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On Semi θ-Perfect Functions

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Abstract. In this paper, we define and investigate a new class of functions called semi θ-perfect functions and also obtain the characterizations of locally *s*-closed spaces in weakly Hausdorff spaces.

1. Introduction

In [2], the present authors have defined and investigated locally *s*-closed spaces. In this paper, we define a new class of functions called semi θ-perfect functions and obtain some properties of semi θ-perfect functions. Also, we obtain further characterizations of locally *s*-closed spaces in weakly Hausdorff spaces.

2. Preliminaries

Let X be a topological space and A a subset of X. The closure of A and the interior of A with respect to X are denoted by $Cl_X(A)$ and $\text{Int}_X(A)$ (simply $Cl(A)$ and $\text{Int}(A)$), respectively. A subset A of X is said to be *semi-open* [4] if there exists an open set U of X such that $U \subset A \subset Cl(U)$. The complement of a semi-open set is said to be *semi-closed*. The *semi-closure* of A, denoted by $sCl(A)$, is defined by the intersection of all semi-closed sets containing A. A subset A is said to be *semi-regular* [1] if it is semi-open and semi-closed. The family of all semi-regular sets of X is denoted by $SR(X)$. By $SR(X, x)$, we denote the family of all semi-regular sets of X containing a point $x \in X$. A point x of X is called a semi *θ-adherent point of* A [1] if $A \cap U \neq \phi$ for every $U \in SR(X, x)$. The set of all semi θ-adherent points of A is denoted by $sCl_\theta(A)$. If $A = sCl_\theta(A)$, then A is said to be *semi θ-closed*. A subset A is said to be *regular open* (resp. *regular closed*) if $\text{Int}(Cl(A)) = A$ (resp. $Cl(\text{Int}(A)) = A$). The *δ-closure* of A [9] is defined by the set of $x \in X$ such that $A \cap U \neq \varnothing$ for any regular open set U of X containing x. If A contains the δ-closure of A, then A is said to be *δ-closed* [9].

Definition 2.1. *A subset A of a topological space X is said to be s-closed relative to X* [1] *if for every cover* $\{V_\alpha : \alpha \in \nabla\}$ *of A by semi-open sets of X, there exists a finite subset* ∇_0 *of* ∇ *such that* $A \subset \cup\{sCl_X(V_\alpha) : \alpha \in \nabla_0\}$. *If* $A = X$, *then the space X is said to be s-closed* [1].

Definition 2.2. *A topological space X is said to be locally s-closed* [2] *if each point of X has an open neighborhood which is an s-closed subspace.*

3. Locally *s*-closed spaces

In this section, we obtain characterizations of locally *s*-closed spaces in a weakly Hausdorff space. A topological space *X* is said to be *weakly Hausdorff* [8] if every point of *X* is the intersection of regular closed sets of *X*.

Lemma 3.1. *If X is a weakly Hausdorff space and A is s-closed relative to X, then A is δ-closed in X.*

Proof. Let $x \in X - A$. For each $a \in A$, there exists a regular closed set F_a such that $a \in F_a$ and $x \notin F_a$. Since $F_a \in SR(X)$ and $A \subset \cup\{F_a : a \in A\}$, there exists a finite subset A_0 of *A* such that $A \subset \cup\{F_a : a \in A_0\}$. Now, put $V = X - \cup\{F_a : a \in A_0\}$. Then *V* is a regular open set containing x and $V \cap A = \phi$. Therefore, $x \notin Cl_\delta(A)$ and hence *A* is δ-closed in *X*.

Lemma 3.2. *Let X be a topological space, B s-closed relative to X and* $A \in SR(X)$, *then* $A \cap B$ *is s-closed relative to X.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \nabla\}$ be a cover of $A \cap B$ by semi-regular sets of *X*. Then $\mathcal{U} \cup \{X - A\}$ is a cover of *B* by semi-regular sets of *X*. Since *B* is *s*-closed relative to *X*, there exists a finite subset ∇_0 of ∇ such that $B \subset (X - A) \cup \{U_\alpha : \alpha \in \nabla_0\}$. This implies that $A \cap B \subset \cup\{U_\alpha : \alpha \in \nabla_0\}$. This completes the proof.

