(H)}{I}{}^{r}_{hk}$, $\overset{(v_1)}{I}{}^{r}_{hk}$, $\overset{(v_2)}{I}{}^{r}_{hk}$, $\overset{(H)}{J}{}^{r}_{hk}$, $\overset{(v_1)}{J}{}^{r}_{hk}$, $\overset{(v_2)}{J}{}^{r}_{hk}$, $\overset{(H)}{H}{}^{r}_{hk}$, $\overset{(v_1)}{H}{}^{r}_{hk}$, $\overset{(v_2)}{H}{}^{r}_{hk}$ *are arbitrary d - tensor fields*.

2. 2–π structures on the fibred of second order acceleration

We consider $F(E)-$linear operator $\overset{(0)}{F} : X(E) \rightarrow X(E)$ defined on the adapted basis $\left(\delta_k, \overset{(1)}{\delta}_k, \overset{(2)}{\delta}_k \right)$ through:

$$\overset{(0)}{F}(\delta_i) = \lambda \cdot F^2 \cdot \overset{(2)}{\delta}_i, \quad \overset{(0)}{F}\left(\overset{(1)}{\delta}_i \right) = 0, \quad \overset{(0)}{F}\left(\overset{(2)}{\delta}_i \right) = -\frac{\lambda}{F^2} \cdot \delta_i \tag{42}$$

where λ is an arbitrary nonzero complex number.

Theorem 2.1. $\overset{(0)}{F}$ *operator has the following characteristies*:

(a) *It is global defined on* $\tilde{E}$;

(b) *It is a tensor field of* $(1,1)$ *type on* $\tilde{E}$ *and depends on* γ *Riemannian structure*;

(c) $\overset{(0)}{F}{}^3 + \lambda^2 \cdot \overset{(0)}{F} = 0$;

(d) $\ker\left(\overset{(0)}{F}\right) = N_1$, $\operatorname{Im}\left(\overset{(0)}{F}\right) = N_0 \oplus V_2$;

(e) $\operatorname{rank}\left(\overset{(0)}{F}\right) = 2n$.

Definition 2.1.

(a) *The tensor fields defined above is called* $2-\pi$ *homogeneous structure of second order on the fibred of second order acceleration*.

(b) $\overset{(0)}{F}$ $2-\pi$ *structure of second order is normal if*:

$$N_{\overset{(0)}{F}}(X,Y) + \lambda^2 \cdot \sum_{i=1}^{n} d\left(\delta y^{(1)i}\right)(X,Y) \cdot \delta_i = 0, \ \forall \ X,\ Y \in X(E) \tag{43}$$

where $N_{\overset{(0)}{F}}$ *represents Nijenhuis tensor associated to* $\overset{(0)}{F}$ *tensor field*.

Definition 2.2. *A* D *linear connection is compatible with* $2-\pi$ *homogeneous structure of second order, if the following condition is achieved*:

$$D_x \overset{(0)}{F} = 0, \ \forall \ X \in X(E) \tag{44}$$

We shall study in the first stage the linear connections compatible with $2-\pi$ homogeneous structure of second order. We can notice that $\overset{(0)}{F}$ can be written as:

$$\overset{(0)}{F} = \lambda \cdot F^2 \cdot \overset{(2)}{\delta}_i \otimes dx^i - \frac{\lambda}{F^2} \cdot \delta_i \otimes \delta y^{(2)i} \tag{45}$$

There can be proved the following result:

Theorem 2.2. *An arbitrary D linear connection is compatible with* $2-\pi$ *homogeneous structure of second order if, and only if the connection coefficients satisfy the following relations*:

$$\overset{(H)i}{L}_{jk} - \overset{(v_2)i}{L}_{jk} - \frac{2}{F^2}\cdot X_k^p \cdot y_p^{(1)} \cdot \delta_j^i = 0 \tag{46}$$

$$\frac{1}{F^4}\cdot \overset{(2)i}{L}_{jk} + \overset{(5)i}{L}_{jk} = 0 \tag{47}$$

$$\overset{(1)i}{L}_{jk} = \overset{(3)i}{L}_{jk} = \overset{(4)i}{L}_{jk} = \overset{(6)i}{L}_{jk} = 0 \tag{48}$$