A subset *A* of a topological space *X* is said to be *S-closed relative to X* [6] if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of *A* by semi-open sets of *X*, there exists a finite subset ∇_0 of ∇ such that $A \subset \cup\{Cl_X(V_\alpha) : \alpha \in \nabla_0\}$. A topological space *X* is said to be *locally S-closed* [6] if each point of *X* has an open neighborhood which is an *S*-closed subspace. A topological space *X* is said to be *extremally disconnected* if the closure of each open set of *X* is open in *X*. The following lemma is an immediate consequence of Theorem 3.2 in [7] since every locally *s*-closed space is locally *S*-closed.

Lemma 3.3. *A locally s-closed weakly Hausdorff space is extremally disconnected.*

Theorem 3.1. *The following properties are equivalent for a weakly Hausdorff space X:*

(1) X is locally s-closed;

(2) for each $x \in X$ and each neighborhood U of x, there exists an open set V in X such that $Cl(V)$ is s-closed relative to X and $x \in V \subset Cl(V) \subset sCl(U)$;

(3) for each $x \in X$ and each regular open neighborhood U of x, there exists an open set V in X such that $Cl(V)$ is s-closed relative to X and $x \in V \subset Cl(V) \subset U$;

(4) for each set C s-closed relative to X and each regular open set U containing C, there exists an open set V in X such that V is s-closed relative to X and $C \subset V \subset sCl(V) \subset U$.

Proof. $(1) \Rightarrow (2)$: Let X be locally s-closed, $x \in X$ and U a neighborhood of *x*. There exists an open set G of X such that $x \in G \subset U$. Since *X* is locally *s*-closed, by Theorem 3.1 of [2] there exists an open set *W* containing *x* which is *s*-closed relative to *X*. Put $V = W \cap G$, then V is an open set containing x and $Cl(V) \subset Cl(W) = W$ because *W* is δ-closed and hence closed, by Lemma 3.1. Since $Cl(V)$ is semi-regular, by Lemma 3.2 $Cl(V)$ is *s*-closed relative to *X*. Moreover, by Lemma 3.3 X is extremally disconnected and hence $Cl(V) \subset Cl(G) = \mathrm{Int}(Cl(G)) = sCl(G) \subset sCl(U)$ [1, Lemma 2.1].

$(2) \Rightarrow (3)$: This is obvious.

$(3) \Rightarrow (4)$: For each $c \in C$, there exists an open set V_c such that $Cl(V_c)$ is *s*-closed relative to X and $c \in V_c \subset sCl(V_c) \subset Cl(V_c) \subset U$. Since $sCl(V_c)$ is semi-regular, by Lemma 3.2 $sCl(V_c)$ is *s*-closed relative to *X*. Since *C* is *s*-closed relative to *X*, there exists a finite subset C_0 of C such that $C \subset \cup\{sCl(V_c) : c \in C_0\}$. Now, put $V = \cup\{sCl(V_c) : c \in C_0\}$, then V is open in *X*, *s*-closed relative to X and $C \subset V \subset sCl(V) \subset U$.

$(4) \Rightarrow (1)$: A point is certainly *s*-closed relative to *X*. Let *x* be any point of *X*. Since *X* is regular open, there exists an open set *V* containing *x* such that *V* is *s*-closed relative to *X*. Therefore, by Theorem 3.1 of [2], *X* is locally *s*-closed.

4. Semi θ-perfect functions

Definition 4.1. *A function $f : X \to Y$ is said to be semi θ-closed [1] if $f(A)$ is semi θ-closed in Y for every semi θ-closed set A of X.*

Theorem 4.1. *A function $f : X \to Y$ is semi θ-closed if and only if for every subset A of X $sCl_\theta(f(A)) \subset f(sCl_\theta(A))$.*

Proof. The proof is straightforward and is thus omitted.

A filterbase F on a topological space *X* is said to be *SR-convergent* to $x \in X$ [1] if for each $V \in SR(X, x)$, there exists $F \in \mathrm{F}$ such that $F \subset V$. A point $x \in X$ is called a *semi θ-adherent point* of a filterbase F on *X* if $x \in [\text{sad}]_\theta \mathrm{F} = \cap\{sCl_\theta(F) : F \in \mathrm{F}\}$. A filterbase F is said to be semi *θ-directed toward* $S \subset X$ if every filterbase subordinate to F has a semi *θ*-adherent point in *S*.

Definition 4.2. *A function* $f : X \to Y$ *is said to be semi θ-perfect if for every filterbase F in* $f(X)$ *SR-converging to* $y \in Y$, $f^{-1}(\mathrm{F})$ *is semi θ-directed toward* $f^{-1}(y)$.

Theorem 4.2. *For a function* $f : X \to Y$, *the following properties are equivalent:*
(1) f is semi θ-closed and $f^{-1}(y)$ *is s-closed relative to X for every* $y \in Y$;
(2) $[\text{sad}]_\theta f(\mathrm{F}) \subset f([\text{sad}]_\theta \mathrm{F})$ *for every filterbase F on X.*

Proof. $(1) \Rightarrow (2)$: Let F be any filterbase on *X* and suppose that $y \notin f([\text{sad}]_\theta \mathrm{F})$. Then for each $x \in f^{-1}(y)$, there exist $S_x \in SR(X, x)$ and $F_x \in \mathrm{F}$ such that $F_x \cap S_x = \phi$. The family $\{S_x : x \in f^{-1}(y)\}$ is a cover of $f^{-1}(y)$ by semi-regular sets of *X*. Since $f^{-1}(y)$ is *s*-closed relative to *X*, there exist points $x_1, x_2, \cdots, x_n$ such that $f^{-1}(y) \subset \cup\{S_{x_i} : 1 \le i \le n\}$. There exists $F \in \mathrm{F}$ such that $F \subset \cap\{F_{x_i} : 1 \le i \le n\}$. Therefore, we have $F \cap [\cup\{S_{x_i} : 1 \le i \le n\}] = \phi$ and hence $f^{-1}(y) \cap sCl_\theta(F) = \phi$. Thus we obtain $y \notin f(sCl_\theta(F))$. Since *f* is semi *θ*-closed, by Theorem 4.1 we have $y \notin sCl_\theta(f(F))$ and hence $y \notin [\text{sad}]_\theta f(\mathrm{F})$.
$(2) \Rightarrow (1)$: First, we show that $f^{-1}(y)$ is *s*-closed relative to *X* for each $y \in Y$. Let F be any filterbase on *X* which meets $f^{-1}(y)$. Then $y \in f(F)$ for each $F \in \mathrm{F}$. Therefore, we have $y \in [\text{sad}]_\theta f(\mathrm{F}) \subset f([\text{sad}]_\theta \mathrm{F})$ and hence $[\text{sad}]_\theta \mathrm{F} \cap f^{-1}(y) \ne \phi$. It follows from Proposition 4.1 of [1] that $f^{-1}(y)$ is *s*-closed relative to *X*. Next, we show that *f* is semi *θ*-closed. Let *A* be any nonempty subset of *X* and F a filterbase on *X* consisted of only *A*. Then, we have $sCl_\theta(f(A)) \subset f(sCl_\theta(A))$. By Theorem 4.1, *f* is semi-*θ*-closed.

Theorem 4.3. *A function* $f : X \to Y$ *is semi θ-perfect if* $[\text{sad}]_\theta f(\mathrm{F}) \subset f([\text{sad}]_\theta \mathrm{F})$ *for every filterbase F on X.*

Proof. Suppose that $[\text{sad}]_\theta f(\mathcal{F}) \subset f([\text{sad}]_\theta \mathcal{F})$ for every filterbase F on *X*. Assume that *f* is not semi θ-perfect. Then, there exists a filterbase F in $f(X)$ such that F SR-converges to a point $y \in Y$ but $f^{-1}(\mathcal{F})$ is not semi θ-directed toward $f^{-1}(y)$. Thus there exists a filterbase G on *X* which is subordinate to $f^{-1}(\mathcal{F})$ and $f^{-1}(y) \cap [\text{sad}]_\theta \mathcal{G} = \phi$. Therefore, we have $y \notin f([\text{sad}]_\theta \mathcal{G})$ and hence $y \notin [\text{sad}]_\theta f(\mathcal{G})$. Thus $y \notin sCl_\theta(f(G_1))$ for some $G_1 \in \mathcal{G}$. Then, there exists $V \in SR(Y, y)$ such that $V \cap f(G_1) = \phi$. Since F SR-converges to *y* and G is subordinate to $f^{-1}(\mathcal{F})$, there exists $G_2 \in \mathcal{G}$ such that $f(G_2) \subset V$. Consequently, we obtain $G_1 \cap G_2 = \phi$. This contradicts that G is a filterbase. This proves that *f* is semi θ-perfect.