$$\overset{(H)i}{F}_{jk} - \overset{(v_2)i}{F}_{jk} - \frac{2}{F^2}\cdot y_k^{(1)} \cdot \delta_j^i = 0 \tag{49}$$

$$\frac{1}{F^4}\cdot \overset{(2)i}{F}_{jk} + \overset{(5)i}{F}_{jk} = 0 \tag{50}$$

$$\overset{(1)i}{F}_{jk} = \overset{(3)i}{F}_{jk} = \overset{(4)i}{F}_{jk} = \overset{(6)i}{F}_{jk} = 0 \tag{51}$$

$$\overset{(H)i}{C}_{jk} - \overset{(v_2)i}{C}_{jk} = 0 \tag{52}$$

$$\frac{1}{F^4}\cdot \overset{(2)i}{C}_{jk} + \overset{(5)i}{C}_{jk} = 0 \tag{53}$$

$$\overset{(1)i}{C}_{jk} = \overset{(3)i}{C}_{jk} = \overset{(4)i}{C}_{jk} = \overset{(6)i}{C}_{jk} = 0 \tag{54}$$

Using the result obtained above, there can proved that:

Theorem 2.3. *A d-linear connection on E is compatible with* $\overset{(0)}{F}$ *structure if and only if there are achieved the following relations*:

$$\overset{(H)i}{L}_{jk} - \overset{(v_2)i}{L}_{jk} - \frac{2}{F^2}\cdot X_k^p \cdot y_p^{(1)} \cdot \delta_j^i = 0 \tag{55}$$

$$\overset{(H)i}{F}_{jk} - \overset{(v_2)i}{F}_{jk} - \frac{2}{F^2}\cdot y_k^{(1)} \cdot \delta_j^i = 0 \tag{56}$$

$$\overset{(H)i}{C}_{jk} - \overset{(v_2)i}{C}_{jk} = 0 \tag{57}$$

In what follows we deal with the problem of normality of $\overset{(0)}{F}$ structure. In [16] there was proved the following:

Theorem 2.4. $\overset{(\mu)}{F}$ *$\mu -$almost $2-\pi$ structure is normal if and only if d-tensor fields X^i_j, Y^i_j, which appear within the coefficients of N nonlinear connection, are solutions of the following system of equations with partial derivatives*:

$$X^i_{jk} = -\underset{(01)}{\overset{(0)}{R}}{}^i_{jk}, \quad \overset{(1)}{X}{}^i_{jk} = \underset{(11)}{\overset{(0)}{B}}{}^i_{jk}, \quad \overset{(2)}{X}{}^i_{jk} = -\underset{(21)}{\overset{(0)}{B}}{}^i_{jk} \tag{58}$$

$$(XY)^i_{jk} = -\underset{(02)}{\overset{(0)}{R}}{}^i_{jk}, \quad \underset{(12)}{(XY)}{}^i_{jk} = -\underset{(12)}{\overset{(0)}{B}}{}^i_{jk}, \quad \underset{(22)}{(XY)}{}^i_{jk} - \underset{(22)}{(XY)}{}^i_{kj} = 0 \tag{59}$$

$$\overset{(1)}{X}{}^i_{[jk]} = -\underset{(12)}{\overset{(0)}{R}}{}^i_{jk}, \quad \mu \text{ is a constant function} \tag{60}$$

Using this theorem, so that $\overset{(0)}{F}$ structure should be normal considering Definition 2.1., we conclude that a necessary condition, is that Finsler function associated to the initial Riemannian space should be constant.

3. $2-x$ homogeneous metrical structures on T^2M

For the beginning, we mention that these can be easily show that the pair $\left(\overset{(0)}{G}, \overset{(0)}{F}\right)$ is a $2-\pi$ metrical structure considering the definition in [16]. The fact that it is determined by a $2-\pi$ homogeneous structure we can call it $2-\pi$ homogeneous metrical structure of the prolongation of second order of Riemannian space. In the following part we like to determine the distinguished linear connections compatible with $2-\pi$ homogeneous metrical structure $\left(\overset{(0)}{G}, \overset{(0)}{F}\right)$. For this target we shall suppose that the following systems of equations have solutions in the set of distinguished tensor fields of second order:

$$\Omega^{ih}_{rj} \cdot X^r_{hk} = \theta^i_{jk} + X^0_k \cdot \delta^i_j \tag{61}$$

$$\Omega^{ih}_{rj} \cdot Y^r_{hk} = \Lambda^i_{jk} + y^1_k \cdot \delta^i_j \tag{62}$$