Corollary 4.1. *If $f : X \to Y$ is a semi θ-closed function such that $f^{-1}(y)$ is s-closed relative to X for each $y \in Y$, than f is semi θ-perfect.*

Theorem 4.4. *If $f : X \to Y$ is a semi θ-perfect function and Y is extremally disconnected, then $[\text{sad}]_\theta f(\mathcal{F}) \subset f([\text{sad}]_\theta \mathcal{F})$ for every filterbase F on X.*

Proof. Suppose that $f : X \to Y$ is a semi θ-perfect function and *Y* is extremally disconnected. Let F be a filterbase on *X* and $y \in [\text{sad}]_\theta f(\mathcal{F})$. Now, put $G = \{V \cap f(F) : V \in SR(Y, y),\ F \in \mathcal{F}\}$. Then, since *Y* is extremally disconnected, G is a filterbase in $f(X)$ which is subordinate to $f(\mathcal{F})$ and SR-converges to *y*. Let us put $\mathcal{H} = \{f^{-1}(G) \cap F : F \in \mathcal{F}, G \in \mathcal{G}\}$. Then H is a filterbase on *X* subordinate to $f^{-1}(\mathcal{G})$. Since *f* is semi θ-perfect, $f^{-1}(\mathcal{G})$ is semi θ-directed toward $f^{-1}(y)$. Therefore, we have $f^{-1}(y) \cap [\text{sad}]_\theta \mathcal{H} \neq \phi$ and hence $y \in f([\text{sad}]_\theta \mathcal{F})$. This shows that $[\text{sad}]_\theta f(\mathcal{F}) \subset f([\text{sad}]_\theta \mathcal{F})$.

Corollary 4.2. *Let Y be an extremally disconnected space. Then, the following properties are equivalent for a function $f : X \to Y$:*

(1) *$f : X \to Y$ is semi θ-perfect;*

(2) *$[\text{sad}]_\theta f(\mathcal{F}) \subset f([\text{sad}]_\theta \mathcal{F})$ for every filterbase F on X;*

(3) *f is a semi θ-closed function such that $f^{-1}(y)$ is s-closed relative to X for each $y \in Y$.*

Proof. This is an immediate consequence of Theorems 4.2, 4.3 and 4.4.

A function $f : X \to Y$ is said to be *weakly continuous* [3] if for each point $x \in X$ and each open set *V* of *Y* containing $f(x)$ there exists an open set *U* containing *x* such that $f(U) \subset Cl(V)$.

Theorem 4.5. *If a function $f : X \to Y$ is weakly continuous and semi θ-perfect and Y is a locally s-closed weakly Hausdorff space, then X is locally s-closed.*

Proof. Let *x* be any point of *X*. Since *Y* is regular open, by Theorem 3.1 there exists an open set *V* of *Y* such that $Cl(V)$ is *s*-closed relative to *Y* and $f(x) \in V \subset Cl(V)$. Since f is weakly continuous, by Theorem 1 of [3] we have $f^{-1}(V) \subset \text{Int}(f^{-1}(Cl(V)))$. Since *Y* is locally *s*-closed weakly Hausdorff, by Lemma 3.3 *Y* is extremally disconnected and hence $Cl(V)$ is open in *Y*. Therefore, by Theorem 4 of [5] we obtain $Cl(f^{-1}(Cl(V))) \subset f^{-1}(Cl(V))$ and hence $x \in f^{-1}(V) \subset \text{Int}(Cl(f^{-1}(Cl(V))))$ $\subset f^{-1}(Cl(V))$. Since f is semi θ-perfect, it follows from Theorem 4.7 of [2] and Corollary 4.2 that $f^{-1}(Cl(V))$ is *s*-closed relative to *X*. Now, put $U = \text{Int}(Cl(f^{-1}(Cl(V))))$. Then, since *U* is semi-regular, by Lemma 3.2 *U* is *s*-closed relative to *X* and an open set containing *x*. Therefore, by Theorem 3.1 of [2] *X* is locally *s*-closed.

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