Theorem 3.1. *The set of the distinguished linear connections compatible with* $\left(\overset{(0)}{G},\overset{(0)}{F}\right)$ $2-\pi$ *homogeneous metrical structure is determined by the following relations*:

$$\overset{(H,0)}{L}{}^i_{jk} = \gamma^i_{jk} + \Omega^{ih}_{rj}\cdot \overset{(H)}{I}{}^r_{hk},\quad \overset{(v_1,0)}{L}{}^i_{jk} = \gamma^i_{jk} + \frac{1}{F^2}\cdot\theta^i_{jk} + \Omega^{ih}_{rj}\cdot \overset{(v_1)}{I}{}^r_{hk} \tag{63}$$

$$\overset{(v_2,0)}{L}{}^i_{jk} = \gamma^i_{jk} + \frac{2}{F^2}\cdot\theta^i_{jk} + \Omega^{ih}_{rj}\cdot\left(\overset{(H)}{I}{}^r_{hk} - \frac{2}{F^2}\cdot\overset{(0)}{X}{}^r_{hk}\right) \tag{64}$$

$$\overset{(H,0)}{F}{}^i_{jk} = \Omega^{ih}_{rj}\cdot \overset{(H)}{J}{}^r_{hk},\quad \overset{(v_1,0)}{F}{}^i_{jk} = -\frac{1}{F^2}\cdot\Lambda^i_{jk} + \Omega^{ih}_{rj}\cdot \overset{(v_1)}{J}{}^r_{hk} \tag{65}$$

$$\overset{(v_2,0)}{F}{}^i_{jk} = -\frac{2}{F^2}\cdot\Lambda^i_{jk} + \Omega^{ih}_{rj}\cdot\left(\overset{(H)}{J}{}^r_{hk} - \frac{2}{F^2}\cdot\overset{(0)}{Y}{}^r_{hk}\right) \tag{66}$$

$$\overset{(H,0)}{C}{}^i_{jk} = \Omega^{ih}_{rj}\cdot\overset{(0)}{H}{}^r_{hk},\quad \overset{(v_1,0)}{C}{}^i_{jk} = \Omega^{ih}_{rj}\cdot\overset{(v_1)}{H}{}^r_{hk},\quad \overset{(v_2,0)}{C}{}^i_{jk} = \Omega^{ih}_{rj}\cdot\overset{(0)}{H}{}^r_{hk} \tag{67}$$

where $\overset{(H)}{I}{}^r_{hk}$, $\overset{(v_1)}{I}{}^r_{hk}$, $\overset{(H)}{J}{}^r_{hk}$, $\overset{(v_1)}{J}{}^r_{hk}$, $\overset{(0)}{H}{}^r_{hk}$, $\overset{(v_1)}{H}{}^r_{hk}$ *are arbitrary d-tensor fields, and* $\overset{(0)}{X}{}^r_{hk}$, $\overset{(0)}{Y}{}^r_{hk}$ *are the solutions of the systems of equations* (61) *and* (62).

Open Problems and Comments

A. There is still open the problem of normality of $\overset{(0)}{F}$ structure, in the meaning of the determination of the basic Riemannian space and of the distinguished fields which enter the composition of nonlinear connection of the prolongation of second order of the mentioned Riemannian space.

B. The local components of the curvature and torsion fields of $D(0)-$metrical connection from Theorem 1.6. can be used for the developement of same theories of field on the geometrical model offered by the prolongation of second order of Riemannian space provided for $\left(\overset{(0)}{G},\overset{(0)}{F}\right)$ $2-\pi$ homogeneous metrical structure introduced in the last section. The same idea can be applied to the previous geometrical model provided for the distinguished connection determined through Theorem 3.1.

C. As task there should be studied the conditions in which the systems (61) and (62) are compatible.

D. There is also to be studied an analogy between the connections determined in this paper and those which can be obtained considering the idea in [6].

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